Chapter 7

More Stochastic-Process Limits

7.1. Introduction

This chapter is a sequel to Chapter 4: We continue providing an overview of established stochastic-process limits. These stochastic-process limits are of interest in their own right, but they also can serve as initial stochastic-process limits in the continuous-mapping approach to establish new stochastic-process limits.

We start in Section 7.2 by considering a different kind of stochastic-process limit: We consider CLT's for sums of stochastic processes with sample paths in D. We apply these CLT's for processes to treat queues with multiple sources, where the number of sources increases in the heavy-traffic limit; see Sections 8.7 and 9.8.

In Section 7.3 we extend the discussion begun in Section 6.3 about stochastic-process limits for counting processes. As indicated before, stochastic-process limits for counting processes follow directly from stochastic-process limits for random walks, provided that the M_1 topology is used. We will apply the stochastic-process limits for counting processes to obtain heavy-traffic limits for the standard queueing models in Chapters 9 and 10.

In Section 7.4 we apply convergence-preservation results for the composition and inverse maps to establish stochastic-process limits for renewal-reward stochastic processes. Renewal-reward stochastic processes are random sums of IID random variables, where the random index is a renewal counting process. When the times between the renewals in the renewal counting process have a heavy-tailed distribution, we need the M_1 topology.

The Internet Supplement contains additional material supplementing Chapter 4: In Section 2.2 of the Internet Supplement we discuss strong approximations, which can be used to establish bounds on the rate of convergence in Donsker's FCLT expressed in terms of the Prohorov metric. In Section 2.3 of the Internet Supplement we extend Section 4.4 by presenting additional Brownian stochastic-process limits with weak dependence. In Section 2.4 of the Internet Supplement we discuss the convergence to general Lévy processes that can be obtained when we consider a sequence of random walks. These limits are applied to queues in Section 5.2 of the Internet Supplement.

7.2. Central Limit Theorem for Processes

In this section we consider a different kind of stochastic-process limit. Instead of considering scaled partial sums of random vectors, we consider CLTs for partial sums of random elements of D, i.e., we consider the limiting behavior of the scaled partial sum

$$\mathbf{Z}_n(t) \equiv n^{-1/2} \sum_{i=1}^{n} [X_i(t) - EX_i(t)], \quad t \geq 0 , \tag{2.1}$$

where $\{X_n : n \geq 1\} \equiv \{\{X_n(t) : t \geq 0\} : n \geq 1\}$ is a sequence of IID random elements of $D \equiv D([0, \infty), \mathbb{R})$.

In fact, we already discussed a special case of IID random elements of D when we considered the Kolmogorov-Smirnov statistic in Section 1.5. Recall that the emprical process associated with a sample of size n from a cdf F can be expressed as

$$F_n(t) \equiv \sum_{i=1}^{n} I_{(-\infty,t]}(Y_i), \quad t \in \mathbb{R} , \tag{2.2}$$

where $\{Y_i\}$ is a sequence of IID real-valued random variables with cdf F.

For the normalized partial sums in (2.1), convergence of finite-dimensional distributions is elementary by the multidimensional CLT in Section 4.3, provided that $E[X_1(t)^2] < \infty$. As indicated in Section 11.6, to establish convergence in distribution in D with an appropriate topology, it only remains to establish tightness. Specifically, we can apply Theorem 11.6.6.

7.2.1. Hahn's Theorem

The following result is due to Hahn (1978). We have converted the original result from $D([0, 1], \mathbb{R})$ to $D([0, \infty), \mathbb{R})$.

Theorem 7.2.1. (CLT for processes in D) *Let $\{X_n : n \geq 1\}$ be a sequence of IID random elements of $D \equiv D([0,\infty),\mathbb{R})$ with $EX_i(t) = 0$ and $E[X_i(t)^2] < \infty$ for all t. If, for all T, $0 < T < \infty$, there exist continuous nondeceasing real-valued functions g and f on $[0,T]$ and numbers $\alpha > 1/2$ and $\beta > 1$ such that*

$$E[(X(u) - X(s))^2] \leq (g(u) - g(s))^\alpha \tag{2.3}$$

and

$$E[(X(u) - X(t))^2(X(t) - X(s))^2] \leq (f(u) - f(t))^\beta \tag{2.4}$$

for all $0 \leq s \leq t \leq u \leq T$ with $u - s < 1$, then

$$\mathbf{Z}_n \Rightarrow \mathbf{Z} \quad in \quad (D, J_1) \ , \tag{2.5}$$

where $\mathbf{Z}_n$ is the normalized partial sum in (2.1)*, $\mathbf{Z}$ is a mean-zero Gaussian process with the covariance function of X_1 and $P(\mathbf{Z} \in C) = 1$.*

Remark 7.2.1. *More elementary conditions.* The canonical sufficient conditions for (2.3) and (2.4) in applications of Theorem 7.2.1 have

$$f(t) = g(t) = Kt, \quad t \geq 0, \quad \alpha = 1 \quad \text{and} \quad \beta = 2 \ ,$$

for some constant K (depending upon T), yielding

$$E[(X(u) - X(s))^2] \leq K(u - s)) \tag{2.6}$$

and

$$E[(X(u) - X(t))^2(X(t) - X(s))^2] \leq K(u - t)^2 \tag{2.7}$$

for $0 \leq s \leq t \leq u \leq T$ with $u - s < 1$.

Note that conditions (2.6) and (2.7) apply to treat the empirical process in (2.2). There

$$E[(X(u) - X(s))^2] = P(s < Y_1 \leq u)$$

and

$$E[(X(u) - X(t))^2(X(t) - X(s))^2] = P(t < Y_1 \leq u \quad \text{and} \quad s < Y_1 \leq t) = 0 \ .$$

We see that condition (2.6) holds whenever the cdf F has a bounded density. However, the different approach in Section 1.5 shows that convergence in (D, J_1) actually holds whenever the cdf F is continuous. Plots of the scaled empirical process for the uniform cdf in Figure 1.8 illustrate Theorem 7.2.1. The limiting Gaussian process in that case is the Brownian bridge. ∎

Remark 7.2.2. *Extensions.* An analog of Theorem 7.2.1 for stable process limits in (D, J_1) when $E[X(t)^2] = \infty$ has been established by Bloznelis (1996). That opens the way for limits with jumps. It remains to develop conditions for the M topologies. Other extensions of Theorem 7.2.1 are contained in Bloznelis (1996), Bloznelis and Paulauskas (2000) and references therein; e.g., see Bass and Pyke (1987). See Araujo and Giné (1980) for CLTs for random elements of general Banach spaces. ■

We now state some consequences of Theorem 7.2.1. We first apply Theorem 7.2.1 to establish a CLT for stochastic processes with smooth sample paths, such as cumulative-input stochastic processes to fluid queues. In that context, a standard model for the input from one source is an on-off model, in which there are alternating random on and off periods. During on periods, input arrives at a constant rate; during off periods there is no input. It is customary to assume that the successive on and off periods come from independent sequences of IID random variables, but we do not need to require that here. A generalization is to allow the source environment be governed by a k-state process instead of a two-state process. When the environment state is j, the input is transmitted at constant rate r_j. For example, the environment process might be a finite-state semi-Markov process; see Duffield and Whitt (1998, 2000). Again we do not require such specific assumptions.

