

# Resolving Conflicting Preferences in School Choice: the “Boston” Mechanism Reconsidered

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**ABSTRACT:** The Boston mechanism is among the most popular school choice procedures in use. Yet, the mechanism has been criticized for its poor incentive and welfare performances, which led the Boston Public Schools to recently replace it with Gale and Shapley’s deferred acceptance algorithm (henceforth, DA). The DA elicits truthful revelation of “ordinal” preferences whereas the Boston mechanism does not; but the latter induces participants to reveal their “cardinal” preferences (i.e., their relative preference intensities) whereas the former does not. We show that cardinal preferences matter more when families have similar ordinal preferences and schools have coarse priorities, two common features of many school choice environments. Specifically, when students have the same ordinal preferences and schools have no priorities, the Boston mechanism Pareto dominates the DA in ex ante welfare. The Boston mechanism may not harm but rather benefit participants who may not strategize well. In the presence of school priorities, the Boston mechanism also tends to facilitate a greater access than the DA to good schools by those lacking priorities at those schools. These results contrast with the standard view, and cautions against a hasty rejection of the Boston mechanism in favor of mechanisms such as the DA.

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## 1 Introduction

Public school choice — the initiative for broadening families’ access to schools beyond their residence area — has broad public support and has been increasingly adopted across the US and abroad.<sup>1</sup> Yet, how to operationalize school choice, i.e., what procedure should be used to assign students to schools, remains hotly debated.

An important debate centers around the procedure known as the “Boston” mechanism, which was used by Boston Public Schools (BPS) until the 2004-2005 school year to assign K-12 pupils to the city schools. Beginning with the seminal article by Atila Abdulkadiroğlu and Tayfun Sönmez (2003), authors recognized problems with the mechanism, and BPS ultimately decided in 2005 to replace the mechanism with the student-proposing deferred acceptance (henceforth DA) mechanism, originally proposed by David E. Gale and Lloyd S. Shapley (1962). While the switch has received some academic support, it was met with resistance from some parents. Most important, the Boston mechanism remains still among the most popular in school choice. It is thus sensible to gain fuller understanding of the two mechanisms before a similar switch is recommended more widely. In this context, the current paper provides a new perspective on the debate and in so doing cautions against hasty rejection of the Boston mechanism, say in favor of the DA.

The criticisms of the Boston mechanism are multi-faceted, but they are traced to its poor incentive property. In the Boston mechanism, the seats of each school are assigned *according to the order students rank that school*; those who rank it first are accepted first, followed by those who rank it second *only* when seats are available, and so forth. Assignments are made among those who rank a school the same in the order of student priorities at that school (ties being broken randomly), but students ranking the school more highly have strict priority at that school ahead of those who don’t. This means that students may not wish to rank schools truthfully. In particular, they may refrain from top-ranking a popular school: Top-ranking such a school will not improve their odds with that school appreciably, but it may rather jeopardize their shot at their second, or even less, preferred school, which could have been available to them

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<sup>1</sup>Government policies promoting school choice take various forms, including interdistrict and intradistrict public school choice as well as open enrollment, tax credits and deductions, education savings accounts, publicly funded vouchers and scholarships, private voucher programs, contracting with private schools, home schooling, magnet schools, charter schools and dual enrollment. See an interactive map at <http://www.heritage.org/research/Education/SchoolChoice/SchoolChoice.cfm> for a comprehensive list of choice plans throughout the US. Korea and Japan are adopting their versions of school choice.

if they have top-ranked it. That strategic ranking may be beneficial presents some difficulties. First, it is not clear how families should strategize their rankings of schools. Second, there is a potential issue of equity since participants who are acting naively or honestly may be disadvantaged by those who are strategically sophisticated.

The DA mechanism avoids the incentive problem by making truthful ranking a dominant strategy for the participants, a property known as “strategy-proofness” (Lester E. Dubins and David A. Freedman 1981; Alvin E. Roth 1982). In the DA, both students and schools rank each other. In the first round students apply to their top-ranked schools, and the schools select from them according to their rankings of students, ties being broken randomly, up to their capacities, but only *tentatively*, and reject the others. In the second round, those rejected by their top choice apply to their second-ranked schools, and schools reselect from those held from the first round and from new applicants, up to their capacities (only based on the school’s ranking of them) again tentatively, and reject the others. This process continues until no students are rejected, at which point the tentative assignment becomes final. Since schools select the students based *solely on schools’ own priorities*, top-ranking even a very popular school under the DA does not sacrifice a student’s chances at less preferred schools in the event she fails to get into her top school.

Clearly, strategy-proofness is an important property to have, but that property alone would not be sufficient. For instance, a pure lottery assignment is also strategy-proof for a trivial reason but would not be considered desirable. The DA scores well on the welfare ground as well, *so long as schools have strict rankings over all students* (in addition to the latter having strict preferences over schools). In that case, the DA produces the so-called *student optimal stable matching* — a matching that is most preferred by every student among all stable matchings (Gale and Shapley 1962).<sup>2</sup> By contrast, the Boston mechanism may produce “any” stable matching in full information Nash equilibrium, that is, *if all participants know all other participants’ preferences as well as their priorities at all schools* (Haluk Ergin and Sönmez 2006).

In reality, however, schools do not have strict priorities over all students. For instance, the BPS gives each student priorities based on whether he/she has a sibling enrolled at a school or whether he/she lives within the walkzone of a school. This means that many students fall in the same priority class. In the DA, any ties among these students must be broken randomly. This makes the assumption of full information particularly problematic. Not only is it unlikely for students to know others’ preferences, but it is simply impossible for them to know others’ — even their own — priorities at schools if they are chosen randomly after students submit their rankings.

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<sup>2</sup>A matching is *stable* if no student or school can do strictly better by breaking off the current matching either unilaterally or by rematching with some other partner without making it worse off.

More importantly, coarse priorities alter the nature of welfare consideration itself. Families tend to value similar qualities about schools (e.g., safety, academic reputation, etc.), which causes them to have similar ordinal preferences. Indeed, the BPS data exhibits strong correlation in students’ preferences over schools. In 2007-2008, only 8 out of 26 schools (at grade level 9) are overdemanded — that is, top-ranked by more participants than the seats available —, whereas an average of 22.21 (std 0.62) schools should have been overdemanded if their preferences had been uncorrelated.<sup>3</sup> Correlated ordinal preferences entail conflicts among participants, and the conflicts cannot be resolved by the school priorities if they are coarse. The standard welfare concept such as Pareto efficiency or student optimal stable matching then loses its relevance; for instance, if all students have the common ordinal preferences and schools have no priorities, then any arbitrary assignment will meet these efficiency standard, and mechanisms become indistinguishable on these criteria. Yet this does not mean that all assignments or all mechanisms are equally desirable. Participants may still differ in their relative preferences intensities over alternative schools, so it is sensible to resolve conflicts based on these intensities (henceforth called **cardinal utilities**). For instance, if a seat is competed by two students, it seems sensible to assign that seat to an individual who would gain more from that seat relative to her next alternative.

