Traffic Models for Wireless Communication Networks

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***Abstract*— In this paper, we introduce a *deterministic fluid model* and two *stochastic traffic models* for wireless networks. The setting is a highway with multiple entrances and exits. Vehicles are classified as *calling* or *noncalling*, depending upon whether or not they have calls in progress. The main interest is in the calling vehicles, but noncalling vehicles are important because they can become calling vehicles if they initiate (place or receive) a call. The deterministic model ignores the behavior of individual vehicles and treats them as a continuous fluid, whereas the stochastic traffic models consider the random behavior of each vehicle. However, all three models use the same two coupled partial differential equations (PDE's) or ordinary differential equations (ODE's) to describe the evolution of the system. The call density and call handoff rate (or their expected values in the stochastic models) are readily computable by solving these equations. Since no capacity constraints are imposed in the models, these computed quantities can be regarded as offered traffic loads. The models complement each other, because the fluid model can be extended to include additional features such as capacity constraints and the interdependence between velocity and vehicular density, while the stochastic traffic model can provide probability distributions.**

Numerical examples are presented to illustrate how the models can be used to investigate various aspects of time and space dynamics in wireless networks. The numerical results indicate that both the time-dependence and the mobility of vehicles can play important roles in determining system performance. Even for systems in steady state with respect to time, the movement of vehicles and the calling patterns can significantly affect the number of calls in a given region of the system. The examples demonstrate that the proposed models can serve as useful tools for system engineering and planning. For instance, we calculate approximate call blocking probabilities.

I. Introduction

UNLIKE a fixed, terrestrial telephone network, a wireless network must support moving customers. Due to customer mobility, both the location and the length of a call in progress affect the network resources required to support the call. Customer mobility is presenting a major challenge to system designers of wireless networks [4], [5]. Since wireless services are becoming more popular, there is an increasing need for mathematical models to help understand system dynamics and analyze the performance of wireless networks.

Motivated primarily by this need, a *Poisson-arrival-location model* (PALM) was introduced in [6], in which customers arrive according to a nonhomogeneous Poisson process and move independently through a general location state space according to a location stochastic process. The PALM is made tractable by assuming that different customers do not interact, although this behavior can be approximated indirectly. Similarly, the PALM can be used as an offered traffic model that serves to approximate important system capacity constraints (e.g., the number of available radio channels) indirectly. This is achieved through applying methods like the modified offered load technique developed in [3] to approximate blocking probabilities in these wireless networks. The PALM also provides a useful framework for representing both time-dependent behavior and customer mobility in wireless communication networks.

The general PALM in [6] is quite abstract. Further specification is needed in order to obtain practical models. Toward this end, a version of the PALM was constructed to study communicating mobiles on a highway in [7]. In this *highway PALM,* vehicles alternate between thinking and calling modes as they move along on a one-way, semi-infinite highway according to a deterministic location function. Two-way traffic and more complicated highway networks are represented by superposing independent versions of these highway PALM's. As with the general PALM, the system is assumed to have no capacity constraints in terms of the number of radio channels available for calls. Thus, the highway PALM can also be regarded as an offered load model, which enables us to characterize key quantities such as the call density, the handoff rate, the call-origination-rate density, and the call-termination-rate density. In [7], it is shown that these quantities are related by two fundamental conservation equations, similar to relations in vehicular traffic theory [2]. These results thus bring together teletraffic theory and vehicular traffic theory. Other researchers have also observed the need for combining these two theories to study wireless networks [8]–[11].

While much more concrete than the general PALM in [6], the highway PALM in [7] is still quite abstract. Further specification is still required in order to produce readily computable performance-related quantities. The purpose of this paper is to develop such a version of the highway PALM that is indeed substantially more tractable, so that all the desired quantities are computable. We obtain such a more tractable highway PALM primarily by making additional Markov assumptions, in the spirit of Section 8 of [6]. Hence we call the main stochastic model introduced here the *Markovian highway PALM.*

In the Markovian highway PALM considered here, in addition to having arrivals occur according to a nonhomogeneous Poisson process, the state of each vehicle is assumed to

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evolve according to a nonstationary continuous-time Markov chain, while the vehicle moves deterministically along the highway. Each vehicle on the highway is classified as either a *calling* or *noncalling* vehicle, depending upon whether or not it has a call in progress. Each calling (noncalling) vehicle becomes a noncalling (calling) vehicle randomly with specified deterministic intensity depending on time and space. Similarly, each calling (noncalling) vehicle leaves the highway with a deterministic intensity depending on time and space. Of course, our main interest is in the calling vehicles; they use the most resources in wireless networks. However, it may be necessary to keep track of locations of noncalling vehicles so that they can receive calls. They are also important as a source of future calling vehicles. There can be dramatic increases in the number of calling vehicles when and where the density of noncalling vehicles is high and the calling rate is high.

In the special case of stationary intensities, our Markovian assumption for each vehicle means that call holding times, think times (before making a call), and highway residence times (before leaving the highway) would be independent exponential random variables with means equal to the reciprocals of the intensities. Without stationarity, these random times can be represented as time-ordered exponentials, as in (8.3) of [6].

It turns out that the densities of the mean numbers of vehicles of each type are described by partial differential equations (PDE's), similar to the ones arising in the classic approach to modeling vehicular traffic [2]. With the Markovian assumption, the two PDE's are coupled due to the calling activities; i.e., a noncalling vehicle becomes a calling vehicle and vice versa, if it initiates (places and receives) or terminates a call. In this Markovian highway PALM, the PDE's relate the derivatives of the expected number of vehicles of each type, while the actual numbers of noncalling and calling vehicles in a given section of the highway have Poisson distributions, due to previous PALM results in [6], [7].

The PDE's can also be interpreted in another way. Instead of characterizing the *expected values* in a *stochastic model*, they can be regarded as characterizing the *actual values* in a *deterministic fluid model*. This deterministic fluid model neglects the behavior of individual vehicles (or customers), but is still capable of capturing the overall dynamics of the system. Calling and noncalling vehicles are treated as two types of continuous fluid. Such a model is appropriate and justifiable if the system has a large number of calling and noncalling vehicles, as discussed in Section 9 of [6]. Indeed, such deterministic differential equation models are common in vehicular traffic theory [2]. (Wright [11] also uses differential equations to capture the vehicle movement in a deterministic way for a highway cellular system. The focus in [11] is on the control strategy, e.g., call throttling, to maintain transmission quality in spread spectrum systems such as CDMA systems.)

In fact, three different models are considered here: the deterministic fluid model, the Markovian highway PALM, and a stochastic generalization of the Markovian highway PALM in which the arrival process need not be Poisson. As noted in Remark 2.3 of [6], the mean value formulas for a general PALM remain valid when the arrival process is *not* Poisson, provided that the arrival rate is still well defined and that successive arrivals do not interact. We use the extra Poisson arrival assumption only to characterize the probability distributions of the quantities of interest; the mean values are determined from the PDE's.

We believe that an interesting feature of this paper is the identification of the three related models. The important point is that all three models lead to the same PDE's. We thus have different possible interpretations of these PDE's. For the deterministic fluid model, we interpret the solution as the actual numbers (which need not be integers), whereas for the stochastic traffic model, the solutions are interpreted as the expected values. With the additional Poisson-arrival assumption in the Markovian highway PALM, we are able to say more about the full probability distributions. For large systems, the three approaches tend to be almost fully consistent, because in the stochastic model, the true distributions will typically cluster relatively tightly about their means, by virtue of the law of large numbers; see Section 9 of [6]. The mean value interpretation in the stochastic models tends to be more general, because it does not require large populations.

In both the deterministic fluid model and the stochastic traffic models, the system quantities of interest such as calling density and handoff rate (or their expected values) are readily computable. Thus, the models can provide insight into the time and space dynamics of mobile customers. Hopefully, this new insight and understanding can contribute to improved design and management of wireless communication networks.

The rest of this paper is organized as follows. In Section II, we develop the deterministic fluid model. The model has two versions: a time-nonhomogeneous model and a time-homogeneous model. The two PDE's become ordinary differential equations (ODE's) in the time-homogeneous model. There are interesting spatial phenomena even in the time-homogeneous case. In Section III, we present some numerical examples to illustrate the time and space dynamics captured by the models. These numerical examples demonstrate that the quantities of interest can readily be computed for our models. In addition, they also show that the proposed models can serve as valuable tools for system engineering and planning.