Let $X(t)$ be the total input over the interval $[0, t]$. Since the input occurs at a random rate, with only finitely many possible rates, the sample paths are Lipschitz with probability one, i.e.,

$$|X(t) - X(s)| \leq K(t - s) \quad w.p.1 \tag{2.8}$$

for all $0 \leq s \leq t$, where K is the maximum possible rate. We are interested in the CLT (2.5) to describe the aggregate input from a large number of sources.

Corollary 7.2.1. (CLT for Lipschitz processes) *If $\{X_n\}$ is a sequence of IID random elements of C satisfying* (2.8)*, then the CLT* (2.5) *holds.*

Proof. It is easy to see that conditions (2.6) and (2.7) hold. Indeed, (2.8) implies that

$$E[(X(u) - X(s))^2] \leq K(u - s)^2 \leq K(u - s)$$

and

$$E[(X(u)-X(t))^2(X(t)-X(s))^2] \leq K(u-t)^2(t-s)^2 \leq K(u-s)^4 \leq K(u-s)^2 \, .$$

■

Hahn (1978) applied Theorem 7.2.1 to establish the following CLTs for Markov processes. For any real-valued random variable Y, let the *essential supremum* be

$$ess\ sup\,(Y) \equiv \inf\{c : P(Y > c) = 0\} \ .$$

Theorem 7.2.2. (CLT for Markov processes) *Let* $\{X_n : n \geq 1\}$ *be a sequence of IID Markov processes with sample paths in* D. *If, for each* T, $0 < T < \infty$, *there exists a continuous nondecreasing real-valued function* g *on* $[0, T]$ *and a number* $\alpha > 1/2$ *such that either*

$$ess\ sup\, E[(X(t) - X(s))^2 | X(s)] \leq (g(t) - g(s))^{\alpha} \tag{2.9}$$

or

$$ess\ sup\, E[(X(t) - X(s))^2 | X(t)] \leq (g(t) - g(s))^{\alpha} \ , \tag{2.10}$$

for $0 \leq s \leq t \leq T$, *with* $t - s < 1$, *then conditions* (2.3) *and* (2.4) *hold for* $X(t) - EX(t)$, *so that the conclusion of Theorem 7.2.1 holds.*

Hahn also observed that Theorem 7.2.2 applies directly to finite-state CTMCs.

Corollary 7.2.2. (CLT for finite-state CTMCs) *If* $\{X_n : n \geq 1\}$ *be a sequence of IID finite-state continuous-time Markov chains determined by an infinitesimal generator matrix* Q, *Then the conditions of Theorem* 7.2.2 *hold, with* $g(t) = t$ *and* $\alpha = 1$ *in* (2.9) *and* (2.10), *so that the conclusion of Theorem* 7.2.1 *holds.*

Theorem 7.2.1 was also applied by Whitt (1985a) to obtain the following CLT for stationary renewal processes.

Theorem 7.2.3. (CLT for stationary renewal processes) *Let* $\{X_n : n \geq 1\} \equiv \{\{X_n(t) : t \geq 0\} : n \geq 1\}$ *be a sequence of IID stationary renewal counting processes with interrenewal-time cdf* F. *If*

$$\overline{\lim_{t \to 0}}\ t^{-1}(F(t) - F(0)) < \infty \ , \tag{2.11}$$

then conditions (2.6) *and* (2.7) *hold, so that the conclusion of Theorem* 7.2.1 *holds.*

Remark 7.2.3. Whitt (1985a) showed that condition (2.11) is necessary for condition (2.7) to hold. Note that condition (2.11) allows an atom at 0 and is satisfied if the cdf F is otherwise absolutely continuous in a neighborhood of 0. Hence condition (2.11) is not very restrictive. ∎

From Theorems 7.2.1–7.2.3, we know that the limit process $\mathbf{Z}$ is a zero-mean Gaussian process with sample paths in C and the covariance function of one of the summands. Hence, to fully characterize the limit process it suffices to determine the covariance function of the original component process.

For example, let us consider the case of Theorem 7.2.3. There it suffices to determine the covariance function of a component stationary renewal process. For a stationary renewal process, now denoted by $\{A(t) : t \geq 0\}$, the covariance function can be computed from the interrenewal-time cdf F, exploiting numerical transform inversion to compute the renewal function; see Chapter 4 of Cox (1962) and Section 13 of Abate and Whitt (1992a). Recall that the renewal function $M(t)$ is the mean number of renewals in $[0, t]$ for the ordinary renewal process. We characterize the covariance function further in the following theorem.

Theorem 7.2.4. (covariance function for a stationary renewal process) *Suppose that A is a stationary renewal counting process (having stationary increments with $A(0) = 0$) with interrenewal-time cdf F having pdf f and mean λ^{-1}. Then, for $t < u$,*

$$Cov(A(t), A(u)) = Var\, A(t) + Cov(A(t), A(u) - A(t)) \ ,$$

where

$$V(t) \equiv Var\, A(t) = 2\lambda \int_0^t [M(s) - \lambda s + 0.5] ds \ , \tag{2.12}$$

$$E[A(t), A(u) - A(t)] = \lambda^3 \int_0^t da \int_0^u db f(a+b) M(u-b) M(t-a) \tag{2.13}$$

and $M(t)$ is the renewal function of the associated ordinary renewal process, having Laplace transform

$$\hat{M}(s) = \frac{\hat{f}(s)}{s[1 - \hat{f}(s)]} \ ,$$

where

$$\hat{f}(s) \equiv \int_0^\infty e^{-st} f(t) dt \quad and \quad \hat{M}(s) \equiv \int_0^\infty e^{-st} M(t) dt \, .$$

Directly, the Laplace transform of $V(t)$ *is*

$$\hat{V}(s) \equiv \int_0^\infty e^{-st} V(t) dt = 2\lambda \left(\frac{\hat{M}(s)}{s} - \frac{\lambda}{s^3} + \frac{1}{2s^2} \right) .$$