The Boston mechanism and the DA differ in the way they resolve conflicts. The DA resolves the conflicts purely by random lotteries, so any two students with the same preferences must be treated the same way (since they report truthfully), regardless of their cardinal utilities. In other words, the outcome of the DA is completely insensitive to the underlying *cardinal* preferences of students. By contrast, the Boston mechanism allows participants to influence how ties are broken, so it has the potential to resolve conflicts based on their cardinal utilities. In fact, the feature of the Boston mechanism often vilified as engendering “gaming” or “strategizing” may be useful for efficient resolution of conflicting interests. These subtleties didn’t go unnoticed by the parents. In the wake of the BPS school redesign, parents noted:

... if I understand the impact of Gale Shapley, and I’ve tried to study it and I’ve met with BPS staff... I understood that in fact the random number ... [has] preference over your choices... (Recording from the BPS Public Hearing, 6-8-05).

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<sup>3</sup>This comparison is based on submitted preferences. Since the DA has been in place since 2005, it is strategy-proof, and since BPS paid significant attention in communicating that feature of the DA to the public, we assume that those submitted preferences are a good approximation of the underlying true preferences. For the counter-factual, we generated 100 different preference profiles by drawing a school as first choice for each student uniformly randomly from the set of schools and compute the number of overdemanded schools given school capacities. Correlation among ordinal preferences, or more technically among multidimensional nonnumeric valued variables, the dimensions of which represent ordinal rankings of the nonnumeric values is not a well-studied topic in Statistics. Developing a correlation statistics and its theory for that problem is beyond the scope of the current work.

I'm troubled that you're considering a system that takes away the little power that parents have to prioritize... what you call this strategizing as if strategizing is a dirty word... (Recording from Public Hearing by the School Committee, 05-11-04).

We argue that the participants' cardinal welfare can be captured well by *ex ante Pareto efficiency*,<sup>4</sup> — this is useful since the welfare evaluation need not involve interpersonal utility comparison — and that, from that perspective, the DA entails a clear and tangible welfare loss relative to the Boston mechanism, given common ordinal preferences and coarse priorities. To illustrate, suppose three students,  $\{1, 2, 3\}$ , are to be assigned to three schools,  $\{s_1, s_2, s_3\}$ , each with one seat. Schools have no intrinsic priorities over students, and students' preferences are represented by the following von-Neumann Morgenstern (henceforth, vNM) utility values, where  $v_j^i$  is student  $i$ 's vNM utility value for school  $j$ :

	$v_j^1$	$v_j^2$	$v_j^3$
$j = s_1$	0.8	0.8	0.6
$j = s_2$	0.2	0.2	0.4
$j = s_3$	0	0	0

Every feasible matching is stable due to schools' indifferences. More importantly, any such assignment is *ex post* Pareto efficient, hence *student optimal stable*, since students have the same ordinal preferences. Yet, their ex ante welfare depends crucially on how the students' conflicting interests are resolved.

To see this, first consider the DA mechanism with random tie breaking. All three students submit true (ordinal) preferences, and they are assigned to the schools with equal probabilities. Hence, the students obtain expected utilities of  $EU_1^{DA} = EU_2^{DA} = EU_3^{DA} = \frac{1}{3}$ .

This assignment is *ex ante* Pareto-dominated by the following assignment: *Assign student 3 to  $s_2$ , and students 1 and 2 randomly between  $s_1$  and  $s_3$* , which yields expected utilities of  $EU_1^B = EU_2^B = EU_3^B = 0.4 > \frac{1}{3}$ . Surprisingly, this latter, Pareto-dominating, assignment arises as the unique equilibrium of the Boston mechanism.<sup>5</sup> Students 1 and 2 have a dominant strategy of ranking the schools truthfully, and student 3 has a best response of (strategically) ranking  $s_2$  as her first choice.<sup>6</sup>

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<sup>4</sup>An assignment is ex ante Pareto efficient if it is Pareto efficient prior to the realization of any random lotteries necessary to break ties, namely it is impossible to reallocate probability shares of different schools in a Pareto improving fashion.

<sup>5</sup>This does not contradict Ergin and Sönmez (2006)'s finding that the Boston mechanism is (weakly) Pareto dominated by the DA, which relies on strict preferences by the schools.

<sup>6</sup>In equilibrium, student 2 will be assigned to  $s_2$ , and students 1 and 2 will be assigned between  $s_1$  and  $s_3$  with equal probabilities, for these students will have lower priority than student 3 at school  $s_2$ .

This example has assumed, for ease of illustration, that participants have complete information about their preferences, but as will be seen, the underlying insight holds much more generally. In our baseline model, we consider a general school choice setting in which participants have common ordinal preferences and schools have no priorities. These latter two assumptions are needed to generate a clear result for the Boston mechanism; it is difficult to analyze the strategic interaction of players in a fully general setting. These two assumptions reflect the salient features of school choice — correlated preferences and coarse school priorities — and serve to isolate their effects in the most transparent form. Some real world problems in fact involve no priorities on the school side. The Supplementary round of the New York City mechanism and the choice procedure of Seoul set to begin in 2010 are two such examples.

Other than these two features, we make no further assumptions. Importantly, we consider more realistic Bayesian setting in which participants have incomplete information about others' preferences. We then focus on Bayesian Nash equilibrium in symmetric strategies — those that specify the same (possibly mixed) action for students with the same von-Neumann Morgenstern (vNM) utilities. The symmetry restriction seems well justified especially when no particular pattern of asymmetry is known a priori. Our results are summarized as follows:

- Generalizing the example, we show that every participant is at least weakly better off in any symmetric equilibrium of the Boston mechanism than in the dominant strategy equilibrium of the DA. This result rests on the intuition that the Boston mechanism allows the participants to communicate their cardinal utilities and resolve the conflicting interests in a more efficient way than the DA.
- An important concern about the Boston mechanism is treatment of those participants who may not be sophisticated in strategizing. We relax our baseline model to consider such naive participants. While strategically sophisticated players do generally better than naive ones with the same vNM values (almost by definition), there is a sense in which that naive players benefit from the presence of strategic players. The latter participants avoid ranking popular schools highly, and this raises the naive participants' odds of getting into those schools. We show that naive participants have a higher chance to attend a popular school under the Boston mechanism than under the DA, and some of them may be better off from the former.
- An important goal of school choice is to provide students in poor neighborhoods with opportunity to attend good schools. This goal will be served best by guaranteeing equal access to all schools regardless of where a child lives. Yet, equal access is compromised by neighborhood priorities which schools award to children living in their proximate neighborhoods. The extent to which the neighborhood priority inhibits the access by students

in failing school areas to good schools differs between the two mechanisms. In the DA, a student need not give up his neighborhood priority to be considered for other (good) school, whereas the Boston mechanism forces the participants to give up their neighborhood priority when ranking other schools highly. In other words, there is a sense in which the inhibitive power of the neighborhood priority is diminished in the Boston mechanism, and this increases access to good schools by those who do not have priority at those schools.