In Section IV, we introduce the Markovian traffic model and discuss its connection with the deterministic fluid model. In Section V, we discuss the additional distributional results that can be obtained when arrivals occur according to a nonhomogeneous Poisson process. In Section VI, we indicate how to approximately represent interactions among vehicles in the stochastic model. The idea is to allow the vehicle velocity to depend on the expected numbers of calling and noncalling vehicles in various regions. Finally, in Section VII, we present our conclusions and discuss future work. An important component of this future work is a study of the approximate representations of vehicle interactions discussed in Section VI.

II. The Deterministic Fluid Model

Our basic setting is a one-way, semi-infinite highway. (As in [7], independent versions of these highways can be superposed to make richer models.) Thus, we can regard the location space as the interval $[0, \infty)$. There are two types of vehicles, calling and noncalling. Each vehicle is assumed to make at most one call at a time and each call occupies one radio channel for the duration of the call. As in [6], [7], we do not impose any capacity constraints, i.e., we assume that there are an infinite number of channels available so that all calls are accepted without blocking. Thus our model can be regarded as a way to quantify the offered load. It is possible to add capacity constraints to the model in various ways, as we illustrate in Section III.

Vehicles of both types at location x and time t move forward on the highway according to a deterministic velocity field $v(x,t)$. To ensure vehicle flow in a single direction, it is assumed that $v(x,t) \geq 0$ for all x and t with $x \geq 0$ and $-\infty < t < \infty$. Additionally, in order to make sure that vehicles flow at all, $v(x,t) > 0$ at least for some $t \geq t_0$ for all times t_0 at each location $x \geq 0$. For full generality, it is also assumed that both calling and noncalling vehicles can enter and leave the highway at any location. (Cases in which vehicles can enter or leave only at finitely many fixed entrances and exits are considered at the end of Section II-A.)

In Section II-A, we first present the model, which captures both the time-dependent behavior (e.g., nonhomogeneous arrivals of vehicles) and vehicle movement on the highway. Then in Section II-B, we simplify the time-nonhomogeneous fluid model into a time-homogeneous model to capture only the spatial dynamics. The PDE's then become ODE's. As can be seen in the numerical examples in Section III, even when vehicles "behave" in a time-independent manner, the space dynamics alone significantly affect system traffic loads.

A. The Time-Nonhomogeneous Deterministic Fluid Model

To begin, we introduce some notation. Let $N(x,t)$ and $Q(x,t)$ be the *number* of noncalling and calling vehicles in location $(0,x]$ at time t, respectively. The model treats vehicles as a continuous fluid; $N(x,t)$ and $Q(x,t)$ are not necessarily integers, but any nonnegative real numbers. In addition, let $n(x,t)$ and $q(x,t)$ be the *noncalling* density and *calling* density at location x and time t, respectively. That is,

$$n(x,t) \equiv \frac{\partial N(x,t)}{\partial x} \text{ and } q(x,t) \equiv \frac{\partial Q(x,t)}{\partial x}.$$

Throughout this section, we assume that all derivatives are well defined.

Furthermore, let $\boldsymbol{C}_n^+(x,t)$ and $\boldsymbol{C}_n^-(x,t)$ be the number of noncalling vehicles *entering* or *leaving* in location $(0,x]$ during time interval $(-\infty,t]$, respectively. Similarly, we use $\boldsymbol{C}_q^+(x,t)$ and $\boldsymbol{C}_q^-(x,t)$ to denote the respective number of calling vehicles entering or leaving in location $(0,x]$ in time $(-\infty,t]$. A noncalling (calling) vehicle is considered to be entering the system, if either: a) it is an actual arrival of a noncalling (calling) vehicle to the highway, or b) it was a calling (noncalling) vehicle existing on the highway but with its call just terminated (started). Likewise, a noncalling (calling) vehicle leaves if it departs from the highway or becomes a calling (noncalling) vehicle by initiating (terminating) a call.

Finally, let us define the rate densities as

$$c_n^+(x,t) \equiv \frac{\partial^2 \boldsymbol{C}_n^+(x,t)}{\partial x \partial t}, \quad c_n^-(x,t) \equiv \frac{\partial^2 \boldsymbol{C}_n^-(x,t)}{\partial x \partial t},$$

$$c_q^+(x,t) \equiv \frac{\partial^2 \boldsymbol{C}_q^+(x,t)}{\partial x \partial t} \text{ and } c_q^-(x,t) \equiv \frac{\partial^2 \boldsymbol{C}_q^-(x,t)}{\partial x \partial t}.$$

Lemma 2.1: The evolution of noncalling and calling vehicles on the highway is governed by the PDE's:

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}[n(x,t)v(x,t)] = c_n^+(x,t) - c_n^-(x,t) \quad (2.1)$$

and

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial}{\partial x}[q(x,t)v(x,t)] = c_q^+(x,t) - c_q^-(x,t) \quad (2.2)$$

for $x \geq 0$ and $-\infty < t < \infty$.

Proof: We first relate $N(x, t+\Delta t)$ to $N(x,t)$. During the time interval $(t, t+\Delta t]$, a certain number of noncalling vehicles in location $(0,x]$ move forward and pass beyond location x. In addition, some new arrivals of noncalling vehicles enter into the system, while some leave from location $(0,x]$ at the time interval. We thus have

$$\begin{aligned} N(x,t+\Delta t) - N(x,t) = &\\ &- n(x,t)v(x,t)\Delta t + \left[\boldsymbol{C}_n^+(x,t+\Delta t) - \boldsymbol{C}_n^+(x,t)\right] \\ &- \left[\boldsymbol{C}_n^-(x,t+\Delta t) - \boldsymbol{C}_n^-(x,t)\right] + o(\Delta t) \quad (2.3)\end{aligned}$$

where $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$, assuming that n and v are continuous in the neighborhood of (x,t). Dividing both sides of (2.3) by Δt, letting $\Delta t \to 0$, and differentiating w.r.t. x yields (2.1). Equation (2.2) is obtained in the same way. □

We remark that (2.2) corresponds to the fundamental conservation equation in (2.7) of [7]. Before showing how (2.1) and (2.2) are coupled due to calling activity, we need additional notation. Let $\boldsymbol{E}_n^+(x,t)$ and $\boldsymbol{E}_n^-(x,t)$ be the number of noncalling vehicles arriving to and departing from the highway in location $(0,x]$ during time interval $(-\infty,t]$, respectively (Case (a) for c_n^+ and c_n^- above). We use $\boldsymbol{E}_q^+(x,t)$ and $\boldsymbol{E}_q^-(x,t)$ to denote the respective number of calling vehicles arriving to and departing from the highway in location $(0,x]$ in time $(-\infty,t]$. The associated rate densities are

$$e_n^+(x,t) \equiv \frac{\partial^2 \boldsymbol{E}_n^+(x,t)}{\partial x \partial t}, \quad e_n^-(x,t) \equiv \frac{\partial^2 \boldsymbol{E}_n^-(x,t)}{\partial x \partial t},$$

$$e_q^+(x,t) \equiv \frac{\partial^2 \boldsymbol{E}_q^+(x,t)}{\partial x \partial t} \text{ and } e_q^-(x,t) \equiv \frac{\partial^2 \boldsymbol{E}_q^-(x,t)}{\partial x \partial t}.$$

Further, let $\beta(x,t)n(x,t)$ and $\gamma(x,t)q(x,t)$ be the rates at which noncalling and calling vehicles actually depart from the highway at location x at time t. In addition, let $\lambda(x,t)n(x,t)$ be the call-origination rate of noncalling vehicles and $\mu(x,t)q(x,t)$ be the call-termination rate of calling vehicles at location x and time t. (In the stochastic model, these are stochastic intensities for individual vehicles; here these are actual deterministic flow rates.)

We note that the rate densities, $c_n^+(x,t)$, $c_n^-(x,t)$, $c_q^+(x,t)$, and $c_q^-(x,t)$ can be expressed in terms of these parameters as follows and we omit the proof.

Lemma 2.2: The four rate densities can be expressed as

$$\text{a) } c_n^+(x,t) = e_n^+(x,t) + \mu(x,t)q(x,t) \quad (2.4)$$

$$\text{b) } c_n^-(x,t) = \beta(x,t)n(x,t) + \lambda(x,t)n(x,t) \quad (2.5)$$

$$\text{c) } c_q^+(x,t) = e_q^+(x,t) + \lambda(x,t)n(x,t) \quad (2.6)$$

$$\text{d) } c_q^-(x,t) = \gamma(x,t)q(x,t) + \mu(x,t)q(x,t). \quad (2.7)$$

We now combine Lemmas 2.1 and 2.2 to obtain the following coupled PDE's characterizing the densities $n(x,t)$ and $q(x,t)$ in our model. These PDE's can be regarded as the deterministic fluid model.

