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Standard Auctions with Financially Constrained Bidders

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We develop a methodology for analyzing the revenue and efficiency performance of auctions when buyers have private information about their willingness to pay and ability to pay. We then apply the framework to scenarios involving standard auction mechanisms. In the simplest case, where bidders face absolute spending limits, first-price auctions yield higher expected revenue and social surplus than second-price auctions. The revenue dominance of first-price auctions over second-price auctions carries over to the case where bidders have access to credit. These rankings are explained by differences in the extent to which financial constraints bind in different auction formats.

1. INTRODUCTION

Auctions are used to sell goods ranging from real estate and works of art to mineral extraction rights and timber harvesting rights. Recently, the U.S. Federal Communications Commission (FCC) sold the rights to offer a range of products known as "Personal Communications Services" (PCS), raising revenue of 9 billion dollars in the process (Cramton (1995*b*)). Auctions also occur frequently in the market for corporate control, in the market for the services of professional athletes, in the privatization of government-controlled firms, and as part of bankruptcy proceedings.

In this paper, we study the performance of auctions when bidders face an increasing marginal cost of expenditure. This feature is present in a number of settings. First, a buyer may be liquidity constrained. In government auctions, for example, sale prices are often high relative to buyers' liquid assets, so buyers must rely on external financing. With capital market imperfections, the marginal cost of financing is increasing.[1] (The cost of financing could simply take the form of interest payments, but it could also involve

1. Capital market imperfections have been the subject of many theoretical and empirical studies. Reasons why internal financing may be less costly than external financing range from transaction costs, tax advantages, agency problems, and costs of financial distress, to asymmetric information. The presence of financial constraints and their heterogeneity across individuals have been documented empirically in individuals' investment decisions (Fazzari, Hubbard, and Petersen (1988*a*, *b*) and Gertler and Gilchrist (1994)), in their consumption decisions (Zeldes (1989)), and in their decisions to become entrepreneurs (Holtz-Eakin, Joulfaian, and Rosen (1994*a*, *b*)).

increased creditor control over the borrower's activities, or downgrading of his debt.[2]) Second, a buyer may incur a rising opportunity cost of diverting resources away from other productive activities when increasing his expenditure. Third, a budget constraint may result from an agent's moral hazard problem. Many organizations delegate acquisition decisions to purchasing units, and budgets are often imposed to control their spending.[3] Finally, budget constraints may be a device to relax competition, as in the case of salary caps in professional sports leagues in the U.S.

The practical importance of buyers' limited purchasing power is highlighted by a range of empirical and anecdotal evidence. In business downturns, the prices of many illiquid assets drop significantly.[4] Financial constraints can explain some bidders' exit decisions in the PCS auctions (see Cramton (1995*a*)) as well as the end-of-day drop in prices at used car auctions (see Genesove (1993), p. 661). The existence of financial constraints is also reflected in auction design. The U.S. government often limits the length and size of mineral leases, and sets some leases aside for sale to small firms.[5] In the PCS auctions, for example, preferences granted to small bidders (installment payments at favourable interest rates and a 40% bidding credit) may have increased the government's revenue by more than 15% (see Ayres and Cramton (1996)). Joint bidding, which is permitted for small firms in the OCS auctions, alleviates their capital constraints (see Hendricks and Porter (1992)). Royalty payments and rental payments, which spread bidders' constraints across periods, are common in mineral rights auctions. These observations motivate our study of the impact of financial constraints in auctions.

Buyers' limited ability to pay presents a qualitatively different problem for a seller. Suppose that a buyer's budget constitutes an absolute spending limit. It might appear that one can simply relabel the buyer's willingness to pay as the minimum of his valuation and budget. An example demonstrates that this is incorrect, however. Suppose that a seller has an object to sell to two buyers. In the first scenario, each buyer has a budget of 2 and a valuation drawn uniformly from $[0, 1]$. In this case, the budget constraint never binds, and the first- and second-price auctions both yield expected revenue of $1/3$, confirming revenue equivalence. In the second scenario, each buyer has a valuation of 2 and a budget drawn uniformly from $[0, 1]$. Although the minimum of the valuation and the budget is distributed the same as in the first scenario, revenue equivalence does not hold here. In this latter case, it is an equilibrium strategy for a bidder to bid his budget in both the first- and second-price auctions (*i.e.* the budget constraint always binds),[6] so the expected revenues are $2/3$ and $1/3$ in the respective auction forms.[7]

2. Several cable TV companies that won PCS licences were reviewed by Moody's Investors Service for possible downgrading of their unsecured debt. See "Moody's—PCS Auction Hazardous To Financial Health," *Newsbytes News Network*, December 19, 1994.

3. Hart (1991) suggests that investors create long-term debt overhang to control borrowers' incentives to invest in bad projects when cash flows are high, which constrains their ability to invest in good projects when cash flows are low.

4. Shleifer and Vishny (1992) cite many examples of the illiquidity costs of distress: the sale of the Campeau retail empire brought about 68% of what an orderly sale would have brought; the rapid sale discount of the Trump Shuttle may have been as much as 50%; and the liquidation sale discount of a typical machine tool manufacturer's assets is estimated to be 50 to 70% off normal prices.

5. The Outer Continental Shelf (OCS) Land Act explicitly limits the size of leases, while allowing consolidation of leases after the bidding is completed, and it limits leases to five and ten years for producing and non-producing tracts, respectively. See Bergsten *et al.* (1987). Setaside sales were used in timber rights auctions and in the PCS auction (Cramton (1995*a*) and McMillan (1994)).

6. Why the bidder bids no more than his budget is justified in Section 3.

7. The same point can be made for the single buyer case. Consider the two scenarios above, but with a single buyer. In the first scenario (without a binding constraint), the optimal strategy for the seller is to make a take-it-or-leave-it offer of $1/2$. In particular, offering a menu of lotteries does not pay (Riley and Zeckhauser (1983)). In the second scenario (with a binding constraint), the seller can raise the same expected revenue by charging $1/2$, but she can do better. While keeping the option of buying at $1/2$, the seller can offer a *lottery*

To see the source of nonequivalence, consider yet another example. Two buyers each value an object at $1/2$ but each has a budget, w, drawn uniformly (and independently) from $[0, 1]$. In a second-price auction, it is weakly dominant for an active bidder to bid the smaller of his budget and $1/2$, which yields expected revenue of 0·292. In a first-price auction, it is equilibrium behavior for a bidder to bid w if $w \in (0, 1/4]$ and to bid $1/2 - 1/16w$ if $w > 1/4$.[8] This strategy yields expected revenue of 0·385.[9] In fact, these standard auctions are not optimal. To see this, consider the all-pay auction in which the good is awarded to the highest bidder but all bidders pay their bids. In this auction form, it is equilibrium behaviour for each bidder to bid $w/2$,[10] which yields the maximum possible expected revenue, $1/2$—the value of the object.