One may take away several broad implications from the current paper. First, we offer a new welfare perspective on school choice — the importance of resolving conflicting interests based on participants’ cardinal utilities. This perspective has been missing in the prior school choice debate because authors have largely focused on “ordinal” notions of welfare such as ex post Pareto efficiency and student optimal stable matching. However, we believe the current “cardinal welfare” perspective is very important in settings such as school choice where participants have similar ordinal preferences.

Second, from this perspective of efficient conflict resolution and more precisely that of ex ante Pareto efficiency, there is a clear sense in which the DA entails welfare loss relative to the Boston mechanism. It is essential to understand that this welfare loss is the “price” paid by the DA for achieving strategy-proofness. This can be easily seen in the above example; the very feature of the Boston mechanism that engenders strategizing (i.e., student 3 lying about her preference) leads to efficient resolution of conflicts in that case. More formally, it is not possible for (symmetric) mechanisms to have both strategy-proofness and ex ante Pareto efficiency in general circumstances (Lin Zhou 1990).

Third, the tradeoff between incentive and cardinal welfare (or ex ante Pareto efficiency) has a policy implication on the design of desirable school choice procedure. As is much emphasized in the prior literature, strategy-proofness is an important property. Somewhat less appreciated, however, is what we highlight: namely, strategy-proofness has its own cost that appears to be important particularly in the school choice problem<sup>7</sup> (with a lot of potential conflicts of interests). This is not to argue that the DA should be rejected in favor of say the Boston mechanism (or “the clock should be turned back” in the case of BPS).<sup>8</sup> Such a conclusion is

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<sup>7</sup>Exceptions are Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak and Roth (Forthcoming), who find that strategy-proofness and student optimal stable matching are not compatible. The welfare cost they identify are ex post inefficiencies and thus differs from the ex ante inefficiencies we focus on. More important, these papers do not deal with the Boston mechanism and thus the tradeoff they focus on has no bearing on the choice between the DA and the Boston.

<sup>8</sup>Incidentally, the clock did turn back in the case of Seattle Public Schools (SPS), which has recently switched from a version of the DA to a version of the Boston mechanism. See <http://www.seattleschools.org/area/newassign/current.assignplan.html> for a more detailed description.

unwarranted, just as it would be unwarranted to reject the Boston mechanism on account of what we know so far. In the end, one could ultimately find strategy-proofness to be so important to tolerate its cost. What is important however is that the decision must be informed on both sides of the tradeoff. More importantly, further work is needed to quantify the benefits and costs associated with strategy-proofness, particularly on the empirical and experimental fronts. More work is also needed to explore ways to balance the tradeoffs between incentives and welfare better than the DA or the Boston mechanism.

## 2 DA vs. Boston in the Baseline Model

We first consider the Bayesian model in which each student (family) knows her own preferences about the schools but does not know about the others' except for the underlying probability distribution. Such a model is realistic, more so than the complete information model in which the agents are assumed to know all other players' preferences. We show that if the students share the same ordinal preferences but may differ in their preference intensities, the Boston mechanism Pareto dominates the DA.<sup>9</sup>

There are  $m \geq 2$  schools,  $S = \{s_1, \dots, s_m\}$  with the index set  $A := \{1, \dots, m\}$ . School  $s_a \in S$  has capacity  $q_a$ . There are  $n \geq 2$  students each of whom draws vNM utility values  $\mathbf{v} = (v_1, \dots, v_m)$  about the schools from a finite set  $\mathcal{V} = \{(v_1, \dots, v_m) \in [0, 1]^m | v_1 > v_2 \dots > v_m \text{ and } g(v_1, \dots, v_m) = 0\}$  with probability  $f(\mathbf{v})$ .<sup>10</sup> The restriction  $g$  may not impose any restriction (if  $g$  is identical to zero) or it could represent some normalization (e.g.,  $g(v_1, \dots, v_m) = \sum_{a \in A} v_a - 1$  or  $1 - v_1 + v_m$ ).<sup>11</sup> The students all have the same ordinal preferences preferring

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<sup>9</sup>The notion of ex ante efficiency and that the Boston mechanism may Pareto dominate the DA from an ex ante efficiency standpoint were first brought to the debate by Abdulkadiroğlu, Yeon-Koo Che, and Yosuke Yasuda (2008) in their model of continuum of students. Subsequently, Antonio Miralles (2008) and Clayton Featherstone and Muriel Niederlee (2008) examined the same issue. Miralles (2008) proposes a variant of the Boston mechanism with round-wise tie breakers and shows that it has similar superior ex-ante efficiency properties as the CADA mechanism proposed by Abdulkadiroğlu, Che, and Yasuda (2008), with a continuum of students with complete information. Both the continuum of agents and the particular tie-breaking rule are essential to his results, whereas our current results are obtained with finite students with incomplete information. Featherstone and Niederlee (2008) study an incomplete information set up. They find that, when student preferences are not correlated and they are uniformly distributed and schools are completely symmetric, truth telling is a Bayesian Nash equilibrium of the Boston mechanism, so the Boston mechanism assigns more students to their first choices. There is little conflict to resolve in such symmetric environments because almost everybody can get his first choice. However, their finding is complementary to ours as we focus on a correlated environment with significant conflict. The subsequent results dealing with the effect of strategic Naivete and of neighborhood priority under Boston mechanism vis-a-vis DA have no analogues in their papers.

<sup>10</sup>The finiteness is assumed only to simplified the existence of the Bayesian equilibrium of the Boston mechanism. The argument for the comparison works for any arbitrary distribution.

<sup>11</sup>In the former case, the sum of the vNM utility values is normalized to be 1, whereas in the latter case,

school  $s_a$  to school  $s_b$  if  $a < b$ . Importantly, though, the students may differ in their relative preference intensities.

We assume that  $\sum_{a \in A} q_a \geq n$ ; namely the total capacities of all schools are large enough to accommodate all students. This is well justified since the public school system ensures that there are enough seats available to all students, and is without loss since some school can be treated as a (common) outside option. Let  $k := \min\{l \mid \sum_{a=1}^l q_a \geq n\}$  be the marginal school. Note  $\sum_{a=1}^{k-1} q_a < n$ . As we will show, no student will be assigned to a school less preferred to this marginal school in both mechanisms. Let  $S'$  be the set of essential schools that would accept non-zero students, i.e.,  $S' = \{s_1, \dots, s_k\}$  and its index set is defined as  $A' := \{1, \dots, k\}$ .