Theorem 2.1: The densities of noncalling and calling vehicles, $n(x,t)$ and $q(x,t)$, satisfy the coupled PDE's:

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}[n(x,t)v(x,t)] = e_n^+(x,t) + \mu(x,t)q(x,t) - [\beta(x,t) + \lambda(x,t)]n(x,t) \quad (2.8)$$

and

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial}{\partial x}[q(x,t)v(x,t)] = e_q^+(x,t) + \lambda(x,t)n(x,t) - [\gamma(x,t) + \mu(x,t)]q(x,t). \quad (2.9)$$

With an additional assumption, the PDE's in (2.8) and (2.9) can be converted into a set of three ordinary differential equations (ODE's), which are easier to solve in some cases. (This is the classical method of characteristics.) For this purpose, let the location x as a time function, $x(t)$, be given by

$$\frac{dx(t)}{dt} = v(x(t),t). \quad (2.10)$$

Equation (2.10) is one of the three ODE's.

Lemma 2.3: Given (2.10), the PDE's (2.8) and (2.9) are equivalent to

$$\frac{dn(x(t),t)}{dt} = e_n^+(x(t),t) + \mu(x(t),t)q(x(t),t) - \left[\frac{\partial v(x,t)}{\partial x} + \beta(x(t),t) + \lambda(x(t),t)\right]n(x(t),t) \quad (2.11)$$

and

$$\frac{dq(x(t),t)}{dt} = e_q^+(x(t),t) + \lambda(x(t),t)n(x(t),t) - \left[\frac{\partial v(x,t)}{\partial x} + \gamma(x(t),t) + \mu(x(t),t)\right]q(x(t),t), \quad (2.12)$$

respectively.

Proof: By the chain rule, we have

$$\frac{dn(x(t),t)}{dt} = \frac{\partial n(x,t)}{\partial x}\cdot\frac{dx(t)}{dt} + \frac{\partial n(x,t)}{\partial t}.$$

Substitute (2.8) and (2.10) into this equation. After some algebraic manipulation, we get (2.11). Equation (2.12) is obtained in the same way. □

Due to the partial derivative of $v(x,t)$ w.r.t. x on the r.h.s. of (2.11) and (2.12), they are ODE's if and only if $v(x,t)$ is not a function of $q(x,t)$ and $n(x,t)$. It is also worth noting that, by choosing some initial time τ with $-\infty < \tau < \infty$ such that $x(\tau) = 0$ as the initial condition for (2.10), $n(x(t),t)$ and $q(x(t),t)$ can be solved for all $\tau \le t < \infty$ from (2.10)–(2.12), e.g., by a Runge–Kutta method. Of course, the solution depends on the initial conditions $n(0,\tau)$ and $q(0,\tau)$. Thus, if τ is selected properly, one can obtain $n(x,t)$ and $q(x,t)$ for the location region and the time interval of interest.

We now suppose that the highway is divided into cells, indexed by $i = 1, 2, 3, \cdots$. For $i > 1$, let the boundary between cell $i-1$ and cell i be located at $x_i - 1$ and $x_0 \equiv 0$. Further, let $Q_i(t)$ be the instantaneous offered load in cell i at time t; that is, the total number of calls in progress in cell i at time t. Let $h_i(t)$ denote the rate of ongoing calls handed off from cell $i-1$ to cell i at time t.

Theorem 2.2: For cell $i \ge 1$, its instantaneous offered load and call handoff rate at time t are

$$Q_i(t) = \int_{x_{i-1}}^{x_i} q(x,t)dx \quad (2.13)$$

and

$$h_i(t) = q(x_{i-1},t)v(x_{i-1},t), \quad (2.14)$$

respectively.

Proof: Equation (2.13) is immediately given by the definition of $q(x,t)$. To consider the rate of handoff calls, let $H_i(t)$ be the number of calls handed off from cell $i-1$ to cell i before time t. Since $H_i(t+\Delta t)$ is equal to $H_i(t)$ plus the calls handed off during the time interval $(t, t+\Delta t]$, we have

$$H_i(t+\Delta t) - H_i(t) = q(x_{i-1},t)[1-\mu(x_{i-1},t)\Delta t]v(x_{i-1},t)\Delta t + n(x_{i-1},t)\lambda(x_{i-1},t)\Delta t v(x_{i-1},t)\Delta t + o(\Delta t). \quad (2.15)$$

By definition, $h_i(t) = dH_i(t)/dt$. Thus, dividing by Δt and taking the limit as $\Delta t \to 0$ in (2.15) yields (2.14). □

We remark that (2.14) corresponds to the conservation equation (2.6) in [7]. It gives a flow rate at a point, which does not actually require that cells be defined; i.e., (2.14) is valid for arbitrary x as well as $x_i - 1$.

We conclude this subsection by commenting on the rate densities of vehicle entering and leaving the highway for the case where vehicles can enter or leave only at entrances/exits at fixed locations, as in real highway systems. Suppose that $\{y_i : i = 1, 2, 3, \cdots\}$ is the location of the ith entrance/exit on the highway. Let us use $\xi_n^i(t)$ and $\xi_q^i(t)$ to denote the external arrival rate of noncalling and calling vehicles at the ith entrance at time t, respectively. Then, we have

$$e_n^+(x,t) = \sum_i \xi_n^i(t)\delta(x - y_i) \quad (2.16)$$

and

$$e_q^+(x,t) = \sum_i \xi_q^i(t)\delta(x - y_i) \quad (2.17)$$

where $\lim_{\epsilon\to 0}\int_{x-\epsilon}^{x+\epsilon}\delta(y)dy = 1$ if $x = 0$ and 0 otherwise.

As for vehicles leaving the highway, we use $p_n^i(t)$ and $p_q^i(t)$ to denote the fraction of noncalling and calling vehicles departing when they pass by the ith entrance/exit at time t, respectively. If these departing vehicles leave at the same velocity as they move forward along the highway, then

$$\beta(x,t) = v(x,t)\sum_i p_n^i(t)\delta(x-y_i) \tag{2.18}$$

and

$$\gamma(x,t) = v(x,t)\sum_i p_q^i(t)\delta(x-y_i). \tag{2.19}$$

B. The Time-Homogeneous Deterministic Fluid Model

We now cease to focus on time dynamics, and assume that the system has reached a steady state with respect to time. As a result, all system variables and parameters become independent of time. For this reason, we simply drop the variable t from our previously defined notation, and use primes to denote derivatives w.r.t. x.

Theorem 2.3: At temporal equilibrium, the densities of noncalling and calling vehicles, $n(x)$ and $q(x)$, at any location $x \geq 0$ satisfy the following ODE's:

$$v(x)n'(x) = e_n^+(x) + \mu(x)q(x) - [\beta(x) + \lambda(x) + v'(x)]n(x) \tag{2.20}$$

and

$$v(x)q'(x) = e_q^+(x) + \lambda(x)n(x) - [\gamma(x) + \mu(x) + v'(x)]q(x). \tag{2.21}$$

Proof: Set all variables in (2.8) and (2.9) to be independent of t and note that

$$\frac{d}{dx}[n(x)v(x)] = v(x)n'(x) + n(x)v'(x)$$

and similarly for $q(x)v(x)$. □

In general, $n(x)$ and $q(x)$ can be solved from the coupled ODE's in (2.20) and (2.21) plus initial conditions. However, under reasonable assumptions, the two ODE's can be combined into one, and $n(x)$ and $q(x)$ can be obtained explicitly. To prove this, we present the following proportionality result.