In this last example, the extent to which budget constraints bind differs across auction forms, which leads to different revenue performances. Buyers with $w < 1/2$, buyers with $w < 1/4$, and no buyers are constrained in the second-price, first-price, and all-pay auctions, respectively. Absent budget constraints, the bidders bid less in a first-price auction than in a second-price auction, since the winner pays his bid in the former but typically pays less than his bid in the latter. Thus, the constraints are less likely to bind in the first-price auction. The constraints are even less likely to bind in the all-pay auction. Bidders tend to make smaller bids there since they forfeit their bids even when they lose.

The main body of this paper generalizes these observations to a model with two-dimensional private information in which buyers are distinguished by one parameter representing their individual valuations of the object ("willingness to pay") and another representing their financial constraints ("ability to pay"). Separating a buyer's ability to pay from his willingness to pay has two major benefits. First, it allows us to capture the role played by the "size" of bidders. While the existing literature is largely silent on this issue, actual auctions are full of examples where bidders' financial strength is more important than their valuations in determining the winning bidder, as noted above. Second, a meaningful study of efficiency is possible with our specification. In the existing literature on symmetric, single-unit auctions, efficiency is not an issue since the good is always allocated to the bidder who values it the most.[11] Here, by contrast, the good may not be allocated to the highest-valuation user, but may instead go to a better-financed bidder with a lower valuation. Moreover, the extent to which a bidder's financial strength affects his equilibrium bid depends on the auction form used. This latter feature yields strict rankings of standard auctions in both expected revenue and social surplus.

that awards the good with probability $1/2$ for a fee of $1/3$. The buyer will still choose to pay $1/2$ if his budget exceeds $1/2$. But, if his budget is between $1/3$ and $1/2$, the buyer will take the lottery. Thus, it is now optimal for the seller to offer a menu of lotteries. Che and Gale (1996*a*) identify an optimal selling mechanism when the single buyer has two-dimensional private information about his valuation and his budget. In sharp contrast to the Riley and Zeckhauser (1983) result, the optimal mechanism typically involves a menu of lotteries.

8. The minimum of v and w still works as a sufficient statistic for the second-price auction, but it does not work for the first-price auction. The equilibrium strategy in the latter is explained as follows. If the other bidder follows the strategy, then the expected payoff is strictly increasing in the bid for bids below $1/4$, so it is optimal to bid one's budget if $w < 1/4$. A bid of $b \geqq 1/4$ wins with probability of $[16(1/2 - b)]^{-1}$, generating an expected pay off of $(1/2 - b)[16(1/2 - b)]^{-1} = 1/16$, so the bidder is indifferent to any bid above $1/4$. This indifference means that it does not pay to deviate from the suggested strategy if $w > 1/4$.

9. This example assumes no reserve price, which is indeed optimal for the seller in a first-price auction. In a second-price auction, however, a reserve price of $1/2$ is optimal, and yields expected revenue of 0·375 which is still less than that in the first-price auction.

10. Suppose that the other bidder follows the strategy. If a bidder bids b, then he wins with probability $2b$, so his expected payoff is $(1/2)(2b) - b = 0$. This indifference means that it does not pay to deviate.

11. If there is a binding reserve price, the good may not be sold. The possibility of an inefficient allocation has been recognized in models of multi-unit auctions. Bundling of goods creates an inefficiency when a buyer purchases a bundle containing a good that he does not value the most (Palfrey (1983)). Inefficiencies also arise in sequential auctions (see Black and de Meza (1992) and Gale and Hausch (1994), for example).

In the next section, we develop a methodology that enables us to compare the expected revenue and social surplus from auctions in a general environment.[12] In the subsequent sections, we apply the methodology to first- and second-price auctions. When budgets constitute absolute spending limits, first-price auctions are shown to yield higher expected revenue and social surplus than second-price auctions. Section 4 generalizes the revenue dominance of first-price auctions to the case where credit is available. Section 5 concludes. The Appendix contains all proofs not in the text.

Relatively little work has been done on the impact of financial constraints in auctions. Maskin (1992) and Shleifer and Vishny (1992) recognized the possibility that a good may not be allocated to the highest-valuation buyer if buyers are financially constrained. Other contributions include Pitchik and Schotter (1988) and Pitchik (1994), who consider sequential auctions of two goods, with a focus on the trend in prices.[13] Neither of these papers considers two-dimensional private information, which we do here.

2. GENERAL FRAMEWORK

A seller has one unit of an indivisible good for sale, which she values at zero. The good could be a consumer durable, a business property, or the right to exploit a natural resource. $N \geqq 2$ risk-neutral buyers compete for the good. A typical buyer values the good at v, and incurs a cost $C(x, w)$ if he pays x, where w represents his budget. Thus, if the bidder wins the object and pays x, he has utility $v - C(x, w)$. For each bidder, w and v are private information. The (w, v) pairs are independently and identically distributed across bidders with a density $f(\cdot, \cdot)$ that is positive and continuously differentiable in the interior of the support, $[\underline{w}, \bar{w}] \times [\underline{v}, \bar{v}]$, where $(\underline{w}, \underline{v}) \geqq (0, 0)$. (Note that v and w may be correlated for the individual bidders.) The cost-of-spending function C: $[0, \infty) \times [\underline{w}, \bar{w}] \rightarrow [0, \infty]$ satisfies the following assumptions.

Assumption 1. $C(x, w)$ is strictly increasing and (weakly) convex in x for $x \leqq \delta(w)$, where $\delta(w) \equiv \sup \{x \in [0, \infty) | C(x, w) < \infty\}$.

Assumption 2. For any w and any $x < \delta(w)$, $\lim_{w' \rightarrow w} C(x, w') = C(x, w)$.

Assumption 3. For any w, $w' > w$, and $x \leqq \delta(w)$, $C_1(x^+, w') \leqq C_1(x^-, w) < M_w$ for some $M_w < \infty$, where $C_1(x^-, w)$ and $C_1(x^+, w)$ denote, respectively, the left- and right-hand derivatives with respect to the first argument, evaluated at (x, w).

Assumption 4. $C(x, \bar{w}) = x$ for $x \leqq \bar{v}$, and $C(x, w) = x$ for $x \leqq \underline{v}$.

Assumption 1 means that the marginal cost of spending is positive and it increases as one spends more. Assumption 2 means that buyers with similar budgets face similar cost functions. The first inequality in Assumption 3 implies that a higher w makes it less costly for a bidder to increase his payment by a given amount. If $C(\cdot, w)$ is differentiable, the condition reduces to a (weak) negative cross partial derivative. The second inequality

12. Our focus is on standard auction forms. Optimal mechanisms with multidimensional private information have been analysed elsewhere. See Armstrong (1996) and Rochet and Chone (1996), for example. Che and Gale (1996*a*) characterizes the optimal sales mechanism in an environment similar to the one considered here, but with a single buyer and absolute budget constraints.