**Gale-Shapley's Deferred Acceptance Algorithm:** It is a dominant strategy for each student to report truthfully, so we focus on such an equilibrium. Each student is then assigned to school  $s_a$ , with probability

$$\hat{P}_a = \begin{cases} \left(\frac{n-q_1}{n}\right)\left(\frac{n-q_1-q_2}{n-q_1}\right) \dots \left(\frac{q_a}{n-q_1-\dots-q_{a-1}}\right) = \frac{q_a}{n} & \text{if } a < k \\ \frac{n-q_1-\dots-q_{k-1}}{n} & \text{if } a = k, \\ 0 & \text{if } a > k, \end{cases}$$

or more succinctly for each  $a \in A'$

$$\hat{P}_a = \frac{\min\{q_a, n - \sum_{b=1}^{a-1} q_b\}}{n}, \quad (1)$$

and  $\hat{P}_a = 0$  for all  $a \in A \setminus A'$ .

**Boston Mechanism:** Let  $\Pi$  be the set of ordinal rankings of  $S$ , and  $\Delta(\Pi)$  the set of probability distributions over  $\Pi$ . A Bayesian strategy is a mapping  $\sigma : \mathcal{V} \rightarrow \Delta(\Pi)$ . We focus on a symmetric strategy where every agent follows the same Bayesian strategy, meaning that they play the same mixed strategy for each realized  $\mathbf{v} \in \mathcal{V}$ .

It is a dominated strategy for any student to put any school  $\{s_k, \dots, s_m\}$  in the top  $k-1$  rankings and put any school  $\{s_{k+1}, \dots, s_m\}$  in his/her top  $k$  rankings. Hence, in any equilibrium in undominated strategies, all seats of schools in  $\{s_1, \dots, s_{k-1}\}$  are assigned, and no seats in schools  $\{s_{k+1}, \dots, s_m\}$  are assigned. A symmetric Bayesian equilibrium with this property exists.<sup>12</sup> Fix *any* such equilibrium  $(\sigma^*, \dots, \sigma^*)$ .

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$v_1 = 1$  and  $v_m = 0$ .

<sup>12</sup>The undominatedness restriction does not cause any problem since we can simply redefine the range to be  $\Delta(\tilde{\Pi})$ , where  $\tilde{\Pi}$  is the set of ordinal rankings within  $S' = \{s_1, \dots, s_k\}$ . Each type  $\mathbf{v}$ -student has finite pure strategies (equal to the number of all possible ordinal rankings within this restricted domain), and her payoff is well defined for each profile of pure strategies. The player's payoff is then linear in a mixed strategy. Treating each type of student as a distinct player, there are only finite players. Hence, the existence of the equilibrium follows from John F. Nash (1950)'s existence theorem.

For any mixed strategy  $\sigma \in \{\sigma^*(\mathbf{v})\}_{\mathbf{v} \in \mathcal{V}}$  used in equilibrium, let  $P_a(\sigma)$  be the probability that a student is assigned to school  $s_a$  if the student employs the strategy  $\sigma$  and all other students play the symmetric equilibrium strategy  $\sigma^*$ . From the above argument,  $P_a(\sigma^*(\mathbf{v})) = 0$  for all  $a \in A \setminus A'$  and all  $\mathbf{v} \in \mathcal{V}$ . For each  $a \in A'$ , we must have

$$\sum_{\mathbf{v} \in \mathcal{V}} n P_a(\sigma^*(\mathbf{v})) f(\mathbf{v}) = \min \left\{ q_a, n - \sum_{b=1}^{a-1} q_b \right\}. \quad (2)$$

To see this, note first that the LHS is the total expected number of students that are assigned to school  $s_a$ . There are  $n$  students and each has  $\mathbf{v}$  with probability  $f(\mathbf{v})$ , and then plays  $\sigma^*(\mathbf{v})$  to get assigned to school  $s_a$  with probability  $P_a(\sigma^*(\mathbf{v}))$ . Summing over possible types gives the expected number of students assigned to school  $s_a$ . The RHS represents the total number of seats at school  $s_a$  that are assigned in equilibrium. Recall that all seats are assigned at school  $s_a$  for  $a < k$ , and no seats are assigned at school  $s_a$  for  $a > k$ , which explains the particular expression on the RHS. Clearly, equation (2) must hold for  $a \in A'$ .

Fix any type  $\tilde{\mathbf{v}} \in \mathcal{V}$  of student. Suppose that student picks the following strategy:  $\tilde{\sigma} := \sum_{\mathbf{v} \in \mathcal{V}} \sigma^*(\mathbf{v}) f(\mathbf{v})$ . That is to say,  $\tilde{\sigma}$  involves playing  $\sigma^*(\mathbf{v})$  with probability  $f(\mathbf{v})$ , i.e., according probability distribution of types that play that strategy. Then, that student will be assigned to school  $s_a \in S'$  with probability

$$P_a(\tilde{\sigma}) \equiv \sum_{\mathbf{v} \in \mathcal{V}} P_a(\sigma^*(\mathbf{v})) f(\mathbf{v}) = \frac{\min\{q_a, n - \sum_{b=1}^{a-1} q_b\}}{n} = \hat{P}_a, \quad (3)$$

where the first equality follows from (2) and the second follows from (1).

Since  $\tilde{\sigma}$  need not be an equilibrium strategy, we must have

$$\sum_{a \in A} \tilde{v}_a P_a(\sigma^*(\tilde{\mathbf{v}})) \geq \sum_{a \in A} \tilde{v}_a P_a(\tilde{\sigma}) = \sum_{a \in A} \tilde{v}_a \hat{P}_a.$$

In other words, the following is true:

**Theorem 1.** *In any symmetric equilibrium of the Boston mechanism, each type of student is weakly better off than she is under the DA with any symmetric tie-breaking.*

**Remark 1.** *While we focus on a Bayesian model since it is more realistic, a similar result holds in a complete information model when the market is large in the sense the size of seats at each school as well as the total population go to infinity while the number of schools remains finite.<sup>13</sup>*

*In fact, the distinction between complete information model and the Bayesian model disappears as the market becomes large in this sense, since all that matters is the aggregate distribution of the participants adopting different strategies. The result is available from the authors.*

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<sup>13</sup>Che and Fuhito Kojima (Forthcoming) consider a similar notion of large market, whereas Mihai Manea (Forthcoming) consider a different notion where the number of objects (schools) here also tends to infinity.