Lemma 2.4: For some $x_0 \geq 0$,

$$\text{if a) } \lambda(x) = \lambda, \mu(x) = \mu, \beta(x) = \gamma(x) \text{ and } \frac{e_q^+(x)}{e_n^+(x)} = \frac{\lambda}{\mu} \quad \text{for all } x \geq x_0, \tag{2.22}$$

and

$$\text{b) } q(x_0) \text{ is finite and } \frac{q(x_0)}{n(x_0)} = \frac{\lambda}{\mu}, \tag{2.23}$$

$$\text{then } \frac{q(x)}{n(x)} = \frac{\lambda}{\mu} \text{ for all } x \geq x_0. \tag{2.24}$$

We actually obtain a stronger proportionality result for a time-dependent setting. The result given as Lemma 2.5 and its proof are presented in the Appendix. Lemma 2.4 can be viewed as a consequence of the time-dependent result since all quantities here are independent of time. We choose to present Lemma 2.4 here because, as explained below, its conditions have a clear physical meaning and are natural for the time-homogeneous model under consideration. When $\lambda(x) = \lambda$ and $\mu(x) = \mu$, vehicles initiate and terminate calls at rates independent of their locations. The condition $\beta(x) = \gamma(x)$ indicates that a vehicle departs from the highway at the same rate, regardless of whether it is a calling or noncalling vehicle. The ratio $e_q^+(x)/e_n^+(x) = q(x_0)/n(x_0) = \lambda/\mu$ means that the proportion of vehicles arriving to the highway at location x which are calling vehicles is identical to that of existing vehicles at location x_0, which in turn is equal to the ratio λ/μ.

If the conditions for the proportionality result are satisfied, it is unnecessary to use two ODE's to describe the movement of noncalling and calling vehicles. Instead, one ODE is sufficient. For this purpose, let $L(x)$ be the total number of vehicles in location $(0, x]$ at steady state and $l(x) \equiv dL(x)/dx$. By definition, $L(x) = N(x) + Q(x)$ and $l(x) = n(x) + q(x)$.

Theorem 2.4: If the conditions in Lemma 2.4 with $x_0 = 0$ are satisfied, then the vehicular density is given by the following ODE:

$$\frac{dl(x)}{dx} = \frac{1}{v}(x)\left\{\left[-\frac{dv(x)}{dx} - \beta(x)\right]l(x) + e_q^+(x)\left[1 + \frac{\mu}{\lambda}\right]\right\}, \tag{2.25}$$

whose solution is

$$l(x) = \frac{e^{-I(x)}}{v(x)}\left\{\left[1 + \frac{\mu}{\lambda}\right]\int_0^x e_q^+(u)e^{I(u)}du + l(0)v(0)\right\}, \tag{2.26}$$

where $I(x) = \int_0^x \frac{\beta(u)}{v(u)}du$. Furthermore,

$$n(x) = \frac{\mu}{\lambda + \mu}l(x) \tag{2.27}$$

and

$$q(x) = \frac{\lambda}{\lambda + \mu}l(x). \tag{2.28}$$

Proof: Based on the fact that $l(x) = n(x) + q(x)$, (2.25) is obtained by substituting conditions a) and b) of Lemma 2.4 into (2.20) and (2.21) and adding them together. Since (2.25) is a linear, first-order ODE, the solution is readily obtained and given by (2.26). The proof is completed by using the proportionality result in Lemma 2.4. □

For example, we can apply Theorem 2.4 to a special case where vehicles arrive only at location $x = 0$ and they never leave the highway. Then, we have $q(0)/n(0) = \lambda/\mu$, $I(x) = 0$, $e_n^+(x) = 0$ and $e_q^+(x) = 0$ for all $x > 0$. The solution in (2.27) and (2.28) becomes

$$n(x) = \frac{n(0)v(0)}{v(x)} \text{ and } q(x) = \frac{q(0)v(0)}{v(x)} \text{ for } x > 0; \tag{2.29}$$

i.e., the vehicular density is inversely proportional to the velocity. It is noteworthy that the arrival rates of noncalling and calling vehicles at location 0 in this special case are $e_q^+(0) = q(0)v(0)$ and $e_n^+(0) = n(0)v(0)$, respectively.

It should be clear that the conditions in Theorem 2.4 are quite natural, so that we often will have the single ODE in (2.25). On the other hand, the interest in explicitly considering two kinds of vehicles is primarily for the situations where these conditions do *not* hold.

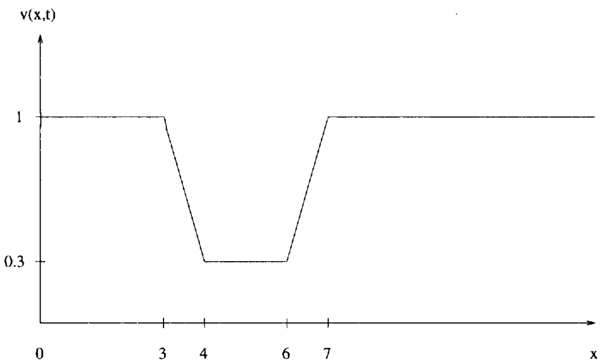

Fig. 1. Vehicle velocity as a function of location for $35 < t < 55$ min.

Once $n(x)$ and $q(x)$ are computed from (2.20) and (2.21) or (2.27) and (2.28), the offered load and call handoff rate for cell i can be obtained from (2.13) and (2.14), with the variable t omitted.

III. Numerical Examples

In this section, we present numerical examples to illustrate the time and space dynamics captured by the deterministic fluid model in Section II. As indicated in Section I, these examples also apply to the stochastic models to be introduced later, but with a different interpretation.

The examples here assume no constraint on the number of available channels. Furthermore, unless stated otherwise, the average think time (time before initiating a call) and call-holding time are 10 and 2 min, respectively. That is, $\lambda(x,t) = 0.1$ and $\mu(x,t) = 0.5$ for all $x \geq 0$ and $-\infty < t < \infty$. The highway has a single entrance at location 0 at which only noncalling vehicle arrive at a constant rate (denoted by α) of 30 cars/min, and vehicles are assumed not to depart from the highway.

We first consider a time-dependent case where the velocity field $v(x,t)$ is a function of location and time. It is assumed that $v(x,t) = 1$ km/min for all $x \geq 0$ when $t \leq 35$ or $t > 55$ min. However, for $35 < t \leq 55$ min, the velocity field is

$$v(x,t) = \begin{cases} 1 & \text{if } x \leq 3 \\ 1 - 0.7(x-3) & \text{if } 3 < x \leq 4 \\ 0.3 & \text{if } 4 < x \leq 6 \\ 0.3 + 0.7(x-6) & \text{if } 6 < x \leq 7 \\ 1 & \text{if } x > 7. \end{cases} \quad (3.1)$$

This velocity field is depicted in Fig. 1. It is U-shaped as a function of location on the highway for $35 < t \leq 55$, so that it can be used to simulate the slowing down of traffic in the time interval due to an accident. For this example, Lemma 2.3 can be applied to convert the PDE's (2.8) and (2.9) into the ODE's (2.10)–(2.12). By using the extrapolated Gragg's modified mid-point method [1], contained in the PORT library developed at AT&T Bell Laboratories, we numerically solve the ODE's with the initial condition of $n(0,t) = \alpha/v(0,t)$ for all $t > 0$ in order to obtain $n(x,t)$ and $q(x,t)$.

Figs. 2–5 show the total vehicular density, the densities of noncalling and calling vehicles, and the call handoff rate as a function of location at time prior (Fig. 2), during (Fig. 3–4), and after (Fig. 5) the traffic accident. For $t = 30$ in Fig. 2, $n(x,t)$ and $q(x,t)$ reach their "equilibrium" solution for sufficiently large x. The reason for this is explained as follows. Recall that all vehicles arriving at $x = 0$ are noncalling vehicles. They start to make calls as they move forward on the highway. As a result, the density of noncalling and calling vehicles decreases and increases, respectively, as x increases. Since vehicles initiate and terminate calls independently at constant rates of $\lambda(x,t) = 0.1$ and $\mu(x,t) = 0.5$, respectively, such decrease and increase of vehicular densities approach an equilibrium at locations farther down the highway. In fact, the ratio $n(x)/q(x)$ in this case tends to equal to $\mu(x,t)/\lambda(x,t)$ for sufficiently large x. Since $v(x,t) = 1$ at $t = 30$, according to (2.14), the density of calling vehicles equals handoff rate, so that their curves shown in the figure coincide.

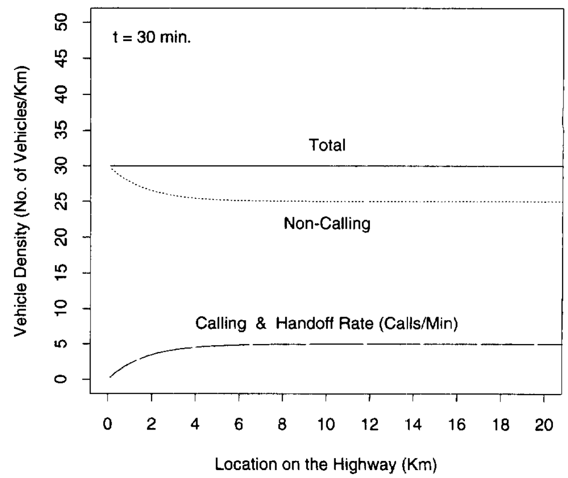

Fig. 2. Vehicle density in the highway system with a traffic accident.