Proof. For (2.12), see p. 57 of Cox (1962). For (2.13), consider the first point to the right of t. It falls at $t+b$ with the stationary-excess (or equilibrium lifetime) pdf $f_e(b) \equiv \lambda F^c(b)$. Conditional on that point being at $t+b$, the first point to the left of t falls at $t-a$ with pdf $f(a+b)/F^c(b)$. Conditional on the loction of these two points at $t-a$ and $t+b$, we can invoke the independence to conclude that the expected value of $A(t)[A(t+u)-A(t)]$ is $\lambda^2 M(t-a)M(u-b)$, where $M(t)$ is the ordinary renewal function (expected number of renewals in $[0,t]$). Integrating over all possible pairs (a,b) gives (2.13). ■

7.2.2. A Second Limit

In many of the CLTs for processes, the component random elements of D have stationary increments. Then the limiting Gaussian process will also have stationary increments in addition to continuous sample paths, so that it is natural to consider an additional stochastic-process limit for the Gaussian process with time scaling; i.e., given $\mathbf{Z}_n \Rightarrow \mathbf{Z}$ as in (2.5), we can consider

$$\mathbf{Y}_n \Rightarrow \mathbf{Y} \quad \text{in} \quad (C, U) , \tag{2.14}$$

where

$$\mathbf{Y}_n(t) \equiv c_n^{-1} \mathbf{Z}(nt), \quad t \geq 0 . \tag{2.15}$$

Alternatively, if the component processes are stationary processes, then the limit process $\mathbf{Z}$ will be stationary, so that we have (2.14) with $\mathbf{Y}_n$ defined by

$$\mathbf{Y}_n(t) \equiv c_n^{-1} \int_0^{nt} \mathbf{Z}(s), \quad t \geq 0 . \tag{2.16}$$

To obtain the second stochastic-process limit, we can often apply Theorem 4.6.2. The second limit allows us to replaces a Gaussian process with a general covariance function by a special Gaussian process – fractional Brownian motion – with the highly structured covariance function in (6.13). As with the first limit, we gain simplicity but lose structural detail by taking the limit. In considerable generality, we see that the large-time-scale behavior of the aggregate process should be like FBM. The second limit provides important new insight when that FBM is not Brownian motion.

We now illustrate by applying Corollary 7.2.1 and Theorem 4.6.2 to establish a double stochastic-process limit in (D, J_1) for the input from many on-off sources with heavy-tailed on-period or off-period distributions. Convergence of the finite-dimensional distributions was established by Taqqu, Willinger and Sherman (1997). As indicated there and in Willinger, Taqqu, Sherman and Wilson (1997), the stochastic-process limit is very helpful to understand the strong dependence and self-similarity observed in network traffic measurements, such as in Leland et al. (1994). The stochastic-process limit shows how high variability (the Noah effect) in the on and off periods can lead to strong positive dependence (the Joseph effect) in the cumulative input process. As indicated in Section 4.2, the very existence of the limit implies that the limit process must be self-similar.

The extension here to weak convergence from only convergence of the finite-dimensional distributions is important for establishing further stochastic-process limits, in particular, heavy traffic limits for queues with input from many on-off sources.

Now we assume that the on periods and off periods come from independent sequences of IID random variables. We let the input rate be 1 during each on period and 0 during each off period. Let the on periods have cdf F_1, ccdf $F_1^c \equiv 1 - F_1$ and finite mean m_1; let the off periods have cdf F_2, ccdf $F_2^c \equiv 1 - F_2$ and finite mean m_2. We assume that the cdf's have probability density functions, although that can be generalized; e.g., see Section VI.1 of Asmussen (1987).

The critical assumtion is that the ccdf's F_1 and F_2 can have heavy tails. Specifically, we assume that the cdf F_i either has finite variance σ_i^2 or that

$$F_i^c(t) \sim c_i t^{-\alpha_i} \quad \text{as} \quad t \to \infty$$

for $1 < \alpha < 2$. (That can be generalized to regularly varying tails; see Taqqu, Willinger and Sherman (1997).)

We now specify the limiting scaling constant. When $\sigma^2 < \infty$, let $\alpha_i = 2$ and $a_i = c_i \Gamma(2 - \alpha_i)/(\alpha_i - 1)$, where Γ is the gamma function. When $\alpha_1 = \alpha_2$, let $\alpha_{min} = \alpha_1$ and

$$\sigma_{lim}^2 \equiv \frac{2(m_1^2 + m_2^2)}{(m_1 + m_2)^3 \Gamma(4 - \alpha_{min})} \ . \tag{2.17}$$

When $\alpha_1 \neq \alpha_2$, let $\alpha_{min} \equiv \alpha_1 \wedge \alpha_2 \equiv \min \alpha_1, \alpha_2$ and

$$\sigma_{lim}^2 \equiv \frac{2m_{max}^2 a_{min}}{(m_1 + m_2)^3 \Gamma(4 - \alpha_{min})} , \tag{2.18}$$

where (min, max) is the pair of indices $(1,2)$ if $\alpha_1 < \alpha_2$ and $(2,1)$ if $\alpha_2 < \alpha_1$.

Let $X(t)$ be the cumulative input from one on-off source over the interval $[0,t]$. We assume that the process X has been initialized so that X has stationary increments. Then

$$EX(t) = m_1/(m_1 + m_2) \quad \text{for all} \quad t \geq 0 .$$

(That can be generalized as well.) We are interested in the limiting behavior of scaled sum of IID versions of this cumulative-input stochastic process, in particular,

$$\mathbf{Z}_{n,\tau} \equiv \tau^{-H} n^{-1/2} \sum_{i=1}^{n} [X_i(\tau t) - m_1 \tau t/(m_1 + m_2)] . \tag{2.19}$$

Theorem 7.2.5. (iterated limit for time-scaled sum of on-off cumulative-input processes) *If $\{X_i : i \geq 1\}$ is a sequence of IID cumulative-input stochastic processes satisfying the assumptions above, then*

$$\mathbf{Z}_{n,\tau} \Rightarrow \sigma_{lim} \mathbf{Z}_H \quad in \quad (D, J_1)$$

as first $n \to \infty$ and then $\tau \to \infty$ for

$$H = (3 - \alpha_{min})/2 , \tag{2.20}$$

$\mathbf{Z}_{n,\tau}$ in (2.19), σ^2 in (2.17) or (2.18) and $\mathbf{Z}_H$ standard FBM.