13. Along the same lines are Palfrey (1980) and Rothkopf (1977), who study bidders with fixed resources to allocate over multiple auctions.

in Assumption 3 implies Lipschitz continuity. We say that a buyer with w is "unconstrained when spending x" if $C(x, w) = x$. Assumption 4 then implies that the wealthiest-possible buyer is unconstrained when spending up to $\bar{v}$, and no buyer is constrained when spending less than the lowest-possible valuation.[14] These assumptions capture the salient features of financial constraints. They hold, for example, when the cost function take the form:

$$C(x,w) = \begin{cases} x, & \text{for } x < w; \\ x + R(x - w), & \text{for } x \geq w, \end{cases}$$

where $R(0) = 0$, $R(y)$ is nondecreasing, (weakly) convex, and Lipschitz continuous for $y > 0$, and $\underline{v} \leqq \underline{w} < \bar{v} \leqq \bar{w}$. With this cost function, the buyer faces no financing costs when spending less than his budget, but he incurs the interest cost $R(y)$ when borrowing y. In particular, if $R(y) \equiv ry$, then the buyer faces a constant (net) borrowing rate of r; and if $R(y) = \infty$ for $y > 0$, then the budget constitutes an absolute spending limit.

Our main purpose is to compare the performance of first- and second-price auctions. However, the methodology for revenue and social surplus comparisons applies to a wider class of auctions forms. More precisely, consider a family, $\mathscr{F}$, of auction rules with the following properties.

P1. The highest bidder receives the object,[15] and the rules of the auction apply identically to all bidders.

P2. There exists a symmetric pure-strategy Bayes–Nash equilibrium in which an active bidder with (w, v) makes a bid $b = B(w, v)$, where $B(\cdot, \cdot)$ is continuous.

P3. The equilibrium bid $B(w, v)$ is increasing in (w, v) for an active type (w, v), and strictly increasing if both w and v rise.

P4. For any equilibrium bid, there exists an unconstrained type with budget $\bar{w}$ that makes the same bid.

These properties facilitate comparison between auction forms. P1 rules out random allocation (apart from ties) and asymmetric treatment of bidders. P2 ensures that an equilibrium for a given auction rule can be characterized by a family of continuous isobid curves in (w, v) space. As will be shown, our social surplus analysis involves comparing the shapes of isobid curves from different auction forms. P3 implies that the isobid curves are negatively sloped and have no interior. Finally, P4 guarantees that there is an unconstrained type on every isobid curve. This last property allows us to focus on such types when calculating expected revenue. These properties are similar to those considered by Riley and Samuelson (1981), but, given our two-dimensional private information and general cost function, P2–P4 are not obviously satisfied. Nonetheless, the first- and second-price auctions are shown to satisfy P1–P4 in the absolute budget case (Section 3). In a general environment with Assumptions 1–4, P2 still needs to be checked for each auction form, but the other properties are satisfied broadly.[16]

14. We use this assumption only in our revenue analysis, where the presence of unconstrained types facilitates the calculation of expected revenue. The assumption is not restrictive since adding an arbitrarily small density of unconstrained types does not affect the expected revenue appreciably.

15. Ties are broken randomly in all auctions that we consider.

16. Auctions in which P1 holds include, for instance, the all-pay auction and any form where the winning bidder pays a positive linear combination of his bid and the highest losing bid. The auction rules can include reserve prices, bidding subsidies, or entry fees. Given P2 and Assumptions 1–4, the weak monotonicity in P3 is satisfied if the payment rises with the bid in the sense of first-order stochastic dominance (see Che and Gale (1995) for details), which holds for all standard auctions listed above. In addition, with P2, the strict monotonicity is also satisfied. Given P2 (with its continuity property) and P3, P4 holds if there exists a type with $\bar{w}$ that makes the lowest equilibrium bid. This latter feature holds for all the above auction forms, except for some auctions involving entry fees. P2 is shown to hold for first- and second-price auctions with absolute budgets (in Section 3) and for the second-price auction in the more general case (Section 4).

A. Social surplus comparison

We first develop a methodology for comparing the social surplus of two auction forms that satisfy the above properties. In so doing, we focus on the valuation of the winning bidder as a measure of social surplus. The winner's valuation is the correct measure if no buyer incurs financing costs (as in Section 3),[17] or if their financing costs are pure transfers to creditors. Even when the financing costs comprise some deadweight loss, selecting a buyer who values the object the most (or a supplier who is most qualified for a job) remains an important concern in government auctions, for example.

Let M be an arbitrary auction rule in $\mathscr{F}$, and fix an equilibrium. The auction rule may involve a reserve price that restricts the set of active bidders. Let $B_M(\cdot,\cdot)$ denote the equilibrium bid function of an active bidder, and let $\hat{v}_M$ be the lowest possible valuation for an active bidder. (By P3 and P4, a type-$(\bar{w}, \hat{v}_M)$ buyer would submit the minimum of the equilibrium bids.) For an active type (w, v), let $\Omega_M(w, v)$ denote the set of types that either do not participate or that bid (weakly) less. This set corresponds to the region on and below the type-(w, v)'s isobid curve in auction M. Also let $\Omega_M^-(w, v) \equiv \{(w', v') \in \Omega_M(w, v) | w' \leqq w\}$ and $\Omega_M^+(w, v) \equiv \{(w', v') \in \Omega_M(w, v) | w' \geqq w\}$. Consider any two auctions, A and B, in $\mathscr{F}$, and fix their associated equilibria.

Definition. Auction A satisfies **the single-crossing property** with respect to auction B if, for any type (w, v) that is active in both auctions,

$$\Omega_A^-(w, v) \subseteq \Omega_B^-(w, v) \text{ and } \Omega_A^+(w, v) \supseteq \Omega_B^+(w, v).$$

The single-crossing property holds **strictly** if, in addition, one of the set inclusions is strict for a positive measure of (w, v).

The single-crossing property implies that isobid curves in auction A cut isobid curves in auction B from below, and at most once (see Figure 1). The following theorem shows that the winner's valuation is (weakly) higher in auction A than in auction B when this property holds. Roughly speaking, the single-crossing property means that A's isobid curves are flatter than those of B in (w, v) space. Therefore, the equilibrium bid functions are more likely to reflect the ordering of buyers' valuations in A than in B. The theorem is proven formally below, but the intuition for the proof can be illustrated clearly by Figure 1. Suppose that a type-(w, v) buyer wins in auction B. Any losing bidder with a lower valuation must then belong to the shaded area. By the single-crossing property, that area is below the isobid curve passing through (w, v) in auction A. Therefore, any losing bidder with a lower valuation in auction B must also lose in auction A, which implies that the winner's valuation can only rise as we shift from auction B to auction A.