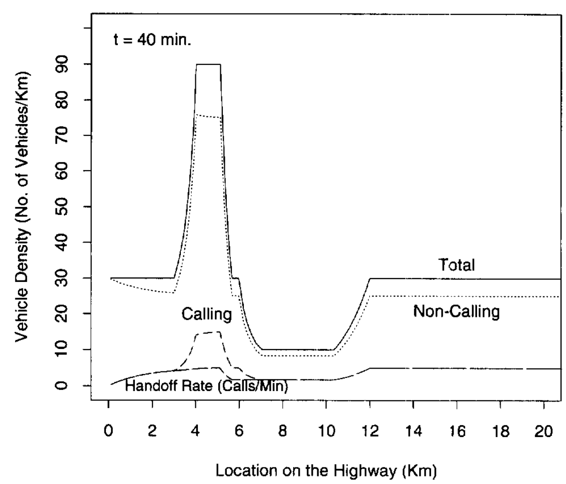

Fig. 3. Vehicle density in the highway system with a traffic accident.

At $t = 40$ in Fig. 3, vehicles start to build up sharply at location $(3,7]$ where velocity is relatively low. Accordingly,

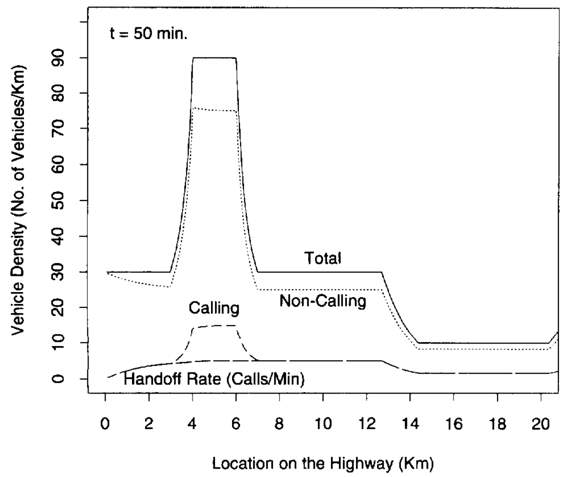

Fig. 4. Vehicle density in the highway system with a traffic accident.

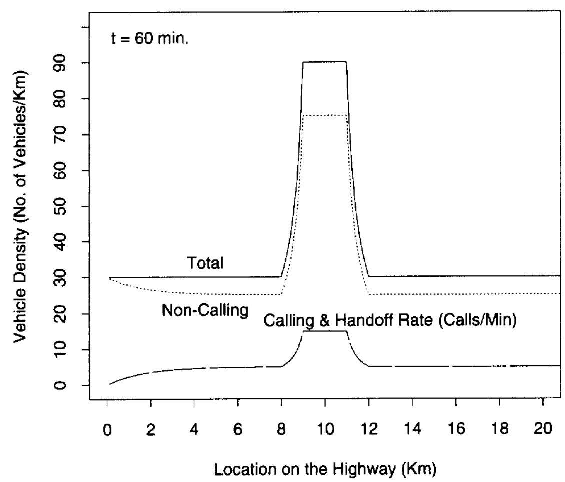

Fig. 5. Vehicle density in the highway system with a traffic accident.

as indicated in the figure, the call density in this region is also higher than elsewhere because of the velocity reduction began at $t = 35$. Note that the vehicular density in location $(6, 12]$ is lower than that beyond location $x \geq 12$. This is because the vehicles that would have been at this location if there were no reduction in velocity starting at $t = 35$ have been trapped in location $(3, 7]$ due to the low velocity. At $t = 50$ in Fig. 4, the vehicular traffic and call density continue to build up in location $(3, 7]$. In addition, the "dip" of vehicular density has shifted to the right from the position shown in Fig. 3, as vehicles continue to move forward on the highway.

Finally, Fig. 5 shows that the whole density curves continue to propagate to the right at $t = 60$, as vehicles resume their original velocity of $v(x, t) = 1$ at all locations after the accident has been cleared at $t = 55$. In particular, those vehicles that were located at location $(3, 7]$ at $t = 55$ now have moved into location $(8, 12]$ at $t = 60$ at a constant velocity of $v(x, t) = 1$ after $t > 55$. Note that these results may

TABLE I
APPROXIMATE BLOCKING PROBABILITIES FOR THE FIXED CELL SIZE

Time	Cell Location	Offered Load (Erlang)	No. of Channels Per Cell	Blocking Prob.
$t = 30$	(2,4]	8.2455	20	0.00023
	(4,6]	9.4716	20	0.00107
	(6,8]	9.8408	20	0.00159
	(8,10]	9.9520	20	0.00178
$t = 50$	(2,4]	11.2459	20	0.00565
	(4,6]	29.4977	20	0.37105
	(4,6]	29.4977	45	0.00179
	(6,8]	13.2327	20	0.02055
	(6,8]	13.2327	25	0.00127
	(8,10]	9.9980	20	0.00187

not closely reflect the vehicle movement following a traffic accident because the model allows vehicles to resume moving at a specific velocity regardless of the high vehicular density. In real situation, with the vehicles close to location 7 first resuming the normal velocity after the accident, the vehicles accumulated in location $(3, 7]$ will slowly "diffuse" to the right. Such diffusion type of vehicle movement can be captured, if the model is augmented with an appropriate relationship between velocity and vehicular density, as discussed in Section VI.

As pointed out earlier, the call density in a region should be treated as its offered load. To illustrate how the offered load results can be used for system engineering and planning purposes, let the highway be served by nonoverlapping cells of fixed size where each cell covers 2 km of the highway. By (2.13), the offered load at a given time is obtained for each cell. Given the number of channels available at a cell, the blocking probability can be approximated by applying the offered load to the Erlang-B formula. This approximation is naturally supported by the stochastic model, indeed the full highway PALM in Section V; also see Sections 5 and 7 of [7] for further discussion. (With the highway PALM, the offered load in each cell has a Poisson distribution. For the finite capacity, a natural approximation is a truncated Poisson distribution, which is the Erlang-B formula.)

Table I presents the blocking probabilities at $t = 30$ and 50, for which the call density has been depicted in Figs. 2 and 4, respectively. As shown in the table, assuming that each cell has 20 channels, the blocking probabilities at $t = 30$ are a fraction of a percent, which are satisfactory. However, due to the traffic congestion caused by the accident, the blocking probability in the cell at location $(4, 6]$ at $t = 50$ increases to 37.1%! In fact, it is found that for the surge of offered load, the cell has to be equipped with 45 channels to maintain the blocking probability satisfactorily low. Similarly, the cell at location $(6, 8]$ also requires five additional channels to handle the offered load adequately. These results show the potential benefit of dynamic channel assignment. They also show that the proposed traffic models can serve as a valuable tool for system engineering and planning.

The second set of examples uses the time-homogeneous fluid model to consider only the space dynamics. The velocity field $v(x)$ in Fig. 1 is used at all time t and all other system parameters remain unchanged. Then, solving the ODE's in

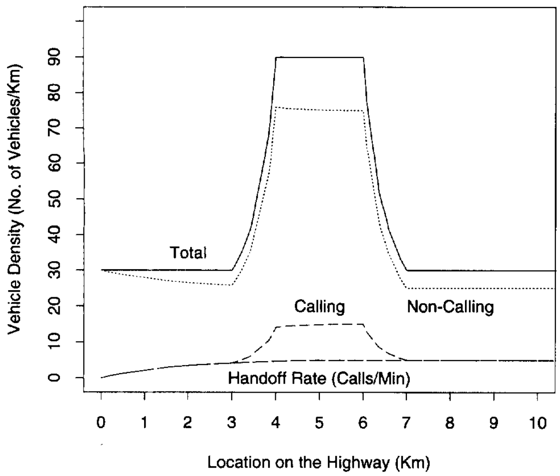

Fig. 6. Vehicle density in temporal steady state (fixed call-origination rate).