Proof. First the CLT in D as $n \to \infty$ follows from Corollary 7.2.1. As a consequence, the limit process, say Y, has paths in C. We establish weak convergence as $\tau \to \infty$ in (C, U) by applying Theorem 4.6.2. Convergence of the finite-dimensional distributions was established by Taqqu, Willinger and Sherman (1997). As an important part of that step, they established (6.23) for H in (2.20). They claim to establish weak convergence in the second limit as $\tau \to \infty$ by applying Theorem 11.6.5 using only (6.23), but their argument at the end of Section 3 has a gap, because it does not control the small-time behavior. However, that gap can be filled quite easily by establishing (6.24). First, by (6.23), there exists t_0 such that

$$VarY(t) \leq 2ct^{2H} \quad \text{for all} \quad t > t_0 \ .$$

However, given t_0 (where, without loss of generality we may assume that t_0 is a positive integer),

$$VarY(t) \leq t_0^2 VarY(t/t_0) \quad \text{for all} \quad t, \quad 0 \leq t \leq t_0 \ ,$$

by writing $Y(t)$ as the sum of t_0 random variables $Y(kt/t_0) - Y((k-1)t/t_0)$, $1 \leq k \leq t_0$. Hence, it suffices to consider $t \leq 1$. However, by the Lipschitz sample-path structure of X_1,

$$|X_1(t)| \leq t \quad w.p.1 \ ,$$

so that

$$VarY(t) = VarX_1(t) \leq t^2 \ ,$$

which is less than t^{2H} for $t \leq 1$. ∎

Theorem 7.2.5 involves an iterated limit, in which first $n \to \infty$ and then afterward $\tau \to \infty$. Mikosch et al. (2001) consider the double limit with $\tau_n \to \infty$ as $n \to \infty$. They show that convergence to FBM still holds when $\tau_n \to \infty$ slowly enough. See Remark 8.7.1 for further discussion.

Theorem 7.2.5 extends to more general source traffic models, such as when the rate process is a finite-state semi-Markov process (SMP), as in Duffield and Whitt (1998, 2000). In that setting we call the SMP environment states *levels*. The self-similarity index H is again determined by the level-holding-time cdf's $F_{i,j}$, giving the distribution of the time spent at level i given that the next level will be j. Assuming that the DTMC governing the state transitions is irreducible, If any of these cdf's has a heavy tail, then we get convergence to FBM with $1/2 < H < 1$. In particular, H again is given by (2.20), where α_{min} is the minimum among the stable indices of the cdf's $F_{i,j}$, assumed to satisfy $1 < \alpha_{min} < 2$.

A related stochastic-process limit for the aggregate input from many sources is in Kurtz (1996); it also leads to FBM under appropriate conditions.

We remark that we encounter difficulties when we try to establish the second limit for the stationary renewal processes treated in Theorem 7.2.3 when $H > 1/2$, because Var $(X(t)) = O(t)$ as $t \to 0$, so that we cannot establish (6.24) for $H > 1/2$. We do get the second limit when $H = 1/2$ though.

Convergence of the finite-dimensional distributions for a more general model was established by Taqqu and Levy (1986).

In this section we have only discussed CLTs for processes that converge to Gaussian processes with continuous sample paths. It is also of interest to establish convergence to stable processes. Such stochastic-process limits (only convergence of finite-dimensional distributions) have been established by Levy and Taqqu (1987, 2000). As indicated before, criteria for weak convergence in D to a stable process have been determined by Bloznelis (1996). More work needs to be done in that area.

7.3. Counting Processes

With queueing models and many other applications, the basic random variables X_n are often nonnegative. For example, in a queueing model X_n may represent a service time, interarrival time, busy period or idle period. With nonnegative random variables, in addition to the partial sums $S_n = X_1 + \cdots + X_n$, $n \geq 1$, with $S_0 = 0$, we are often interested in the associated *counting process* $N \equiv \{N(t) : t \geq 0\}$, defined in (3.1) of Section 6.3. For example, the arrival and service counting processes are used to establish heavy-traffic limits for the standard single-server queue.

When the random variables X_n are nonnegative, we can think of the partial sums S_n as points on the positive halfline $\mathbb{R}_+$. Then $N(t)$ counts the number of points in the interval $[0, t]$. The two processes $\{S_n : n \geq 0\}$ and $\{N(t) : t \geq 0\}$ thus serve as equivalent representations of a *stochastic point process*. When $\{X_n : n \geq 1\}$ is a sequence of IID random variables, the counting process N is also called a *renewal process*.

As indicated in Section 6.3, the partial sums $\{S_n : n \geq 0\}$ and the counting process $\{N(t) : t \geq 0\}$ are inverse processes. Fortunately, we are able to exploit to inverse relation to great advantage for establishing limit theorems. Under minimal regularity conditions, we are able to show that CLTs and FCLTs hold for $\{N(t)\}$ if and only if they hold for $\{S_n\}$, without making any direct probability assumptions about the sequence $\{X_n\}$. These equivalence results are applications of the continuous-mapping approach, which we carefully develop in Chapter 13. Since these limits are also frequently applied in the continuous-mapping approach, we state the key results here.

Before proceeding, however, we point out that, even though the nonnegativity condition on the summands X_j is often natural, it is actually not required to obtain limits for the counting processes from associated limits for the partial sums. Without the nonnegativity, we can go from a CLT or FCLT for partial sums to an associated CLT or FCLT for the associated sequence of successive maxima M_n, where

$$M_n \equiv \max\{S_1, \ldots, S_n\}, \quad n \geq 1 .$$

It turns out that limits for S_n imply corresponding limits for M_n. Then M_n itself can be regarded as a partial-sum process with nonnegative steps, so that we can apply the results in this section to M_n. We then obtain limits for the associated counting process, defined as

$$N(t) \equiv \max\{k : M_k \leq t\}, \quad t \geq 0 .$$

The details are in Chapter 13.

7.3.1. CLT Equivalence

We now return to the case of nonnegative random variables. We first state an equivalence result for CLTs; it is Theorem 3.5.1 in the Internet Supplement, which extends Theorem 6 of Glynn and Whitt (1988) and Theorem 4.2 of Massey and Whitt (1994a). Note that there are no direct probability assumptions on the basic sequence $\{X_k\}$ and the limit is arbitrary. Also note that space scaling is done by a regularly varying function with index p, $0 < p < 1$, which covers the standard scaling by $\sqrt{n}$ and is consistent with the CLT for IID random variables in the domain of attraction of a stable law with index α, $1 < \alpha \leq 2$, in Section 4.5. (See Appendix A for more on regularly varying functions.)

Theorem 7.3.1. (CLT equivalence for partial sums and counting processes) *Suppose that $\{X_n : n \geq 1\}$ is a sequence of nonnegative random variables, $m > 0$ and ψ is a regularly varying real-valued function on $(0, \infty)$ with index p, $0 < p < 1$. Then*

$$\psi(n)^{-1}(S_n - mn) \Rightarrow L \quad in \quad \mathbb{R} \quad as \quad n \to \infty ,$$

where $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$, if and only if

$$\psi(t)^{-1}(N(t) - m^{-1}t) \Rightarrow -m^{-(1+p)}L \quad in \quad \mathbb{R} \quad as \quad t \to \infty ,$$

where $N(t) \equiv \max\{k \geq 0 : S_k \leq t\}$, $t \geq 0$.