### 3 Does the Boston Mechanism Harm Naive Players?

The appeal of the strategy-proof mechanisms such as the DA and others (e.g., the top trading cycles mechanism) is that participants' strategic sophistication becomes irrelevant since ranking schools according to their true preferences is their dominant strategy. By contrast, the Boston mechanism may expose strategically naive participants. Indeed, Abdulkadiroğlu et al. (2006) provide a potential evidence that some players may have behaved naively and suffered as a consequence under the Boston mechanism. They find that as much as 20% of the applicants ranked two overdemanded schools as their first and second choices.<sup>14</sup> These applicants could never get admitted by their second choice schools, so they would have done better by using their second rank for some other school. The evidence is not conclusive, though, since ex post suboptimal behavior does not mean that their behavior was necessarily suboptimal ex ante. Their behavior may as well have been optimal if they put sufficiently high chance, quite possibly rationally, to the event that these schools are not overdemanded. Nevertheless, the concern about the potential strategic exploitation of strategically naive participants was an important consideration in the redesign of the BPS program.<sup>15</sup>

A theoretical justification of this view is given by Parag Pathak and Sönmez (2008), who argue that strategically sophisticated participants exploit naive ones in the Boston mechanism, to such an extent that *the former effectively enjoys a higher priority over the latter at every school except for the latter's most preferred*. While naive players are generally expected to do worse, the particular sense and extent to which they are exploited is striking. A closer look reveals, however, that this characterization rests crucially on the two modeling features: *strict school priorities* and *complete information by strategic players*. Given these assumptions, each strategic player knows exactly who her competitors are and what their priorities are at each school. So, if a strategic player realizes that she has no shot at her favorite school but that her competitor at the next best school is a naive player and that school is the naive player's second most preferred, say, then the former will exploit the latter by simply top-ranking that school under the Boston mechanism. Therefore, there is a clear sense in which a naive player is harmed by strategic player when schools have strict priorities and (strategic) players have complete information. The welfare effect of strategic play can be formalized precisely in our common ordinal preference domain.

**Proposition 1.** *With complete information, common ordinal preferences and strict school pri-*

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<sup>14</sup>A school is said to be overdemanded if more applicants top-rank the school than the seats available at that school.

<sup>15</sup>BPS Superintendent Payzant noted: "A strategy-proof algorithm levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well." (Superintendent Payzant's Memorandum to the School Committee - May 25, 2005)

orities under the Boston mechanism, if a naive student  $i^*$  becomes strategically sophisticated, that student becomes weakly better off but every other student, strategic as well as naive, becomes weakly worse off. If  $i^*$  becomes strictly better off, then some student becomes strictly worse off.

**Remark 2.** Under general preferences, Pathak and Sönmez (2008) obtain a similar result but for only strategically sophisticated players. The restriction to common ordinal preferences allows us to strengthen this comparative statics result. The proof of this result appears in the Appendix.

Clearly, this conclusion depends sensitively on the assumption of strict school priorities and complete information. Absent complete information, a strategic player cannot be sure who she will face as competitors, so she cannot target naive players for manipulation. Hence, a naive player need not be the victim of the strategic behavior. On the contrary, a naive player may actually benefit from a strategic play. Given non-strict school priorities, ties are broken randomly, so it is impossible for the strategic player to know the priorities of her competitors. Hence, a strategic player may end up forgoing a spot at her favorite school even though she would have gotten it had she ranked it truthfully. That spot will then go to another participant; and a naive player may as well be the beneficiary. In fact, there is a clear sense in which naive players benefit from the presence of strategic behavior when schools have coarse priorities and participants have similar ordinal preferences. In that case, strategic players tend to avoid popular schools, and this increases the chance for naive players to get into their favorite schools (likely be the popular schools given correlated ordinal preferences), which they will rank truthfully as first choice.

To illustrate, consider our example in Introduction, except that now each school has quota of 2, and there are two students of each type, one naive and one strategically sophisticated. In other words, there are total of six seats and six students. Under the DA, every student ranks truthfully, and the assignment is uniform, just as before, so each student receives expected payoff of  $1/3$ . Next consider the Boston mechanism. Naive students (there are three, one for each type) all rank schools truthfully, namely  $s_1 - s_2 - s_3$  in that order. One can also see that the strategic students rank the same as before; that is, type 1 and 2 students rank  $s_1 - s_2 - s_3$ , and the type 3 student ranks  $s_2 - s_1 - s_3$ . Consequently, strategic type 3 gets assigned to school  $s_2$  for sure and receives the expected payoff of 2. All others, strategic and naive, get assigned to the schools with probabilities  $(P_{s_1}, P_{s_2}, P_{s_3}) = (0.4, 0.2, 0.4)$ . It is true that naive students lose priority at school  $s_2$  to the strategic type 3 student; but they enjoy a higher probability of assignment to school  $s_1$  due to that strategic player. As seen by Proposition 1, this latter benefit never arises in the complete information with strict school priorities.

Type 1 and 2 students, strategic as well as naive, receive expected payoff of 0.36 and type 3 naive student gets 0.32. The naive type 3 student is worse off under the Boston ( $0.32 < 1/3$ ), but the two naive type 1 and 2 students are better off under the Boston ( $0.36 > 1/3$ ). Indeed,

naive type 1 and type 2 students benefit from the presence of the strategic type 3 student who refrains from top-ranking  $s_1$ . If both of type 3 students were naive, then the assignment would be the same as the DA, so all four remaining students (including two naive students) would be worse off; and if the two type 3 students were both strategic, all four students would be better off.

The positive externalities that strategic players confer to naive players do not arise in the model of complete information and strict priorities, as seen by the above Proposition. But they arise generally. Consider our general Bayesian model. Suppose now that each type  $\mathbf{v} \in \mathcal{V}$  of student is naive with probability  $x \in (0, 1)$ . This does not change the analysis of the DA. The outcome of the Boston mechanism is affected by the presence of naive students.