Theorem 1. *If $\hat{v}_A \leqq \hat{v}_B$ and auction A satisfies the single-crossing property with respect to auction B, then A yields higher social surplus than B (with probability one). Auction A yields strictly higher expected social surplus than B if the single-crossing property holds strictly.*

Proof. Suppose that a type-(w, v) bidder is the winner in auction B. That bidder must also be active in auction A, since $\Omega_A(\bar{w}, \hat{v}_A) \subseteq \Omega_B(\bar{w}, \hat{v}_B)$, by the single-crossing property. It then suffices to show that, with probability one, the type (w,v) cannot lose to a bidder with $v' < v$ in auction A, which then implies that the winner in A has a higher

17. When borrowing is impossible, the buyers do not incur any financing costs. See Section 3.

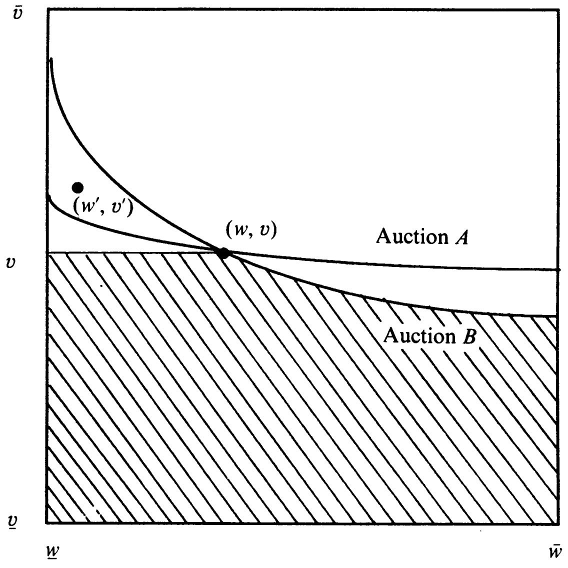

FIGURE 1

valuation than the winner in B.[18] Consider a non-winning bidder in B with (w', v'), $v' < v$. By definition, $(w', v') \in \Omega_B(w, v)$. If he participates, then $B_B(w', v') < B_B(w, v)$ with probability one, since the isobid curve has no interior, by P3. If $w' < w$, then $B_A(w', v') < B_A(w, v)$, by the strict monotonicity in P3. The same strict ranking holds with probability one if $w' \geqq w$, since $\Omega_B^+(w, v) \subseteq \Omega_A^+(w, v)$, by the single-crossing property. Thus, with probability one, the type-(w', v') bidder does not win.

The converse may not hold. That is, the winner in auction A, (say, (w', v') in Figure 1) may lose in auction B to a bidder with a lower valuation (say, (w, v) in Figure 1). This occurs with positive probability if the single-crossing property holds strictly, which proves the second statement. ||

B. Revenue comparison

The expected revenue from auctions can also be ranked unambiguously in many cases. Select an arbitrary auction M in $\mathscr{F}$, and fix an equilibrium. We focus on active bidders with $(\bar{w}, v)$, for any $v \geqq \hat{v}_M$. Define the probability that a randomly selected bidder either does not participate or bids lower than a type-$(\bar{w}, v)$ bidder does:

$$F_M(v) \equiv \int_{\bar{\Omega}_M(v)} f(w, s) dw ds,$$

where $\bar{\Omega}_M(v) \equiv \Omega_M(\bar{w}, v)$. By P2 and P3, $F_M(\cdot)$ is continuous and strictly increasing, with $F_M(\bar{v}) = 1$.

Since the bid alone determines the expected payment, all types on a given isobid curve make the same expected payment. Moreover, for any bid, there exists a type with budget $\bar{w}$ that makes the same bid, by P4. Therefore, we can express the seller's expected revenue,

18. In principle, bidders with different valuations could tie in both auction forms, but such an event occurs with zero probability.

R_M, in terms of the payments from the types with budget $\bar{w}$. Specifically, we calculate the expected revenue by integrating those types' payments over the associated isobid curves. Let $t_M(w, v)$ be the expected payment that a type-(w, v) bidder makes in equilibrium, and let $\bar{t}_M(v) \equiv t_M(\bar{w}, v)$. Then, the expected revenue under auction M is

$$R_M \equiv N \int_{\underline{v}}^{\bar{v}} \int_{\underline{w}}^{\bar{w}} t_M(w, v) f(w, v) dw dv = N \int_{\hat{v}_M}^{\bar{v}} \bar{t}_M(v) dF_M(v). \tag{1}$$

We now calculate $\bar{t}_M(v)$. Suppose that all other bidders use $B_M(\cdot, \cdot)$. Then, a type-$(\bar{w}, v)$ bidder chooses (*i.e.*, imitates the strategy of type) $\tilde{v} \geqq \hat{v}_M$ to maximize his net surplus

$$vF_M(\tilde{v})^{N-1} - \bar{t}_M(\tilde{v}). \tag{2}$$

Note that the expected *cost* of his payment is equal to the expected payment, since the bidder is not constrained.[19] In equilibrium, imitating another type cannot pay, so $\tilde{v} = v$. Let $\bar{U}_M(v)$ denote the resulting equilibrium expected utility for the bidder. Since the bidder with $(\bar{w}, \hat{v}_M)$ must be indifferent to participating, $\bar{U}_M(\hat{v}_M) = 0$. For $v \geqq \hat{v}_M$, the Envelope Theorem and integration yield

$$\bar{U}_M(v) = \int_{\hat{v}_M}^{v} F_M(t)^{N-1} dt. \tag{3}$$

Combining (2) and (3) yields a type $(\bar{w}, v)$'s expected payment:

$$\begin{aligned} \bar{t}_M(v) &= vF_M(v)^{N-1} - \int_{\hat{v}_M}^{v} F_M(t)^{N-1} dt \\ &= \hat{v}_M F_M(\hat{v}_M)^{N-1} + \int_{\hat{v}_M}^{v} t dF_M(t)^{N-1}. \end{aligned} \tag{4}$$

Substituting (4) into (1) and integrating by parts, the seller's expected revenue is

$$\begin{aligned} R_M &= N\hat{v}_M[1 - F_M(\hat{v}_M)]F_M(\hat{v}_M)^{N-1} + \int_{\hat{v}_M}^{\bar{v}} v dF_M^{(2)}(v) \\ &= \bar{v} - \hat{v}_M F_M(\hat{v}_M)^N - \int_{\hat{v}_M}^{\bar{v}} F_M^{(2)}(v) dv, \end{aligned} \tag{5}$$

where $F_M^{(2)}(\cdot)$ is the distribution function of the second order statistic of N random variables drawn independently from $F_M(\cdot)$. This expression for the expected revenue resembles that of the standard case without financial constraints (see, for example, Milgrom (1989)). *In fact, the expected revenue is precisely the same as in a hypothetical model where bidders are not constrained but have valuations drawn from the distribution function* $F_M(\cdot)$. An important difference, of course, is that $F_M(\cdot)$ is endogenously determined by the equilibrium bidding strategies here, which is why nonequivalence may arise in the presence of financial constraints. When different auction forms induce different bidding strategies, it is as if bidders' valuations are drawn from different distribution functions in the hypothetical model. The next result follows naturally.