7.3.2. FCLT Equivalence

Next we present an equivalence result for FCLTs allowing double sequences $\{X_{n,k}\}$. Thus, there is a sequence of partial sums $\{S_{n,k} : k \geq 1\}$ and an associated counting process $\{N_n(t) : t \geq 0\}$ for each n, so that the result here also apply to the convergence of sequences of random walks to general Lévy processes in Section 2.4 of the Internet Supplement. (Theorem 7.3.1 above does not extend to double sequences.)

Again no direct probability assumptions are made on the basic sequences $\{X_{n,k} : k \geq 1\}$ and the limit process can be anything. Finally, note that we use the M_1 topology. As we have seen in previous sections of this chapter, it is usually possible to establish the FCLT for partial sums in (3.1) below in the stronger J_1 topology, but nevertheless as discussed in Section 6.3, the FCLT for the counting process only holds in the M_1 topology when the limit process $\mathbf{S}$ for the normalized partial sums has sample paths containing positive jumps, as occurs with the stable Lévy motion limit in Section 4.5.

To state the result we use the composition function, mapping x, y into $x \circ y \equiv x(y(t))$, $t \geq 0$.

Theorem 7.3.2. (FCLT equivalence for partial sums and counting processes) *Suppose that* $\{X_{n,k} : k \geq 1\}$ *is a sequence of nonnegative random variables for each* $n \geq 1$, $c_n \to \infty$, $n/c_n \to \infty$, $m_n \to m$, $0 < m < \infty$ *and* $\mathbf{S}(0) = 0$. *Then*

$$\mathbf{S}_n \Rightarrow \mathbf{S} \quad in \quad D([0,\infty), \mathbb{R}, M_1) , \tag{3.1}$$

where $S_{n,k} \equiv X_{n,1} + \cdots + X_{n,k}$, $k \geq 1$, $S_{n,0} \equiv 0$ *and*

$$\mathbf{S}_n(t) \equiv c_n^{-1}(S_{n,\lfloor nt \rfloor} - m_n nt \rfloor), \quad t \geq 0 , \tag{3.2}$$

if and only if

$$\mathbf{N}_n \to -m^{-1}\mathbf{S} \circ m^{-1}\mathbf{e} \quad in \quad D([0,\infty), \mathbb{R}, M_1) , \tag{3.3}$$

where $N_n(t) \equiv \max\{k \geq 0 : S_{n,k} \leq t\}$, $t \geq 0$, *and*

$$\mathbf{N}_n(t) \equiv c_n^{-1}(N_n(nt) - m_n^{-1} nt), \quad t \geq 0 . \tag{3.4}$$

If the limits in (3.1) *and* (3.3) *hold, then*

$$(\mathbf{S}_n, \mathbf{N}_n) \Rightarrow (\mathbf{S}, \mathbf{N}) \quad in \quad (D, M_1)^2 . \tag{3.5}$$

Theorem 7.3.2 comes from Section 7 of Whitt (1980), which extends related results by Iglehart and Whitt (1971) and Vervaat (1972). Theorem

7.3.2 is proved by applying the continuous-mapping approach with the the inverse map x^{-1} defined in (5.5) in Section 3.5, using linear centering; see Section 13.8. Specifically, the result is implied by Corollary 13.8.1.

We now show how the FCLT equivalence in Theorem 7.3.2 can be applied with previous FCLTs for partial sums to obtain FCLTs for counting processes. We start with Brownian motion limits.

Corollary 7.3.1. (Brownian FCLT for counting processes) *Suppose that the conditions of Theorem* 7.3.2 *hold with* $c_n = \sqrt{n}$. *If the FCLT for partial sums in* (3.1) *holds with* $\mathbf{S} = \sigma\mathbf{B}$, *where* $\mathbf{B}$ *is standard Brownian motion, then*

$$\mathbf{N}_n \Rightarrow m^{-3/2}\sigma\mathbf{B} \quad in \quad (D, J_1)$$

for $\mathbf{N}_n$ *in* (3.4) *with* $c_n = \sqrt{n}$.

Proof. Apply Theorem 7.3.2, noting that the J_1 and M_1 topologies are equivalent when the limit has continuous sample paths and

$$-m^{-1}\sigma\mathbf{B} \circ m^{-1}e \stackrel{\mathrm{d}}{=} m^{-3/2}\sigma\mathbf{B}\,. \quad \blacksquare$$

Now we consider FCLTs with stable Lévy motion limits. Recall that $S_\alpha(\sigma, \beta, \mu)$ denotes a stable law with index α, scale parameter σ, skewness parameter β and shift parameter μ; see Section 4.5. We first consider the case $1 < \alpha < 2$. When the summands are IID nonnegative random variables, the skewness parameter will be $\beta = 1$.

Corollary 7.3.2. (stable Lévy FCLT for counting processes when $\alpha > 1$) *Suppose that the conditions of Theorem* 7.3.2 *hold with* $c_n = n^{1/\alpha}$, $1 < \alpha < 2$. *If the FCLT for the partial sums in* (3.1) *holds with* $\mathbf{S}$ *a stable Lévy motion with* $\mathbf{S}(1) \stackrel{\mathrm{d}}{=} \sigma S_\alpha(1, \beta, 0)$, *then*

$$\mathbf{N}_n \Rightarrow -m^{-(1+\alpha^{-1})}\mathbf{S} \quad in \quad (D, M_1)$$

for $\mathbf{N}_n$ *in* (3.4) *with* $c_n = n^{1/\alpha}$ *and*

$$-m^{-(1+\alpha^{-1})}\mathbf{S}(1) \stackrel{\mathrm{d}}{=} m^{-(1+\alpha^{-1})}\sigma S_\alpha(1, -\beta, 0)\ .$$

Proof. Apply Theorem 7.3.2, noting that

$$-m^{-1}\mathbf{S} \circ m^{-1}\mathbf{e} \stackrel{\mathrm{d}}{=} -m^{-(1+\alpha^{-1})}\mathbf{S}$$

and

$$-S_\alpha(\sigma,\beta,0) \stackrel{\mathrm{d}}{=} S_\alpha(\sigma,-\beta,0) \ . \quad \blacksquare$$

We now present FCLTs for counting processes that capture the Joseph effect.

Corollary 7.3.3. (FBM FCLT for counting processes) *Suppose that $\{X_n\}$ is a stationary sequence of nonnegative random variables with mean $m = EX_n$ and $Var(X_n) < \infty$. If $\{X_n - m\}$ satisfies the conditions of Theorem 4.6.1, which requires that $Y_n \geq 0$ and $a_j \geq 0$ in* (6.6)*, then*

$$\mathbf{N}_n \Rightarrow -m^{-1}\mathbf{S} \circ m^{-1}\mathbf{e} \quad in \quad (D, M_1) \ , \tag{3.6}$$

where

$$\mathbf{N}_n(t) \equiv c_n^{-1}(N(nt) - m^{-1}nt), \quad t \geq 0 \ , \tag{3.7}$$

for c_n in (6.16) and $\mathbf{S}$ is standard FBM.