**Theorem 2.** *(i) In any symmetric Bayesian equilibrium of the Boston mechanism with naive students, all strategic participants are at least weakly better off under the Boston mechanism than under the DA. (ii) Suppose a strategic player manipulates with positive probability. Then, every naive player is assigned to each of top  $j$  schools,  $\{s_1, \dots, s_j\}$ , for some  $j \in A$ , with weakly higher probability and to some school in that set with strictly higher probability under the Boston mechanism than under the DA. (iii) If strategic students with type  $\mathbf{v}$  rank the schools truthfully in equilibrium, then naive students with the same preference type  $\mathbf{v}$  are (at least weakly) better off from the Boston mechanism than the DA.*

*Proof.* The Pareto dominance for the strategic players can be proven by the same argument as before. The second statement is shown as follows. Let  $j$  be the smallest index in  $A$  such that there exists some type of a strategic player that does not rank school  $s_j$  as  $j$ -th. (Call the type “manipulating” type.) By definition, the manipulation involves ranking  $j$  lower than  $j$ -th position (i.e., ranking it  $l$ -th for some  $l > j$ ). Since each player, both strategic and naive, ranks school  $s_{j'}$  at the  $j'$ -th position for  $j' < j$ , she is assigned to  $s_{j'}$ , for  $j' < j$ , with the same probability under the Boston mechanism as under the DA. When the manipulating type player is rejected by all schools  $s_1, \dots, s_{j'}$  (which occurs with positive probability), a naive player will have a higher priority than such a player and the same priority as the other strategic player. Hence, a naive player will have higher probability of assignment to school  $s_j$  under the Boston mechanism than under the DA. This completes proof of (ii). The third statement follows easily. Since a strategic player with  $\mathbf{v}$  ranks the schools truthfully in equilibrium and is weakly better off from Boston than from DA, the naive students with the same  $\mathbf{v}$  must be also weakly better off from the Boston. ■

**Remark 3.** *Pathak and Sönmez (2008) obtain a result similar to Theorem 2-(i). Their result holds given selection of a Pareto dominant equilibrium under the Boston mechanism but for general preferences. The current result holds in any symmetric Bayesian Nash equilibrium but for common ordinal preferences. A more significant difference is that they assume strict school*

*priorities and complete information on the part of strategic players, whereas the current model assumes no school priority and incomplete information.*

## 4 Neighborhood Priority and Access to Good Schools

Neighborhood priority is a common practice in school choice programs. For instance, students who live within 1 mile from an elementary school, within 1.5 miles from a middle school, and within 2 miles from a high school are given priority in attending those schools in Boston. On the other hand, one of the major goals of public school choice is to provide equal access to good schools for every student, especially for those in poor neighborhoods with failing schools.<sup>16</sup> This goal is compromised by neighborhood priority.

The extent to which the neighborhood priority inhibits the access by students in failing schools to good schools differs between the two mechanisms. Under the DA, it is a dominant strategy to report preferences truthfully regardless of one's or others' priorities at schools. In other words, one does not need to give up his neighborhood priority to compete for other schools. This is in sharp contrast to what happens under the Boston mechanism. When a student does not rank his neighborhood school as first choice under the Boston, he loses his neighborhood priority at that school to those who rank it higher in their choice list. Similarly, if he ranks his neighborhood school as first choice, then he gives up competition at the other schools. In either case, another student would be able to improve her odds at that school or some other school. That feature of the Boston mechanism provides strategic opportunities at good schools for students living within the proximity of failing schools.

We illustrate this point by modifying our example as follows. There are six students to be assigned to three schools,  $\{s_1, s_2, s_3\}$ , each with two seats. Each school  $s_a$  is located in neighborhood  $a = 1, 2, 3$ . There are two students living in each neighborhood  $a$ , one of whom living within the walk zone of school  $s_a$ , the other in the extended neighborhood. We will refer to the one in the walk zone as neighborhood  $a$  student, and the other one in the extended neighborhood as neighborhood  $a^x$  student. The neighborhood  $a$  student is entitled to neighborhood priority at  $s_a$ . As before, the students have identical ordinal preferences,  $s_1 \succ s_2 \succ s_3$ . However their preference intensity for their neighborhood school may be greater. To capture this feature, we assume that cardinal preferences of the students are represented by the following vNM utility values, where  $v_j^a$  and  $v_j^{a^x}$  are the vNM utility value of students in the neighborhood

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<sup>16</sup>This goal is aimed in different ways in other forms of choice as well. For example, charter schools tend to locate in disadvantaged neighborhoods (see Caroline M. Hoxby and Sonali Murarka (2009) for the case of New York). No Child Left Behind is an attempt at the federal level that also aims to provide more choices for students in schools that do not meet state standards for at least two consecutive years. For further explanation, see <http://www.ed.gov/nclb/overview/intro/4pillars.html>

$a$  for  $s_j$ :

	$v_j^1, v_j^{1^x}$	$v_j^2, v_j^{2^x}$	$v_j^3, v_j^{3^x}$
$j = s_1$	0.75	0.55	0.55
$j = s_2$	0.25	0.45	0.25
$j = s_3$	0	0	0.20

Let  $P_i^M = (P_{i1}^M, P_{i2}^M, P_{i3}^M)$ , where  $P_{ij}^M$  is the probabilistic assignment of student  $i = a, a^x$ ,  $a = 1, 2, 3$ , to school  $s_j$  under mechanism  $M = DA, B, B$  being a mnemonic for the Boston mechanism. Also let  $EU_i^M$  denote her expected utility under  $P_i^M$ .

Under the DA assignment the assignment probabilities are given by

$$P_1^{DA} = (1, 0, 0), P_2^{DA} = (\frac{1}{5}, \frac{4}{5}, 0), P_3^{DA} = (\frac{1}{5}, \frac{3}{10}, \frac{1}{2}), \text{ and}$$

$$P_{a^x}^{DA} = (\frac{1}{5}, \frac{3}{10}, \frac{1}{2}) \text{ for } a = 1, 2, 3.$$

At the unique equilibrium of the Boston mechanism, all students except for the neighborhood  $2^x$  students submit true preferences  $s_1 - s_2 - s_3$  while neighborhood  $2^x$  student report  $s_2$  as her first choice. At this equilibrium, we have

$$P_1^B = (1, 0, 0), P_2^B = (\frac{1}{4}, \frac{3}{4}, 0), P_3^B = (\frac{1}{4}, \frac{1}{12}, \frac{2}{3}),$$

$$P_{2^x}^B = (0, 1, 0), \text{ and } P_{a^x}^B = (\frac{1}{4}, \frac{1}{12}, \frac{2}{3}) \text{ for } a = 1, 3.$$

Comparing the two mechanisms, neighborhood 1 student is indifferent, neighborhood  $1^x$  student is worse off but all other students, those with priority at the worst neighborhood and those without any priority, are better off under the Boston, as seen by expected utilities under two mechanisms:  $EU_1^{DA} = 0.75 = EU_1^B$ ,  $EU_2^{DA} = 0.470 < EU_2^B = 0.475$ ,  $EU_3^{DA} = 0.285 < EU_3^B \approx 0.292$ ,  $EU_{1^x}^{DA} = 0.225 > EU_{1^x}^B \approx 0.208$ ,  $EU_{2^x}^{DA} = 0.245 < EU_{2^x}^B = 0.450$ , and  $EU_{3^x}^{DA} = 0.285 < EU_{3^x}^B \approx 0.292$ .