19. While this approach requires the existence of unconstrained types, the underlying assumption (Assumption 4) is not restrictive to the extent that adding an arbitrarily small density of unconstrained types does not affect the expected revenue much, given P2.

Theorem 2. *If $\hat{v}_A = \hat{v}_B = \hat{v}$ and $F_A(v) \leqq F_B(v)$ for all $v \geqq \hat{v}$, then auction A yields (weakly) higher expected revenue than auction B. The ranking is strict if there exists an interval of v on which $F_A(v) < F_B(v)$.*

Proof. Since $F_A(\cdot)$ stochastically dominates $F_B(\cdot)$,

$$R_A = \bar{v} - \hat{v}F_A(\hat{v})^N - \int_{\hat{v}}^{\bar{v}} F_A^{(2)}(v)dv \geqq \bar{v} - \hat{v}F_B(\hat{v})^N - \int_{\hat{v}}^{\bar{v}} F_B^{(2)}(v)dv = R_B.$$

The two equalities follow from (5), and the inequality follows since $F_A^{(2)}(v) \leqq F_B^{(2)}(v)$ for all $v \geqq \hat{v}$.[20] Clearly, $R_A > R_B$ if $F_A(v) < F_B(v)$ for an interval of v. ||

Since flatter isobid curves indicate that bidders are hampered less by financial constraints, one would expect competition to be more fierce when isobid curves are flatter. This intuition is confirmed below.

Corollary. *If auction A satisfies the single-crossing property with respect to auction B, then A yields (weakly) higher expected revenue than B. The ranking is strict if the single-crossing property holds strictly.*

Proof. By definition, the single-crossing property implies $\Omega_A^-(\bar{w}, v) \subseteq \Omega_B^-(\bar{w}, v)$ for any active type $(\bar{w}, v)$, which implies that $F_A(v) \leqq F_B(v)$. Thus, the result follows from Theorem 2. The strict ranking is analogous. ||

3. AUCTIONS WITH BUDGET-CONSTRAINED BIDDERS

In the next two sections, we use the results from the preceding section to compare first- and second-price sealed-bid auctions. In the former, the highest bidder wins and pays his bid; in the latter, the highest bidder wins and pays the second-highest bid (or the reserve price if no other bidder enters). First-price auctions are frequently used to sell government rights, such as mineral extraction rights, and to let procurement contracts for the U.S. Department of Defense and Department of Transportation, for example. Second-price auctions are used in the strategically equivalent form—open oral auctions—to sell timber-harvesting rights.

In this section, we focus on bidders who face absolute limits on what they can spend. (We consider a more general case in the next section.) Specifically, a type-(w, v) bidder incurs a cost

$$C(x, w) = \begin{cases} x & \text{for } x \leq w, \\ \infty & \text{for } x > w, \end{cases}$$

when he spends x. It is useful to examine this special case since we can characterize the equilibrium and show its existence in each auction form here. Furthermore, both revenue and social surplus comparisons are possible. We assume that $\underline{v} \leqq \underline{w} < \bar{v} \leqq \bar{w}$, which is sufficient for Assumptions 1–4 to hold. The second inequality makes the presence of budget constraints meaningful.

20. For a cumulative distribution function $K(\cdot)$, the distribution function for the second order statistic is $K(v)^N + N[1-K(v)]K(v)^{N-1}$. The function $k^N + N(1-k)k^{N-1}$ is increasing for $0 < k < 1$. Thus, we have the ranking of the distributions of the second order statistics.

Before proceeding, we must deal with the possibility that a buyer bids more than his budget and then reneges on his bid. We assume that the seller does not award the object to a buyer who reneges, and that she also imposes a small penalty on him.[21] In a first-price auction, the winner pays his bid, so it would never be optimal to bid more than one's budget, given such a response from the seller. If the bid wins, the penalty makes the buyer's net surplus negative, while there is no gain if the bid does not win. Now consider a second-price auction, and suppose that a bidder wins with a bid above his budget. Either the bidder would have won anyway at the same price (i.e. the second-highest bid is below the winner's budget) or he wins at a price above his budget, which results in a negative surplus. Once again, it is a dominated strategy to bid above one's budget.

A. Second-price auctions

Let the reserve price be $r_s \geq \underline{v}$. Then, only buyers with min $\{w_i, v_i\} \geqq r_s$ will participate. We show that it is a dominant strategy for a participating buyer to bid min $\{w_i, v_i\}$. If $v_i > w_i$, then the budget constraint does not bind, so it is dominant strategy to bid u_i, following Vickrey (1961). If $v_i > w_i$, then the same argument implies that bidding w_i dominates bidding strictly less. Since bidding above one's budget is also dominated, the dominant strategy is to bid one's budget in that case. Graphically, the strategy is represented by a family of Leontief isobid curves with kinks on the 45 degree line (see a typical isobid curve in Figure 3). The bidding strategy clearly satisfies P1–P4.

Consider a bidder with $(\bar{w}, v)$, $v \geqq r_s$. The set of types that do not participate or that submit a lower bid is

$$\bar{\Omega}_s(v) \equiv \{(w', v') \mid \min \{w', v'\} \leqq v\}.$$

Let $G(w, v)$ denote the probabilty that $w' < w$ *or* $v' < v$

$$G(w, v) \equiv \int_{\underline{v}}^{\bar{v}} \int_{w}^{\bar{w}} f(\tilde{w}, \tilde{v}) d\tilde{w} d\tilde{v}.$$

A randomly selected buyer is in $\bar{\Omega}_s(v)$ with probability $F_s(v) \equiv G(v, v)$.

B. First-price auctions

Let $r_f \geqq \underline{v}$ be the reserve price. Buyer i participates if and only if min $\{w_i, v_i\} \geqq r_f$. We consider equilibrium bid functions of the form $B_f(w, v) = \min \{w, b_f(v)\}$ for some continuous, strictly increasing function $b_f(\cdot)$. (We later show that any symmetric equilibrium must take this form, given a mild regularity condition.) The bidding strategy again entails Leontief isobid curves, with unconstrained bidders adopting the strategy $b_f(\cdot)$ (see Figure 3). We now characterize $b_f(\cdot)$.