Proof. Apply Theorems 7.3.2 and 4.6.1, noting that $-\mathbf{S} \stackrel{\mathrm{d}}{=} \mathbf{S}$. $\blacksquare$

Corollary 7.3.4. (LFSM FCLT for counting processes) *Suppose that $\{X_n\}$ is a stationary sequence of nonnegative random variables with finite mean $m = EX_n$. If $\{X_n - m\}$ also satisfies the conditions of Theorem 4.7.2, which with the nonnegativity requires that $Y_j \geq 0$ and $a_j \geq 0$ in (6.6), then the stochastic-process limit in* (3.6) *holds with $\mathbf{S}$ being LSFM in* (7.10) *with $\beta = 1$ and $\mathbf{N}_n$ in* (3.7) *with c_n in* (7.12).

We have yet to state general equivalence theorems that cover FCLTs with stable Lévy motion limits having index $\alpha \leq 1$. When $\alpha = 1$, we have space scaling by n, but a translation term that grows faster than n. The following covers the special case of $\alpha = 1$.

Theorem 7.3.3. (FCLT equivalence to cover stable limits with index $\alpha =$ 1) *Suppose that $\{X_{n,k} : k \geq 1\}$ is a sequence of nonnegative random variables for each n, $c_n \to \infty$, $m_n \to \infty$, $nm_n/c_n \to \infty$ and $\mathbf{S}(0) = 0$. Then*

$$\mathbf{S}_n \Rightarrow \mathbf{S} \quad in \quad D([0,\infty), \mathbb{R}, M_1)$$

where $S_{n,k} \equiv X_{n,1} + \cdots + X_{n,k}$, $k \geq 1$, $S_{n,0} \equiv 0$ and

$$\mathbf{S}_n(t) \equiv c_n^{-1}[S_{n,\lfloor nt \rfloor} - m_n nt], \quad t \geq 0 \ , \tag{3.8}$$

if and only if

$$\mathbf{N}_n \Rightarrow -\mathbf{S} \quad in \quad D([0,\infty), \mathbb{R}, M_1) \ , \tag{3.9}$$

where

$$\mathbf{N}_n(t) \equiv c_n^{-1}[m_n N_n(nm_n t) - nm_n t], \quad t \geq 0 \ .$$

Proof. We can apply Theorem 7.3.2 after we express (3.8) in the appropriate form: Letting $x_n(t) \equiv S_{n,\lfloor nt \rfloor}/nm_n$ and scaling space by nm_n/c_n, we obtain

$$\mathbf{S}_n(t) = \frac{nm_n}{c_n}\left[\frac{S_{n,\lfloor nt \rfloor}}{nm_n} - t\right], \quad t \geq 0 \ .$$

Then $x_n^{-1}(t) \approx N_n(nm_n t)/n$ and (3.9) essentially follows from Theorem 7.3.2. ∎

We now state a FCLT equivalence theorem to cover the case of stable Lévy motion limits with $\alpha < 1$. In the standard framework with IID random variables, the random variables have infinite mean. In this case, there is no translation term. We now use the inverse map without centering to characterize the limit process. Specifically, we apply Theorem 13.6.1.

Theorem 7.3.4. (FCLT equivalence to cover stable limits with $\alpha < 1$) *Suppose that* $\{X_{n,k} : k \geq 1\}$ *is a sequence of nonnegative random variables such that* $S_{n,k} \equiv X_{n,1} + \cdots + X_{n,k} \to \infty$ *w.p.1 as* $k \to \infty$ *and* $N_n(t) \equiv \max\{k \geq 0 : S_{n,k} \leq t\} \to \infty$ *w.p.1 as* $t \to \infty$ *for each* n. *Also suppose that* $c_n \to \infty$ *as* $n \to \infty$. *Then*

$$\mathbf{S}_n \Rightarrow \mathbf{S} \quad in \quad (D, M_1)$$

with $\mathbf{S}^{-1}(0) = 0$ *for*

$$\mathbf{S}_n(t) \equiv c_n^{-1} S_{n,\lfloor nt \rfloor}, \quad t \geq 0 \ ,$$

if and only if

$$\mathbf{N}_n \Rightarrow \mathbf{S}^{-1} \quad in \quad (D, M_1)$$

with $\mathbf{S}(0) = 0$ *for*

$$\mathbf{N}_n(t) \equiv n^{-1} N_n(c_n t), \quad t \geq 0 \ .$$

Since the process sample paths are nondecreasing, the M_1 convergence in Theorem 7.3.4 is equivalent to convergence of the finite-dimensional distributions. Theorem 7.3.4 is easy to apply because

$$\{\mathbf{S}^{-1}(s) \geq t\} = \{\mathbf{S}(t) \leq s\} \ ,$$

so that

$$P(\mathbf{S}^{-1}(s) \geq t\} = P(\mathbf{S}(t) \leq s)$$

for all positive s and t; see Lemma 13.6.3 in Section 13.6. When $\mathbf{S}(t)$ has a stable law with index α, $0 < \alpha < 1$, the marginal distributions are easy to compute by numerical transform inversion.

7.4. Renewal-Reward Processes

We now apply the convergence-preservation results for the composition and inverse maps established in Chapter 13 to obtain FCLTs for renewal-reward processes. Renewal-reward processes are random sums of IID random variables, where the random index is an independent renewal counting process. The results for the renewal process alone follow from the previous section.

Let $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ be independent sequences of IID random variables, where Y_n is nonnegative. Let $S_n^x \equiv X_1 + \cdots + X_n$ and $S_n^y \equiv Y_1 + \cdots + Y_n$ be the associated partial sums, with $S_0^x \equiv S_0^y \equiv 0$. Let N be the renewal counting process associated with $\{Y_n\}$, i.e.

$$N(t) \equiv \max\{k \geq 0 : S_k^y \leq t\}, \quad t \geq 0\,.$$

The renewal-reward process is the random sum

$$Z(t) \equiv \sum_{i=1}^{N(t)} X_i, \quad t \geq 0\,. \tag{4.1}$$

The random variable $Z(t)$ represents the cumulative input of required work in a service system during the interval $[0, t]$ when customers with random service requirements X_n arrive at random times S_n^y.

We assume that X_1 and Y_1 have finite means $m \equiv EX_1$ and $\lambda^{-1} \equiv EY_1 > 0$. Thus we have the SLLNs: $n^{-1}S_n^x \to m$, $n^{-1}S_n^y \to \lambda^{-1}$ w.p.1 as $n \to \infty$ and $t^{-1}N(t) \to \lambda$, $t^{-1}Z(t) \to \lambda m$ w.p.1 as $t \to \infty$.