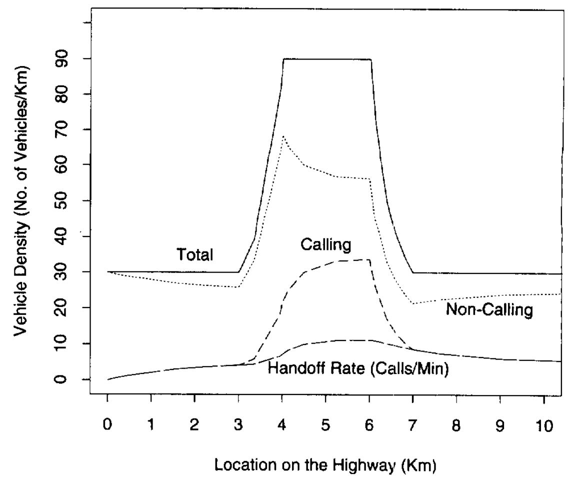

Fig. 7. Vehicle density in temporal steady state (call-origination rate × velocity = constant).

(2.20) and (2.21) yields $n(x)$ and $q(x)$ and they are portrayed in Fig. 6. It is interesting to note that this figure is practically identical to Fig. 4 in location $(0, 10]$. This is so because, in the time-dependent case, the system has almost reached its steady state at $t = 50$ with the accident started at $t = 35$. Of course, this time-homogeneous example does not have the dip of vehicular densities after location 12 in Fig. 4.

Fig. 7 considers a case that uses all system parameters of the case shown in Fig. 6, except that the call-origination rate $\lambda(x)$ is inversely proportional to $v(x)$. In particular, we stipulate that $\lambda(x)v(x) = 0.1$ for all $x \geq 0$. This setting can be applied to study the phenomenon that people are likely to make phone calls when the vehicle velocity is reduced, e.g., when they are trapped in a traffic jam. Because of an increased value of $\lambda(x)$ at location $3 \leq x \leq 7$, the call density in this region in Fig. 7 can be more than two times of that shown in Fig. 6, where $\lambda(x)$ is identical at all locations. These results clearly reveal the importance of space dynamics (i.e., vehicle velocity and density) and the customers' calling behavior in determining the offered load and thus the performance of wireless systems.

Finally, we consider a location-dependent call-origination rate that decays exponentially in location. Specifically, $\lambda(x) = 1 + 0.9e^{-x/2}$. Such a form of call-origination rate function can be used to approximate the calling activity of spectators departing from a stadium after a sporting event. In such situations, vehicles are likely to initiate calls when they begin to leave the stadium (i.e., close to location 0). Fig. 8 assumes $v(x) = 0.77$ for all $x \geq 0$, which is the velocity averaged over the location $(0, 10]$ for the velocity field shown in Fig. 1. Although the settings for Figs. 7 and 8 have the same average velocity, their call densities and handoff rates are very different. Hence, the space dynamics and calling patterns are shown again to have significant impacts on the traffic loads even for systems in temporal steady state.

We have also considered other examples for time-homogeneous cases where the highway has multiple entrances and exits, although they are not given here. In these cases, the ODE's are solved for segments of the highway between two

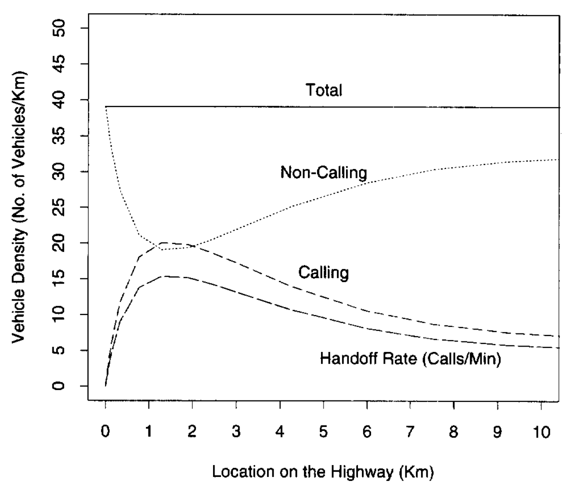

Fig. 8. Vehicle density in temporal steady state (exponential call-origination rate).

successive entrances/exits. Based on the vehicular densities at the end of one segment (i.e., just before an entrance/exit), the probability of a vehicle leaving from the exit, and the flow of vehicles entering from the entrance, we can obtain the initial conditions for the next segment. Then, solving the ODE's with these initial conditions yields the vehicular densities in the next segment of the highway.

IV. The Stochastic Traffic Model

In contrast to the deterministic fluid model introduced above, the stochastic traffic model captures the stochastic fluctuations. Specifically, the model considers the random calling status of each individual vehicle as it moves along on the highway. However, it turns out that the PDE's and ODE's which govern the expected values are identical to those of the deterministic fluid model. Hence, the numerical examples that we have just considered apply equally well to the stochastic model. In Figs. 2–8, we must simply replace the actual values on the y-axis by expected values.

The stochastic model has the same highway setting. Unless stated otherwise, the same notation is used as for the deterministic fluid model. Using the same definitions, $N(x,t)$, $Q(x,t)$, $\boldsymbol{C}_n^+(x,t)$, $\boldsymbol{C}_n^-(x,t)$, $\boldsymbol{C}_q^+(x,t)$, $\boldsymbol{C}_q^-(x,t)$, $\boldsymbol{E}_n^+(x,t)$, $\boldsymbol{E}_n^-(x,t)$, $\boldsymbol{E}_q^+(x,t)$, and $\boldsymbol{E}_q^-(x,t)$ become integer-valued random variables in the stochastic model. Now, the densities $n(x,t)$ and $q(x,t)$ are defined as the partial derivatives of expected values; i.e.,

$$n(x,t) \equiv \frac{\partial E[N(x,t)]}{\partial x} \text{ and } q(x,t) \equiv \frac{\partial E[Q(x,t)]}{\partial x},$$

respectively, where $E[Y]$ denotes the expected value of Y. Correspondingly, we let the rate densities be the second partial derivatives of expected values; i.e.,

$$c_n^+(x,t) \equiv \frac{\partial^2 E[\boldsymbol{C}_n^+(x,t)]}{\partial x \partial t}, \; c_n^-(x,t) \equiv \frac{\partial^2 E[\boldsymbol{C}_n^-(x,t)]}{\partial x \partial t},$$

$$c_q^+(x,t) \equiv \frac{\partial^2 E[\boldsymbol{C}_q^+(x,t)]}{\partial x \partial t}, \; c_q^-(x,t) \equiv \frac{\partial^2 E[\boldsymbol{C}_q^-(x,t)]}{\partial x \partial t},$$

$$e_n^+(x,t) \equiv \frac{\partial^2 E[\boldsymbol{E}_n^+(x,t)]}{\partial x \partial t}, \; e_n^-(x,t) \equiv \frac{\partial^2 E[\boldsymbol{E}_n^-(x,t)]}{\partial x \partial t},$$

$$e_q^+(x,t) \equiv \frac{\partial^2 E[\boldsymbol{E}_q^+(x,t)]}{\partial x \partial t} \text{ and } e_q^-(x,t) \equiv \frac{\partial^2 E[\boldsymbol{E}_q^-(x,t)]}{\partial x \partial t}.$$

For the stochastic model, we introduce additional notation. Let $\alpha(t)$ be the total arrival rate of vehicles arriving to the highway at time t. By definition,

$$\alpha(t) \equiv \frac{\partial}{\partial t}\{E[\boldsymbol{E}_n^+(\infty,t)] + E[\boldsymbol{E}_q^+(\infty,t)]\}. \tag{4.1}$$

We make the following assumptions for the stochastic model:

1) Vehicles arrive to the highway according to a pair of two-dimensional stochastic jump processes $\boldsymbol{E}_n^+(x,t)$ and $\boldsymbol{E}_q^+(x,t)$ with nondecreasing sample paths having only unit jumps and deterministic intensity functions $e_n^+(x,t)$ and $e_q^+(x,t)$, where the total arrival rate $\alpha(t)$ in (4.1) is integrable over all $-\infty < t < \infty$.
2) Vehicles move forward on the highway at a deterministic velocity specified by the velocity field $v(x,t)$.
3) The state of each vehicle after it arrives evolves as a nonstationary continuous-time Markov chain, while it moves deterministically down the highway. The Markov chains of different vehicles are conditionally stochastically independent given their arrival times. (The Markov chains are not unconditionally independent due to dependence induced through the arrival times, but once we condition upon the arrival times, there is no dependence left.) A calling vehicle becomes a noncalling vehicle and vice versa (due to call termination and initiation) randomly with intensity $\mu(x,t)$ and $\lambda(x,t)$, respectively. In addition, a calling (noncalling) vehicle leaves the highway randomly with intensity $\gamma(x,t)(\beta(x,t))$.
4) There are no capacity constraints. That is, each cell has an infinite number of channels such that no call blocking occurs.