Consider a bidder with $(\bar{w}, v)$, for any $v \geqq r_f$. Since $\bar{v} \leqq \bar{w}$, such a bidder must be unconstrained. In equilibrium, the set of types that do not participate or that submit lower

21. The PCS auction included a bid withdrawal penalty equal to the difference between the withdrawn bid and the eventual winning bid, if the bid was withdrawn before the close of the auction, and an additional penalty of 3% of the minimum of the winning bid and the defaulting bid, if the bidder defaulted after the close of the auction (Cramton (1995*a*) and McMillan (1994)). The possibility of reneging also arises in standard auctions without budget constraints.

bids than the type-$(\bar{w}, v)$ bidder is

$$\bar{\Omega}_f(v) \equiv \{(w', v') \mid \min\{w', b_f(v')\} \leq b_f(v)\}.$$

A randomly selected buyer is in this set with probability $F_f(v) \equiv G(b_f(v), v)$. The problem facing a bidder with $(\bar{w}, v)$ is the same as if all bidders were unconstrained, with valuations drawn from the distribution $F_f(\cdot)$. The standard result without budget constraints (for example, Riley and Samuelson (1981)) then implies that a symmetric equilibrium bid function must satisfy

$$b_f(v) = v - \frac{\int_{r_f}^{v} F_f(s)^{N-1} ds}{F_f(v)^{N-1}} \quad \text{for } v \geqq r_f. \tag{6}$$

Existence of $b_f(\cdot)$ is not immediate, however, since $F_f(\cdot)$ depends on $b_f(\cdot)$ itself. The following technical assumption ensures the existence of a unique equilibrium bid function.

Assumption 5. $(N-1)w + (G(w, v))/(G_1(w, v))$ is strictly increasing in w for all $v \in (\underline{v}, \bar{v})$.[22]

Lemma 1. *Given Assumption* 5, *there exists a unique, symmetric equilibrium in which bidders with* $\min\{w, v\} \geqq r_f$ *employ the bid function* $B_f(w, v) = \min\{w, b_f(v)\}$, *where* $b_f(v)$ *satisfies* (6), *and is continuous and strictly increasing.*

Given Lemma 1, P1–P4 are satisfied. To fix ideas about this bidding strategy, let there be two buyers, each of whom has (w, v) drawn uniformly from $[0, 1]^2$, and assume no reserve price. Then, an unconstrained type's equilibrium bidding strategy solves the differential equation

$$b_f'(v) = \frac{(1-v)(v - b_f(v))}{1 - (1-v)(v + 1 - 2b_f(v))},$$

with $b_f(0) = 0$. Its numerical solution is graphed in Figure 2*a* (see the curve at the bottom), along with the corresponding equilibrium bidding surface in Figure 2*b*. For comparison, Figure 2*a* also depicts the equilibrium bidding strategy when a bidder's valuation is drawn uniformly from [0, 1] but no bidder faces a binding constraint $(b(v) = v/2)$. Clearly, the presence of budget constraints makes even the unconstrained buyers less aggressive.

C. Revenue and social surplus comparisons

Our results from section 2 can be used to rank the performances of alternative auctions. Equation (6) implies that $b_f(v) < v$ for all $v > r_f$. This guarantees that the first-price auction satisfies the strict single-crossing property with respect to the second-price auction (see Figure 3). The following ranking is immediate.

Proposition 1. *Given a second-price auction with reserve price* $r_s \in [\underline{v}, \bar{v})$, *a first-price auction with the same reserve price yields (weakly) higher social surplus, strictly higher expected social surplus and strictly higher expected revenue.*

22. This assumption is satisfied for common distribution functions such as the uniform distribution (i.e., (w, v) is uniform on $[0, 1]^2$, which implies that w is uniform given v).

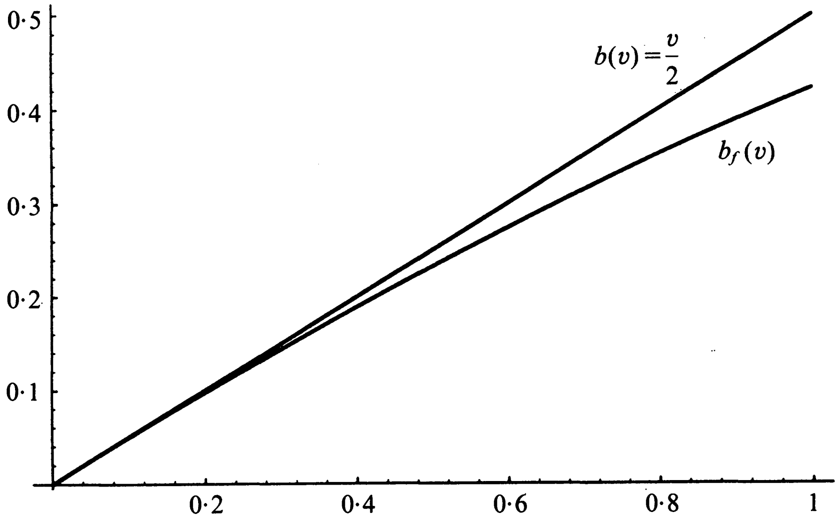

FIGURE 2a

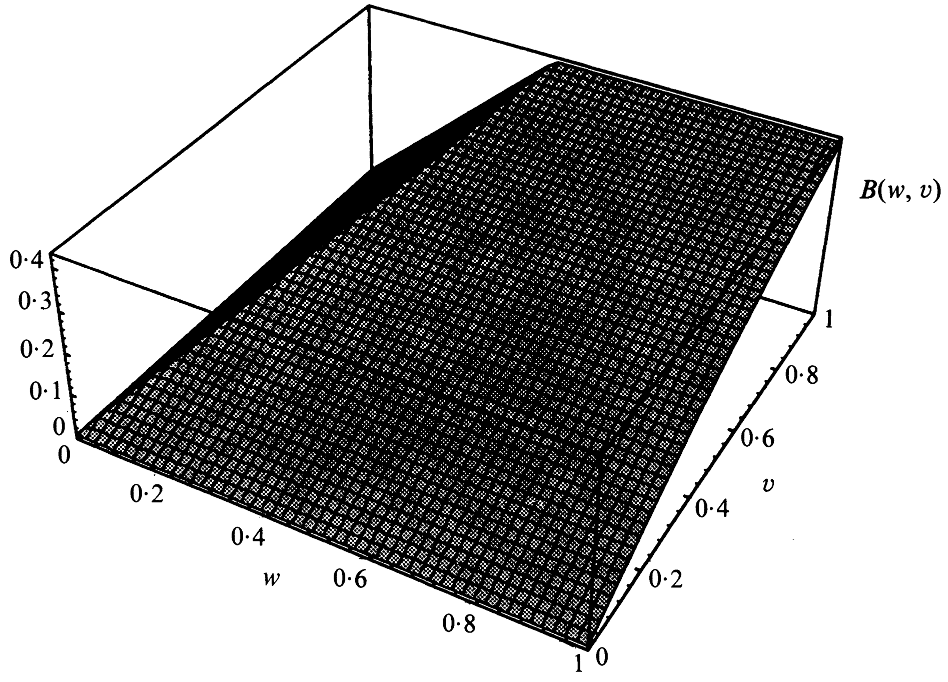

FIGURE 2b

Proof. The reserve price $r_f = r_s$ entails $\hat{v}_f = \hat{v}_s$ in a first-price auction. Since $b_f(v) < v$ for $v > r_f$, the first-price auction satisfies the strict single-crossing property with respect to the second-price auction. Thus, the results follow from Theorem 1 and the Corollary. ||