We are interested in the FCLT refinements. We assume that X_1 and Y_1 either have finite variances σ_x^2 and σ_y^2 or are in the normal domain of attraction of stable laws. In the case of infinite variances, let

$$P(|X_1| > t) \sim \gamma_x t^{-\alpha_x} \tag{4.2}$$

$$P(X_1 > t) \sim \frac{(1+\beta_x)}{2} P(|X_1| > t) \tag{4.3}$$

$$P(Y_1 > t) \sim \gamma_y t^{-\alpha_y} \tag{4.4}$$

as $t \to \infty$, where $1 < \alpha_x < 2$ and $1 < \alpha_y < 2$. (Recall that $Y_n \geq 0$ w.p.1.)

We form the normalized process

$$\mathbf{Z}_n(t) \equiv n^{-1/\alpha}(Z(nt) - \lambda m n t), \quad t \geq 0\,, \tag{4.5}$$

where $\alpha = \min\{\alpha_x, \alpha_y\}$ with $\alpha_x \equiv 2$ if $\sigma_x^2 < \infty$ and $\alpha_y \equiv 2$ if $\sigma_y^2 < \infty$.

The starting point for obtaining FCLTs for $Z(t)$ are FCLTs for S_n^x and S_n^y, which involve the scaled stochastic processes

$$\mathbf{S}_n^x(t) \equiv n^{-1/\alpha_x}(S_{\lfloor nt \rfloor}^x - mnt), \quad t > 0 , \tag{4.6}$$

and

$$\mathbf{S}_n^y(t) \equiv n^{-1/\alpha_y}(S_{\lfloor nt \rfloor}^y - \lambda^{-1}nt), \quad t > 0 . \tag{4.7}$$

The associated scaled process for the renewal counting process $N(t)$ is

$$\mathbf{Y}_n(t) \equiv n^{-1/\alpha_y}(N(nt) - \lambda nt), \quad t \geq 0 . \tag{4.8}$$

Connections between limits for $\mathbf{S}_n^y$ and $\mathbf{Y}_n$ follow from Section Section 13.8; an overview was given in Section 7.3.

Theorem 7.4.1. (renewal-reward FCLT with finite variances) *If the random variables X_1 and Y_1 have finite variances σ_x^2 and σ_y^2, then*

$$\mathbf{Z}_n \Rightarrow \sigma \mathbf{B} \quad \textit{in} \quad (D, J_1) ,$$

where $\mathbf{Z}_n$ is in (4.5) *with $\alpha = 2$, $\mathbf{B}$ is standard Brownian motion and*

$$\sigma^2 \equiv \lambda \sigma_x^2 + m^2 \lambda^3 \sigma_y^2 . \tag{4.9}$$

Proof. We apply Corollary 13.3.2. For that purpose, let $(\mathbf{X}_n, \mathbf{Y}_n)$ in (3.8) be defined by $\mathbf{X}_n = \mathbf{S}_n^x$ in (4.6) and $\mathbf{Y}_n$ in (4.8). The convergence $\mathbf{X}_n \Rightarrow \mathbf{U}$, where $\mathbf{U} \stackrel{\mathrm{d}}{=} \sigma_x \mathbf{B}$ follows from Donsker's theorem, Theorem 4.3.2. The convergence $\mathbf{Y}_n \Rightarrow \mathbf{V}$, where $\mathbf{V} \stackrel{\mathrm{d}}{=} \lambda^{3/2} \sigma_y \mathbf{B}$ follows from Donsker's Theorem and Corollary 13.8.1, as indicated in Corollary 7.3.1. By independence and Theorem 11.4.4, we obtain the joint convergence $(\mathbf{X}_n, \mathbf{Y}_n) \Rightarrow (\mathbf{U}, \mathbf{V})$ in (3.8) from the two marginal limits. Since the sample paths of $(\mathbf{U}, \mathbf{V})$ are continuous, condition (3.9) automatically holds. Finally, the limit in (3.10) simplifies, because

$$\sigma_x \mathbf{B}_1 \circ \lambda \mathbf{e} + m \lambda^{3/2} \sigma_y \mathbf{B}_2 \stackrel{\mathrm{d}}{=} (\lambda \sigma_x^2 + m^2 \lambda^3 \sigma_y^2)^{1/2} \mathbf{B}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are independent standard Brownian motions. ∎

It is often insightful to replace variances by dimensionless parameters describing the variability independent of the scale. Thus, let c_x^2 and c_y^2 be the *squared coefficients of variation* or SCVs, defined by $c_x^2 \equiv \sigma_x^2/m^2$ and $c_y^2 \equiv \lambda^2 \sigma_y^2$. We can alternatively express σ^2 in (4.9) as

$$\sigma^2 = \rho m (c_x^2 + c_y^2) , \tag{4.10}$$

where $\rho \equiv \lambda m$.

The limit process $\sigma \mathbf{B}$ in Theorem 7.4.1 has continuous sample paths. In contrast, when either X_1 or Y_1 has a heavy-tailed distribution, the limit process has discontinuous sample paths.

Theorem 7.4.2. (renewal-reward FCLT for heavy-tailed summands) *Suppose that* (4.2) *and* (4.3) *hold with* $1 < \alpha_x < 2$, *and that either* $\sigma_y^2 < \infty$ *or* $\sigma_y^2 = \infty$ *and* (4.4) *holds with* $\alpha_x < \alpha_y < 2$. *Then*

$$\mathbf{Z}_n \Rightarrow \sigma \mathbf{S}_\alpha \quad in \quad (D, J_1) \ ,$$

where $\mathbf{Z}_n$ *is in* (4.5), $\alpha = \alpha_x$, $\mathbf{S}_\alpha$ *is a stable Lévy motion with* $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1, \beta_x, 0)$ *for* β_x *in* (4.3) *and*

$$\sigma^\alpha \equiv (\gamma_x \lambda / C_\alpha) \tag{4.11}$$

for γ_x *in* (4.2) *above and* C_α *in* (5.14) *of Section* 4.5.

Proof. We again apply Corollary 13.3.2 and Theorem 11.4.4, with $\mathbf{X}_n = \mathbf{S}_n^x$ in (4.6) and $\mathbf{Y}_n$ in (4.8). From Theorems 4.5.2 and 4.5.3, we get $\mathbf{X}_n \Rightarrow \mathbf{U}$, as needed in the condition of Corollary 13.3.2, with $\mathbf{U} \equiv \delta \mathbf{S}_\alpha$, where $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1, \beta_x, 0)$ for β_x in (4.3) and

$$\delta = (\gamma_x / C_\alpha)^{1/\alpha}$$

for γ_x in (4.2) and C_α in (5.14). With the space scaling by $n^{-\alpha}$, we get $\mathbf{Y}_n \Rightarrow \mathbf{V}$ for $\mathbf{V} = 0\mathbf{e}$. Since $\mathbf{V} = 0\mathbf{e}$, condition (3.9) holds trivially. Hence we get $\mathbf{Z}_n \Rightarrow \mathbf{U} \circ \lambda \mathbf{e} \stackrel{\mathrm{d}}{=} \lambda^{1/\alpha} \mathbf{U}$ from Corollary 13.3.2. Hence σ is as in (4.11). ∎

Even though the limit process $\sigma \mathbf{S}_\alpha$ in Theorem 7.4.2 has discontinuous sample paths, we can use the J_1 topology on D. That is no longer true for the next two theorems.