As in [6] and [7], we can construct $Q(x,t)$ by stochastic integration as follows. For $j \geq 1$, let $T_s^+(j)$ and $T_s^-(j)$ be the time when a vehicle arriving to the highway at time s initiates and terminates its jth call, respectively. Then, we have

$$s \leq T_s^+(1) \leq T_s^-(1) \leq T_s^+(2) \leq T_s^-(2) \leq \cdots$$

and

$$Q(x,t) = \int_{-\infty}^{t} 1_{\{L_s(t) \in (-\infty,x] \times \{1\}\}} dA(s)$$

where $A(t)$ counts the number of vehicles arriving to the highway up to time t, 1_B is an indicator function such that $1_B = 1$ if B is true and 0 otherwise, and the location process $L_s(t)$ specifies the position and calling status of the vehicle that arrived at time s. That is, $L_s(t) = (x,k)$ where x is the position on the highway and

$$k = \begin{cases} 1 \text{ if } t \in \cup_{j=1}^{\infty} [T_s^+(j), T_s^-(j)) \\ 0 \text{ otherwise.} \end{cases} \tag{4.3}$$

In this context, an analog of Lemma 2.1 holds, which is a natural extension of the conservation equation (2.7) of [7].

Lemma 4.1: In the stochastic traffic model, the densities of noncalling and calling vehicles, $n(x,t)$ and $q(x,t)$, satisfy (2.1) and (2.2).

Proof: To consider the expected values, (2.3) is replaced by

$$\begin{aligned} E[N(x,t+\Delta t)] - E[N(x,t)] = \\ -n(x,t)v(x,t)\Delta t + \{E[\boldsymbol{C}_n^+(x,t+\Delta t)] - E[\boldsymbol{C}_n^+(x,t)]\} \\ - \{E[\boldsymbol{C}_n^-(x,t+\Delta t)] - E[\boldsymbol{C}_n^-(x,t)]\} + o(\Delta t). \end{aligned} \tag{4.4}$$

The proof is completed by following the same argument in the proof of Lemma 2.1. □

Similarly, we have the following, for which we omit the proof.

Lemma 4.2: The results in Lemma 2.2, namely (2.4) to (2.7), hold for the stochastic traffic model.

We can combine Lemmas 4.1 and 4.2 to obtain an analog of Theorem 2.1.

Theorem 4.1: In the stochastic traffic model, the densities of noncalling and calling vehicles, $n(x,t)$ and $q(x,t)$, at any location $x \geq 0$ and time $-\infty < t < t$ satisfy (2.8) and (2.9).

Since the PDE's for the stochastic model are identical to those for the deterministic fluid model, Lemma 2.3 remains applicable to convert them into ODE's.

Now suppose that the highway is divided into cells at some locations $\{x_0(\equiv 0), x_1, x_2, \cdots\}$, $Q_i(t)$ is the number of ongoing calls (i.e., offered load) in cell i at time t, and $H_i(t)$ is the number of calls handed off from cell $i-1$ to cell i before time t. The following is an analog of Theorem 2.2; it follows from Theorem 3.1 of [7].

Theorem 4.2: In the stochastic traffic model,

a) For each cell $i \geq 1$ and at any given time $-\infty < t < \infty$, $Q_i(t)$ is a stochastic process with mean

$$E[Q_i(t)] = \int_{x_{i-1}}^{x_i} q(x,t)dx. \tag{4.5}$$

b) For each cell $i \geq 1$, $H_i(t)$ is a stochastic process as a function of t with rate

$$h_i(t) \equiv \frac{dE[H_i(t)]}{dt} = q(x_{i-1},t)v(x_{i-1},t). \tag{4.6}$$

Proof: Part a) is elementary, so we only discuss b). To consider handoff calls, let $\tau(s,x)$ denote the time when a vehicle that arrives to the highway at time s reaches the location x. We assume $\tau(s,x)=\infty$ if the vehicle will never reach location x. $H_i^j(t)$ is defined as the number of jth ongoing calls handed off from cell $i-1$ to cell i before time t. Then,

$$H_i^j(t) = \int_{-\infty}^{t} 1_{\{T_s^+(j)\leq\tau(s,x_i-1)<T_s^-(j)\}}dA(s).$$

Clearly, $H_i(t)$ is the sum of $H_i^j(t)$ over all j. The rate of $E[H_i(t)]$ as given in (4.6) is obtained by considering only the expected values and by following the approach in Theorem 2.2. □

V. The Markovian Highway PALM

We obtain the full Markovian highway PALM simply by assuming, in addition to the assumptions of Section IV, that $\boldsymbol{E}_q^+(x,t)$ and $\boldsymbol{E}_n^+(x,t)$ are independent two-dimensional Poisson counting processes. Let $\boldsymbol{E}_q^+(B)$ and $\boldsymbol{E}_n^+(B)$ be random measures associated with the stochastic counting processes $\boldsymbol{E}_q^+(x,t)$ and $\boldsymbol{E}_n^+(x,t)$; i.e., $\boldsymbol{E}_{q/n}^+(B)$ counts the number of arrivals in the set B where B is a set of (x,t) pairs in $[0,\infty)\times\boldsymbol{R}$. The Poisson assumption means that the numbers of arrivals $\boldsymbol{E}_q^+(B_i)$ and $\boldsymbol{E}_n^+(B_i)$ of calling and noncalling vehicles in disjoints subsets $B_i, 1\leq i\leq n$, of $[0,\infty)\times\boldsymbol{R}$ are mutually independent random variables with Poisson distributions determined by the deterministic intensity functions $e_q^+(x,t)$ and $e_n^+(x,t)$, respectively; e.g.,

$$P(\boldsymbol{E}_n^+(B)=k) = \frac{\gamma^+(B)^k e^{-\gamma^+(B)}}{k!} \tag{5.1}$$

where

$$\gamma^+(B) = \int_B\int e_n^+(u,v)dudv. \tag{5.2}$$

For example, for $B=[0,x]\times(-\infty,t]$ and $\gamma^+(x,t)\equiv\gamma^+(B)$ for this B,

$$\gamma^+(x,t) = \int_0^x\int_{-\infty}^t e_n^+(u,v)dudv. \tag{5.3}$$

Alternatively (equivalently), we can have a one-dimensional nonhomogeneous Poisson arrival process $A(t)$ with arrival rate function $\alpha(t)$ as in (4.1) and let an arrival at time t occur in the spatial location x with density

$$f_t(x) = \frac{e_n^+(x,t)+e_q^+(x,t)}{\alpha(t)}, \qquad x\geq 0. \tag{5.4}$$

By (4.1), $\int_0^\infty f_t(x)dx = 1$ for all t. Given that this arrival occurs at location x, we make it a calling vehicle with probability $e_q^+(x,t)/[e_q^+(x,t)+e_n^+(x,t)]$ and a noncalling vehicle otherwise.

With this extra Poisson assumptions, the PALM results in [6], [7] imply the following. (See Theorem 3.1 of [7].)

Theorem 5.1: If, in addition to the assumptions in Section IV, $\boldsymbol{E}_n^+(x,t)$ and $\boldsymbol{E}_q^+(x,t)$ are independent two-dimensional Poisson counting processes, then the stochastic processes $\{Q(x,t): x\geq 0\}$ and $\{H_i(t): -\infty<t<\infty\}$ are Poisson processes. Moreover, $Q_i(t)$ for $i\geq 1$ are mutually independent Poisson random variables with means in (4.5).

Unlike Theorem 3.1 of [7], the processes $\boldsymbol{C}_q^+(x,t)$ and $\boldsymbol{C}_q^-(x,t)$ are not two-dimensional Poisson processes, because calls can enter and leave more than once. We also point out that the Poisson property for $Q_i(t)$ and $H_i(t)$ holds without the Markov assumption for each vehicle made in Section IV. However, without these Markov assumptions, the PDE's in (2.8) and (2.9) are no longer valid. In this case, $q(x,t)$ cannot be easily computed because the model has to keep track of the residual think time and call-holding time for each vehicle; see [7].

If only the location dynamics at temporal steady state are of interest, the arrival process can be a time-homogeneous Poisson process. All results of the Markovian PALM presented above remain valid simply because a stationary Poisson process is a special case of time-homogeneous Poisson process. In this case, as for the deterministic fluid model in Section II-B, the densities for noncalling and calling vehicles, $n(x)$ and $q(x)$, can be solved from (2.20) and (2.21), or given by (2.27) and (2.28), respectively.