The revenue and surplus rankings can be explained by differences in the extent to which budget constraints bind. In the second-price auction, the winning bidder pays less

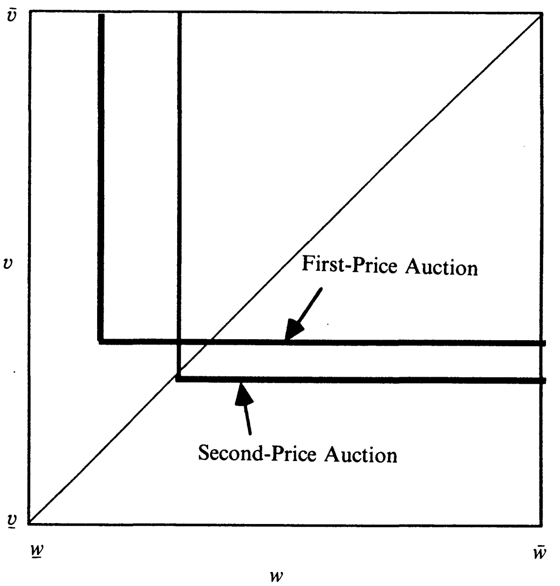

FIGURE 3

than his bid (with probability one), whereas the winner pays his bid in the first-price auction. Therefore, absent budget constraints, the bid is higher in the second-price auction than in a first-price auction. This means that the budget constraint is more likely to bind in the second-price auction. In Figure 3, the region of budget-constrained types is strictly larger in the second-price auction than in the first-price auction. (The same intuition held in the example in the Introduction, where the set of budget-constrained types was smaller in the first-price auction, given the same reserve price.) If more types are budget constrained, then the bids are less likely to reflect bidders' valuations. Thus, the good is less likely to be allocated to a bidder who values it more, which makes the social surplus lower in the second-price auction. An increased likelihood of bidders' being budget constrained reduces the seller's expected revenue, too. Since the budget constraint limits a bid more in the second-price auction, that auction form yields smaller expected revenue.

The above result is directly relevant for government auctions, where both social surplus and revenue appear to be important concerns (see McMillan (1994)). In particular, the result is consistent with the infrequent use of oral auctions there.[23] In private auctions, by contrast, a seller may not care about social surplus, so she may simply choose the reserve price to maximize her expected revenue. Interestingly, the same social surplus ranking holds even in this case, since, as we demonstrate below, the seller chooses a (weakly) lower reserve price in a first-price auction than in a second-price auction. The intuition is that, while raising the reserve price has no effect on the equilibrium bids in a second-price auction, it raises the unconstrained types' equilibrium bids in a first-price auction. The latter effect makes budget constraints tighter in the first-price auction, so the seller has a smaller incentive to raise the reserve price there.

23. One exception to this rule occurs with the sale of timber harvesting rights, where open oral auctions are used. The U.S. Bureau of Land Management experimented with first-price sealed-bid auctions and found that average winning bids were higher. It reverted to using open oral auctions, however, because of a strong preference on the part of the industry (Mead *et al.* (1981)). Open oral auctions are still commonly employed in many private auctions, possibly because bidders' valuations are affiliated (see Milgrom and Weber (1982)).

Proposition 2. *A revenue-maximizing seller chooses a weakly lower reserve price in a first-price auction than in a second-price auction (more precisely, if r_s is optimal in a second price auction, then there exists $r_f \leqq r_s$ that is optimal in a first-price auction). Given the optimal reserve prices, the rankings of Proposition* 1 *hold.*

4. AVAILABILITY OF CREDIT

Most buyers have access to some form of credit or to other sources of funds. The general cost-of-expenditure function introduced in Section 2 corresponds to situations where bidders face financing costs that are increasing in their payment. We show that one of the main results of Section 3 carries over to general cost functions: a first-price auction revenue-dominates a second-price auction. Theorem 2 again allows us to obtain the ranking without specifying the equilibria completely.

A. Second-price auctions

Suppose that the seller employs a reserve price $r_s \in [w, \bar{v})$. A type-(w, v) buyer participates if and only if $v \geqq C(r_s, w)$. For an active bidder with (w, v), it is now a dominant strategy to bid the highest b that satisfies $v \geqq C(b, w)$.[24] A higher bid could result in winning at a price above one's valuation, while a lower bid could result in foregoing a win at a price below one's valuation, with no countervailing benefit if it does not change who wins. With this equilibrium strategy, the basic properties in Section 2 are all satisfied.

Lemma 2. *Given Assumptions* 1–4, *a second-price auction satisfies* P1–P4.

Consider a type-$(\bar{w}, v^*)$ bidder, for any $v^* \geqq r_s$. We now characterize the set of types that bid lower or do not enter. By Assumption 4, $C(b, \bar{w}) = b$ for any $b \leqq \bar{v}$, so a type $(\bar{w}, v^*)$ has a dominant strategy of bidding v^*. Now consider an arbitrary bidder with (w, v). If $v \leqq C(v^*, w)$, then he cannot beat the type-$(\bar{w}, v^*)$ bidder. More precisely,

$$(w, v) \in \bar{\Omega}_s(v^*) \text{ if } v \leqq C(v^*, w). \tag{7}$$

We use (7) for the revenue comparison.

B. First-price auctions

The participation decision is analogous for a first-price auction. If the seller employs a reserve price $r_f \in [\underline{w}, \bar{v})$, a type-$(w, v)$ buyer participates if and only if $v \geqq C(r_f, w)$. Suppose that there exists an equilibrium in which active bidders adopt a continuous bidding strategy, $B_f(\cdot, \cdot)$. Let $p(b)$ denote the probability that a bid b wins in equilibrium. For all (w, v) such that $v \geqq C(r_f, w)$, the equilibrium bid function must satisfy

$$B_f(w, v) \in \arg\max_b p(b)[v - C(b, w)]. \tag{8}$$

It is convenient for later analysis to establish the following lemma.