This example captures a plausible scenario in which students have stronger preferences for schools in their neighborhood but there is no predictable pattern in their cardinal utilities for schools outside their neighborhoods. In particular, neighborhood 2,  $2^x$  and neighborhood 3 and  $3^x$  students value  $s_1$  the same. Therefore, there is no strong welfare ground for any of them to be assigned to that school. In fact, as discussed above, assigning neighborhood 3 and  $3^x$  students to  $s_1$  may be more desirable from a policy point of view. In this example, the neighborhood 2 student guarantees her neighborhood school by giving up her competitiveness at  $s_1$ , which in turn opens up a strategic opportunity for neighborhood 3 and  $3^x$  students to improve their odds at  $s_1$ .

This observation — that the Boston mechanism improves the access of priority-disadvantaged students to good schools outside their neighborhood — can be generalized as follows: Consider

our general Bayesian model in which each of  $n$  students draws his vNM values from  $\mathcal{V}$  according to probability distribution  $f$ . We further assume  $n > q_1 + q_2$ , meaning there are at least two good schools in the sense of those being demanded more than their quotas. Suppose that  $n_a \geq 0$  students are given neighborhood priority at school  $s_a \in \{s_1, \dots, s_m\}$ . Each student has priority at no more than one school and  $n \geq \sum_a n_a$ . Assume that  $\{n_1, \dots, n_m\}$  is common knowledge. Also define  $g = \min\{a : n_a < q_a\}$ , which is the index of the most preferred school that can serve to all of its neighborhood children. Every other more preferred school  $s_{\tilde{a}}$ ,  $\tilde{a} < g$ , has at least as many neighborhood students as its capacity, i.e.  $q_{\tilde{a}} \leq n_{\tilde{a}}$ . A symmetric Bayesian strategy then specifies the same (mixed) action for students with the same vNM value  $\mathbf{v} \in \mathcal{V}$  and same priority standing. Then the following characterizations hold.

**Theorem 3.** *Consider any symmetric Bayesian equilibrium of the Boston mechanism. (i) If  $g > 1$  and  $q_{\hat{a}} < n_{\hat{a}}$  for some  $\hat{a} < g$ , then every student with priority at  $s_a$ ,  $a > g$ , or no priority at any school has a strategy that guarantees a weakly higher probability of being assigned to  $s_{\tilde{a}}$  for every  $\tilde{a} \leq g$  and a strictly higher probability of being assigned to  $s_{\tilde{a}}$  for some  $\tilde{a} \leq g$  in comparison with the DA. (ii) If  $g = 1$ , every student with priority at  $s_a$ ,  $a \geq 3$ , or no priority at any school has a strategy that guarantees a strictly higher probability of being assigned to  $s_{\tilde{a}}$  for some  $\tilde{a} = 1, 2$  in comparison with the DA.*

*Proof.* First consider the DA mechanism. Since it is a dominant strategy to report ordinal rankings truthfully, the DA will assign only students with priority at  $s_{\tilde{a}}$  to  $s_{\tilde{a}}$  for all  $\tilde{a} < g$  and it will assign all students with priority at  $s_g$  to  $s_g$ . Therefore, the probability that a student with priority at  $s_a$ ,  $a > g$ , or no priority at any school is assigned school  $s_{\tilde{a}}$ ,  $\tilde{a} < g$ , is zero under the DA, and her probability of assignment to school  $s_g$  is  $\frac{q_g - n_g}{\sum_{a' < g} (n_{a'} - q_{a'}) + n - \sum_{a \leq g} n_a}$ , since  $n_{a'} - q_{a'}$  of students with priority at  $s_{a'}$  for  $a' < g$ , and all other students will compete for the  $q_g - n_g$  seats left at school  $s_g$ . First we prove (i). Consider the Boston mechanism and any school  $s_{\tilde{a}}$ ,  $\tilde{a} < g$ . Suppose first that some type  $\mathbf{v}$  student with priority at  $s_{\tilde{a}}$  ranks school  $s_{a'}$ , for some  $a' \neq \tilde{a}$ , as first choice with positive probability in equilibrium. In that case, since there is a positive probability that every student with priority at  $s_{\tilde{a}}$  is of type  $\mathbf{v}$ , a student with priority at  $s_a$ ,  $a > g$ , or no priority at any school will be assigned to  $s_{\tilde{a}}$  with positive probability if she ranks  $s_{\tilde{a}}$  as first choice. Recall that the probability that such a student is assigned  $s_{\tilde{a}}$  by the DA is zero, so statement (i) holds in this case. Suppose next that, for each  $\tilde{a} < g$ , all students with priority at  $s_{\tilde{a}}$  rank school  $s_{\tilde{a}}$  as first choice with probability 1 in equilibrium. In that case, if a student without priority at any  $s_a$  with  $a \leq g$ , ranks school  $s_g$  as first choice, she will be assigned  $s_g$  with the probability of at least  $\frac{q_g - n_g}{n - \sum_{a \leq g} n_a} > \frac{q_g - n_g}{\sum_{a' < g'} (n_{a'} - q_{a'}) + n - \sum_{a \leq g} n_a}$ . The inequality follows since  $q_{\hat{a}} < n_{\hat{a}}$  for some  $\hat{a} < g$ . This completes the proof of (i). Next we prove (ii). If every student ranks  $s_1$  as first choice with probability 1 in equilibrium, then a student with priority at  $s_a$ ,  $a \geq 3$ , or no priority at any school can guarantee assignment at  $s_2$  by ranking

it as first choice. That probability is smaller than 1 under the DA since  $n > q_1 + q_2$ . If some type of student ranks  $s_1$  lower in his choice list with positive probability, then by ranking  $s_1$  as first choice, a student with priority at  $s_a$ ,  $a \geq 3$ , or no priority at any school can guarantee assignment at  $s_1$  with a larger probability in comparison to the DA. That follows from the fact that every student ranks  $s_1$  as first choice under the DA. This completes the proof of (ii). ■

When school priorities are strict and students have the same ordinal preferences, the Nash equilibrium outcome of the Boston mechanism is unique and it coincides with the unique stable matching of the economy, which in turn implies that there is no randomness or uncertainty in equilibrium. Strategic opportunities characterized in this Theorem arise under coarse school priorities and incomplete information. This effect is not present under the DA since students submit their ordinal rankings truthfully whether school priorities are strict or coarse and regardless of the information structure.