Theorem 7.4.3. (renewal-reward FCLT for a heavy-tailed renewal process) *Suppose that* (4.4) *holds for* $1 < \alpha_y < 2$ *and that either* $\sigma_x^2 < \infty$ *or* $\sigma_x^2 = \infty$ *and* (4.2) *holds with* $\alpha_y < \alpha_x < 2$. *Then*

$$\mathbf{Z}_n \Rightarrow \sigma \mathbf{S}_\alpha \quad in \quad (D, M_1)$$

where $\mathbf{Z}_n$ *is in* (4.5), $\alpha = \alpha_y$, $\mathbf{S}_\alpha$ *is stable Lévy motion with* $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1, -1, 0)$ *and*

$$\sigma^\alpha \equiv m^\alpha \lambda^\alpha \gamma_y \lambda / C_\alpha$$

for γ_y *in* (4.4) *above and* C_α *in* (5.14) *of Section* 4.5.

Proof. We again apply Corollary 13.3.2 and Theorem 11.4.4 with $\mathbf{X}_n = \mathbf{S}_n^x$ in (4.6) and $\mathbf{Y}_n$ in (4.8). Using the scaling function $n^{-1/\alpha}$, we obtain $\mathbf{U} = 0\mathbf{e}$. We obtain the FCLT for the renewal counting process from Theorems 4.5.2, 4.5.3, 7.3.2 and Corollary 7.3.2, making use of the M_1 topology. The limit process $\mathbf{V}$ then is

$$\mathbf{V} \stackrel{\mathrm{d}}{=} -\delta\lambda\mathbf{S}'_\alpha \circ \lambda e$$

where $\mathbf{S}'_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1,1,0)$ and $\delta = (\gamma_y/C_\alpha)^{1/\alpha}$. Thus

$$m\mathbf{V} \stackrel{\mathrm{d}}{=} m\lambda^{1+\alpha^{-1}}(\gamma_y/C_\alpha)^{1/\alpha}\mathbf{S}_\alpha$$

where $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1,-1,0)$, by virtue of (5.9) in Section 4.5. ■

We now treat the case in which the two processes have heavy tails with the same index.

Theorem 7.4.4. (renewal-reward FCLT for heavy-tailed summands and renewal process) *Suppose that* (4.2)–(4.4) *hold with* $1 < \alpha_x < 2$ *and* $\alpha_y = \alpha_x$. *Then*

$$\mathbf{Z}_n \Rightarrow \sigma\mathbf{S}_\alpha \quad in \quad (D, M_1) \ ,$$

where $\mathbf{Z}_n$ *is in* (4.5), $\alpha = \alpha_x = \alpha_y$, $\mathbf{S}_\alpha$ *is a stable Lévy motion with* $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1,\beta',0)$ *for*

$$\beta' \equiv \frac{\gamma_x\beta_x - m^\alpha\lambda^\alpha\gamma_y}{\gamma_x + m^\alpha\lambda^\alpha\gamma_y},$$

$$\sigma^\alpha \equiv \frac{\lambda}{C_\alpha}(\gamma_x + m^\alpha\lambda^\alpha\gamma_y),$$

γ_x *in* (4.2), γ_y *in* (4.4) *and* C_α *in* (5.14) *of Section* 4.5.

Proof. We again apply Corollary 13.3.2 and Theorem 11.4.4 with $(\mathbf{X}_n, \mathbf{Y}_n)$ defined as in the previous theorems. As in Theorem 7.4.2, we get $\mathbf{X}_n \Rightarrow \mathbf{U}$ for $\mathbf{U} \stackrel{\mathrm{d}}{=} \delta\mathbf{S}_\alpha$ with $\delta = (\gamma_x/C_\alpha)^{1/\alpha}$. As in Theorem 7.4.3, we get $\mathbf{Y}_n \Rightarrow \mathbf{V}$ for $\mathbf{V} \stackrel{\mathrm{d}}{=} \eta S_\alpha$ with $\mathbf{S}_\alpha(1) \stackrel{\mathrm{d}}{=} S_\alpha(1,-1,0)$ and $\eta = \lambda^{1+\alpha^{-1}}(\gamma_y/C_\alpha)^{1/\alpha}$. Since $\mathbf{U}$ and $\mathbf{V}$ are independent processes without any fixed discontinuities, condition (3.9) holds. Hence we get $\mathbf{Z}_n \Rightarrow \mathbf{Z}$, where

$$\mathbf{Z} \stackrel{\mathrm{d}}{=} \delta\mathbf{S}^1_\alpha \circ \lambda e + m\eta\mathbf{S}^2_\alpha$$

where $\mathbf{S}_\alpha^1$ and $\mathbf{S}_\alpha^2$ are two independent α-stable Lévy motions with $\mathbf{S}_\alpha^1(1) \stackrel{\mathrm{d}}{=} S_\alpha(1, \beta_x, 0)$ and $\mathbf{S}_\alpha^2(1) \stackrel{\mathrm{d}}{=} S_\alpha(1, -1, 0)$. Thus we obtain

$$\begin{aligned} \sigma^\alpha &= \delta^\alpha \lambda + m^\alpha \eta^\alpha \\ &= \frac{\gamma_x}{C_\alpha}\lambda + m^\alpha \frac{\gamma_y}{C_\alpha}\lambda^\alpha \lambda \\ &= (\lambda/C_\alpha)(\gamma_x + m^\alpha \lambda^\alpha \gamma_y) \\ \beta' &= \frac{\delta\lambda^{1/\alpha}\beta_x + m\eta(-1)}{\delta\lambda^{1/\alpha} + m\eta} \end{aligned}$$

using scaling relations (5.9) and (5.10) in Section 4.5. ∎

We used the strong (mostly independence) assumptions to get

$$(\mathbf{X}_n, \mathbf{Y}_n) \Rightarrow (\mathbf{U}, \mathbf{V}) \quad \text{in} \quad D^2 \tag{4.12}$$

for $\mathbf{X}_n = \mathbf{S}_n^x$ in (4.6) and $\mathbf{Y}_n$ in (4.8). Given (4.12) without the other specific assumptions we would still get limits for $\mathbf{Z}_n$ in (4.5) by applying the convergence-preservation results for composition and inverse. Chapter 4 contains alternative FCLTs that can be employed.