VI. Approximating Vehicle Interactions in the Stochastic Models

A key to tractability in the stochastic traffic models in Sections IV and V is the assumption that vehicles (customers) do not interact; i.e., the nonstationary Markov location processes for different vehicles are conditionally independent, given their arrival times. This means that different vehicles cannot interact.

However, there is no such interaction restriction in the deterministic fluid model. For example, vehicle velocity commonly depends on vehicular density, with velocity decreasing as the density increases. This feature is easily incorporated into the deterministic fluid model via the PDE's in (2.8) and (2.9). For example, one possible model has velocity decrease linearly with density [2], [9], [10]. Velocity can reach its maximum value $v_m(x,t)$, when the vehicular density is close to zero at location x and time t. On the other hand, the velocity reduces to zero if the density is increased to a critical value (i.e., bumper-to-bumper density). That is, for some critical density $l_c > 0$,

$$v(x,t) = v_m(x,t)\left[1-\frac{n(x,t)+q(x,t)}{l_c}\right]. \tag{6.1}$$

This formula can easily be inserted into (2.8) and (2.9).

In a full stochastic analog of this phenomenon, the actual distribution of vehicles at any time would be random, so that both the vehicular density, however defined, and the resulting vehicle velocity defined by (6.1) also become random. We would need some sort of stochastic differential equation, corresponding to a complicated interacting stochastic particle system.

To be more concrete, we might count the random number of vehicles in the interval $(x, x+1]$ and let this be the "random density" at x, say $\boldsymbol{D}(x,t)$. We could then let the random velocity at location x and time t be

$$\boldsymbol{V}(x,t) = v_m(x,t)\left[1 - \frac{\boldsymbol{D}(x,t)}{l_c}\right]. \tag{6.2}$$

Obviously, (6.2) is a stochastic analog of (6.1).

Unfortunately, however, the stochastic model with (6.2) seems very difficult to analyze. Our idea is to approximately capture the behavior of complicated stochastic dynamics given by (6.2) by substituting appropriate relations among deterministic quantities in the deterministic analog (6.1). Indeed, we can apply (6.1) directly to the stochastic model if we interpret $n(x,t)$ and $q(x,t)$ as the densities of expected values. Then the deterministic velocity is allowed to depend on the expected values via $n(x,t)$ and $q(x,t)$. As for the deterministic model, (6.1) can be inserted into (2.8) and (2.9) and solved. Just as for the deterministic model, this can have a significant influence upon $v(x,t)$, $n(x,t)$, and $q(x,t)$.

Now we can construct a stochastic model according to the assumptions of Section IV using this resulting deterministic velocity field $v(x,t)$. By Theorem 4.1, the densities of the mean values, $n(x,t)$ and $q(x,t)$, in this stochastic model will be of the appropriate form. Moreover, if we let the arrival process have the Poisson property as assumed in Section V, then we will be able to deduce the Poisson distributional results as well. The resulting Markovian highway PALM based on (6.1) then is a candidate approximation for the stochastic model based on (6.2). Of course, it remains to evaluate the quality of such approximations.

As emphasized in [7], even though the PALM models cannot directly model vehicle interactions, they often can capture essential features. As illustrated by the examples in Section III, the stochastic model cannot directly represent vehicles slowing down in response to other vehicles in front of them slowing down, but we can represent the vehicles slowing down at a certain time and space under the assumption that an accident occurs there.

Similarly, we can indirectly capture the dependence of velocity upon vehicular density as in (6.1), if we make (6.1) apply to expected values. From the perspective of the deterministic fluid model, we can think of the stochastic model as an enhancement to be able to deduce approximate distributional conclusions.

VII. Conclusions and Future Work

We have presented a deterministic fluid model, a stochastic traffic model, and a Markovian highway PALM for a portion of a wireless network along a highway. Vehicles can enter and leave the system at multiple entrances and exits, and they are classified as noncalling and calling vehicles, depending on whether or not they have calls in progress. The deterministic model treats each type of vehicle as a continuous fluid, while the stochastic models consider the behavior of each individual vehicle. All models use the same two coupled PDE's or ODE's to describe the evolution of the system. The call density and call handoff rate are readily computable by solving the equations. The two kinds of models are complementary, because additional features such as the interdependency between velocity and vehicular density can be easily included in the fluid model, whereas the stochastic traffic models also give probability distributions.

Numerical examples were presented to illustrate the computability of our results and investigate various aspects of the time and space dynamics of wireless networks. The numerical results indicate that both the time-dependent behavior and the mobility of vehicles play important roles in determining the system performance. Our results show that even for systems in temporal steady state, the movement of vehicles and the calling patterns can significantly affect the offered loads in a given region of the system. Therefore, the models will be useful tools to examine various phenomena in wireless networks. Furthermore, our numerical examples also show how the proposed models can be used to compute approximate blocking probabilities, as illustrated in Section III. Thus, the models have the potential for evolving into tools for planning and engineering wireless networks.

This work can be extended in several areas. First, the quality of the approximation approach to capturing the interdependence between velocity and vehicular density via expected values in Section VI should be evaluated. Second, the models assume no capacity constraints. As shown in Section III, it is desirable to apply approximation techniques with the models to develop suitable approximate approaches to quantifying the blocking probability for new call attempts and handoff calls. (A start was also made in [7].) These approximation techniques should be further enhanced to consider retries of blocked calls. Lastly, the call blocking is actually affected by the channel-assignment strategy in use. These new models could be used to study the tradeoffs of various assignment strategies.

Appendix
A Time-Dependent Proportionality Result

Lemma 2.5: If for all x and t, we have

$$\lambda(x,t) = \lambda, \mu(x,t) = \mu, \beta(x,t) = \gamma(x,t), \frac{e_q^+(x,t)}{e_n^+(x,t)} = \frac{\lambda}{\mu}$$

and $\{x(t)|t \geq t_0\}$ satisfying (2.10), then

$$\frac{q(x(t_0),t_0)}{n(x(t_0),t_0)} = \frac{\lambda}{\mu} \text{ implies that } \frac{q(x(t),t)}{n(x(t),t)} = \frac{\lambda}{\mu} \text{ for all } t \geq t_0.$$

Proof: Let $\Phi(x,t) \equiv \lambda n(x,t) - \mu q(x,t)$. Substituting this and our hypotheses in (2.8) and (2.9) yields

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}[n(x,t)v(x,t)] = e_n^+(x,t) - \beta(x,t)n(x,t) - \Phi(x,t)$$

and

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial}{\partial x}[q(x,t)v(x,t)] = e_q^{+}(x,t) - \beta(x,t)n(x,t) + \Phi(x,t).$$

Combining these equations gives us

$$\frac{\partial \Phi(x,t)}{\partial t} + v(x,t)\frac{\partial \Phi(x,t)}{\partial x} = - \left[\lambda + \mu + \beta(x,t) + \frac{\partial v(x,t)}{\partial x}\right]\Phi(x,t). \quad \text{(A1)}$$

By the chain rule and the fact that $\{x(t)|t \geq t_0\}$ satisfying (2.10), we have

$$\frac{d\Phi(x(t),t)}{dt} = \frac{\partial \Phi(x(t),t)}{\partial t} + v(x(t),t)\frac{\partial \Phi(x(t),t)}{\partial x}.$$

Putting (A1) into the above equation now yields

$$\frac{d\Phi(x(t),t)}{dt} = -\left[\lambda + \mu + \beta(x,t) + \frac{\partial v(x,t)}{\partial x}\right]\Phi(x,t).$$

Hence for all $t \geq t_0$, we have

$$\Phi(x(t),t) = \Phi(x(t_0),t_0)\exp\left\{-\int_{t_0}^{t}\left[\lambda + \mu + \beta(x(s),s) + \frac{\partial v(x(s),s)}{\partial x}\right]ds\right\}.$$

From this, it follows that $\Phi(x(t_0),t_0) = 0$ implies $\Phi(x(t),t) = 0$ for all $t \geq t_0$, which proves the lemma.

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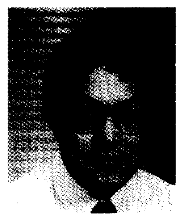

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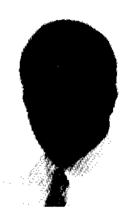

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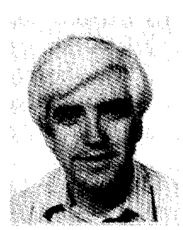

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