## 5 Conclusion

The Boston mechanism and its variants are widely used in school choice programs in the US, including Seattle Public Schools, WA, Cambridge, MA, Providence, RI, Fort Collins and Denver, CO, Charlotte-Mecklenburg, NC, Miami-Dade and Tampa-St. Petersburg, FL. Examples of the Boston mechanism from around the world include the assignment of city schools in Seoul set to begin in 2010, elementary and middle school admissions in Japan, and college admissions in China and Germany. On the other hand, the matching literature on school choice seems to reject the Boston mechanism. The standard view is that the Boston mechanism has a serious deficiency in both incentives and welfare. Although its incentive property is well understood, the welfare assessment of Boston mechanism is not as clear-cut as may have been thought of.

Our welfare assessment of the Boston mechanism so far has been shaped largely by models that make unrealistic assumptions such as complete and strict priorities on the part of schools and complete information on the part of students. In such models, the issue of how divergent interests are coordinated according to school priorities — captured by such notions as ex post Pareto efficiency or student optimal stable matching — figures prominently in welfare evaluation. Such evaluation could serve as a reasonable approximation, if not perfect, of truth, either if schools have near-complete priorities over students or if students have divergent preferences. The real-life school choice environment seems far from this latter stylization, however. In practice, families tend to have similar preferences about schools, and schools have at best coarse priorities. In such an environment, ex post efficiency and student optimal stable matching are of little help in differentiating alternative mechanisms. Rather, the issue of how a mechanism resolves conflicts based on cardinal welfare — captured by ex ante Pareto efficiency — looms

prominent. What we have shown is that, from this perspective, the Boston mechanism possesses several desirable features that other alternatives such as the DA lack.

Our results should not be seen as an unqualified endorsement of the Boston mechanism. The lack of strategy-proofness remains a significant drawback of the Boston mechanism that may ultimately make it unacceptable. Nevertheless, the current paper has shown a clear sense of tradeoff in the choice between DA and the Boston mechanism. Informing the school choice debate of this tradeoff is the most important purpose of this paper. Resolving this tradeoff ultimately necessitates quantifying both sides of the tradeoff, which will require much more work on the theoretical, computational, empirical as well as experimental front. Also needed are the attempts to explore a mechanism that balances the tradeoffs better than the existing mechanisms. They remain ongoing and future research.

## 6 Appendix: Proof of Proposition 1

*Proof.* Assume that students have the same ordinal preferences  $s_1 \succ s_2 \succ \dots \succ s_m$  and schools have strict priorities  $\pi = (\pi_{s_1}, \dots, \pi_{s_m})$ , where  $\pi_{s_a}$  is school  $a$ 's priorities, represented by an ordered list of all students. Since the students have the same ordinal rankings, every such economy  $(\succ, \pi)$  has a unique stable matching, which can be obtained by the following procedure: *Assign the top  $q_1$  students in  $\pi_{s_1}$  to  $s_1$ ; given the assignments at  $s_1, \dots, s_{k-1}$ , assign the top  $q_k$  unassigned students in  $\pi_{s_k}$  to  $s_k$ . That matching is also Pareto efficient.* Given a set of sophisticated students  $M$  and naive students  $N$  and  $(\succ, \pi)$ , let  $(\succ, \tilde{\pi})$  be the associated augmented economy à la Pathak and Sönmez (2008):  $\tilde{\pi}_{s_1} = \pi_{s_1}$  and for every  $s \neq s_1$ ,  $\tilde{\pi}_s$  ranks all sophisticated students at the top according to  $\pi_s$  then all naive students below according to  $\pi_s$ . The augmented economy  $(\succ, \tilde{\pi})$  has a unique stable matching  $\mu$ . Therefore,  $\mu$  is the unique complete information Nash equilibrium of the Boston mechanism in the economy  $(\succ, \pi)$  with sophisticated students  $M$  and naive students  $N$  (Proposition 1, Pathak and Sönmez, 2008). Suppose that some  $i^* \in N$  becomes sophisticated. Let  $(\succ, \tilde{\pi}^*)$  be the associated augmented economy and  $\mu^*$  be the unique stable matching of  $(\succ, \tilde{\pi}^*)$ , which is also the unique complete information Nash equilibrium outcome of the Boston mechanism in the economy  $(\succ, \pi)$  with sophisticated students  $M \cup \{i^*\}$  and naive students  $N \setminus \{i^*\}$ . Then by construction,  $i^*$  improves his standing at every school  $s \neq s_1$  in  $\tilde{\pi}^*$  in comparison to  $\tilde{\pi}$ . If  $\mu^*(i^*) = \mu(i^*)$ , then  $\mu^*(i) = \mu(i)$  for every  $i \neq i^*$ , which follows immediately from the construction of  $\mu$  and  $\mu^*$ . If  $\mu^*(i^*) \neq \mu(i^*)$ , then  $\mu^*(i^*) \succ \mu(i^*)$ , since  $i^*$  improves his standing at every school but  $s_1$  in  $\tilde{\pi}^*$ . Since  $\mu$  and  $\mu^*$  are Pareto efficient,  $\mu^*(i^*) \succ \mu(i^*)$  implies that there exists  $i_1 \in M \cup N \setminus \{i^*\}$  such that  $\mu(i_1) = \mu^*(i^*) \neq \mu^*(i_1)$  and  $\mu(i_1) \succ \mu^*(i_1)$ . Then either  $\mu^*(i_1) = \mu(i^*)$  or there exists  $i_2 \in M \cup N \setminus \{i^*, i_1\}$  such that  $\mu(i_2) = \mu^*(i_1) \neq \mu^*(i_2)$  and  $\mu(i_2) \succ \mu^*(i_2)$ . In general, given  $\{i^*, i_1, \dots, i_k\}$ ,  $k \geq 1$ , such that  $\mu(i_{l+1}) = \mu^*(i_l) \neq \mu^*(i_{l+1})$ ,  $\mu(i_{l+1}) \succ \mu^*(i_{l+1})$  for all  $l = 1, \dots, k - 1$  and  $\mu^*(i_{l+1}) \neq \mu(i^*)$ ,

Pareto efficiency of  $\mu$  and  $\mu^*$  implies that there exists  $i_{k+1} \in M \cup N \setminus \{i^*, i_1, \dots, i_k\}$  such that  $\mu(i_{k+1}) = \mu^*(i_k) \neq \mu^*(i_{k+1})$  and  $\mu(i_{k+1}) \succ \mu^*(i_{k+1})$ . Continuing this iteration, by finiteness we obtain some  $K$  such that  $\mu^*(i_K) = \mu(i^*)$ . Then for every  $i \in \{i_1, \dots, i_K\}$ ,  $\mu(i) \succ \mu^*(i)$ , i.e.  $i$  becomes strictly worse off at the unique complete Nash equilibrium of the Boston mechanism when  $i^*$  becomes sophisticated. For every  $i \in M \cup N \setminus \{i^*, i_1, \dots, i_K\}$ ,  $\mu(i) = \mu^*(i)$ . This completes the proof. ■

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