24. The inequality is necessary since we do not require that $C(\cdot, w)$ be continuous. Since $C(x, w)$ is convex and increasing for $x \in [0, \delta(w)]$, a discontinuity can only occur at $\delta(w)$.

Lemma 3. *If there exists an equilibrium with a continuous bidding strategy $B_f(\cdot, \cdot)$, then P3 and P4 are satisfied. Furthermore, $p(\cdot)$ is strictly increasing and differentiable in the interior of the range of equilibrium bids.*

Consider an unconstrained type $(\bar{w}, v^*)$, for any $v^* \in (r_f, \bar{v})$. If he bids $b^* \in (r_f, v^*)$ in equilibrium, then, by Lemma 3,

$$p'(b^*)(v^* - b^*) - p(b^*) = 0. \tag{9}$$

We now characterize the set of types that bid less than a type-$(\bar{w}, v^*)$ bidder. Consider an arbitrary type-(w, v) bidder with $\delta(w) \geqq b^*$. If he bids (weakly) less than b^* in equilibrium, then[25]

$$p'(b^*)(v - C(b^*, w)) - C_1(b^{*+}, w)p(b^*) \leqq 0. \tag{10}$$

Since both $p'(b^*)$ and $p(b^*)$ are strictly positive, by Lemma 3, (9) and (10) imply that

$$v \leqq C(b^*, w) + (v^* - b^*)C_1(b^{*+}, w). \tag{11}$$

Equilibrium bidding strategies are characterized as in Subsection 3.B. The inequality in (11) determines the set $\bar{\Omega}_f(v^*)$, and $F_f(v^*) = \int_{\bar{\Omega}_f(v^*)} f(w, v)dwdv$. The unconstrained bidders' equilibrium bidding strategy $b_f(\cdot)$ is then implicitly defined by (6), using the new distribution function $F_f(\cdot)$. Bidding strategies for constrained types are characterized by (10), with $p(b_f(\cdot)) = F_f(\cdot)^{N-1}$. We omit the details of the equilibrium characterization, since the necessary condition (11) will be sufficient for our revenue comparison.[26]

C. Revenue Comparison

We are now ready to show the revenue dominance of the first-price auction over the second-price auction. Set $r_f = r_s$. In light of Theorem 2, it then suffices to show that $\bar{\Omega}_f(v^*) \subseteq \bar{\Omega}_s(v^*)$ for all $v^* \in [r_f, \bar{v})$. Consider an arbitrary $(w, v) \in \bar{\Omega}_f(v^*)$. As shown above, if a type $(\bar{w}, v^*)$ bids b^*, then (w, v) must satisfy (11). Suppose first that $b^* < \delta(w)$. Then, convexity of $C(\cdot, w)$ implies

$$C(b^*, w) + (v^* - b^*)C_1(b^{*+}, w) \leqq C(v^*, w). \tag{12}$$

Together with (11), (12) yields $v \leqq C(v^*, w)$. Now suppose that $b^* = \delta(w)$. By (9), $v^* > b^*$, which implies that $C(v^*, w) = \infty \geqq v$. Either way, $v \leqq C(v^*, w)$, and (7) implies that $(w, v) \in \bar{\Omega}_s(v^*)$, which proves that $\bar{\Omega}_f(v^*) \subseteq \bar{\Omega}_s(v^*)$. This latter result implies the stochastic dominance relationship required in Theorem 2. Therefore, the following revenue ranking is obtained.

Proposition 3. *Given a second-price auction with reserve price $r_s \in [\underline{v}, \bar{v})$, a first-price auction with the same reserve price yields higher expected revenue than the second-price auction. The ranking is strict if $C(b, w)$ is strictly convex at some equilibrium bid $b < \delta(w)$, for some w.*

Proof. The weak revenue ranking follows immediately from the stochastic dominance property shown above. The strict ranking is obtained as follows. If $C(b, w)$ is strictly

25. For completeness, let $C_1(b^{*+}, w) \equiv \infty$ if $b^* = \delta(w)$.

26. The social surplus cannot be ranked unambiguously here because the single-crossing property may not hold uniformly with the general cost function.

convex at some equilibrium bid $b < \delta(w)$, for some w, then there exists $v^* \in (\underline{v}, \bar{v})$ such that $C(b, w) + (v^* - b)C_1(b^+, w) < C(v^*, w)$. This implies that $F_f(v^*) < F_s(v^*)$, following the above argument. Now, by continuity of the equilibrium bid functions in both auction forms, there exists an interval containing v^* in which $F_f(v) < F_s(v)$. The strict ranking then follows by Theorem 2. ||

As before, differences in how the winner's marginal cost of spending rises with his bid yield the ranking. The intuitition can be further understood by comparison with the case of risk-averse bidders (see Holt (1980), Riley and Samuelson (1981), and Maskin and Riley (1984)). In these papers, a bidder has concave von Neumann-Morgenstern utility over his net surplus.[27] Consequently, the risk associated with the difference in net surplus generates the revenue ranking. In a first-price auction, a risk-averse bidder has an incentive to smooth his net surplus (between winning and losing). He accomplishes this by bidding more aggressively. By contrast, bids are unaffected by risk aversion in a second-price auction. In our model, a bidder with a convex payment function has an incentive to smooth his *payment* (between winning and losing). Thus, bidders are less aggressive in both auction forms, but more so in the second-price auction because the winner's payment is random.[28]

5. CONCLUDING REMARKS

We have studied the performance of first-price and second-price auctions when buyers face financial constraints. Our finding that second-price auctions yield lower expected revenue and lower social surplus than first-price auctions suggests that the former should not be used when such constraints are present, all else equal. This finding is consistent with the rare use of oral auctions in U.S. government auctions (see footnote 23).

The methodology that we have developed can be applied to a wide range of auction forms for which equilibrium isobid curves can be characterized. All-pay auctions, and other arrangements that require losers to pay, such as entry fees, can be studied with a slight modification of our methodology. All-pay auctions can relax budget constraints by spreading the payments across bidders, so they tend to perform better than the standard auctions.[29] The use of entry fees in first-price auctions has a similar effect: if a reserve price is replaced with an appropriate entry fee, budget constraints are alleviated since many bidders each pay a small amount rather than a single winning bidder paying a large amount. Thus, the expected revenue increases with the use of entry fees (Che and Gale (1996b)).

Recognition of the practical importance of financial constraints is another contribution of this paper. The insights contained in this paper and the new perspectives that are

27. Because that literature focuses on one-dimensional private information, the social surplus ranking is not an issue. The standard specification is of the form $u(v-x)$ where x is the payment. Maskin and Riley (1984) consider specifications that include $u(v-\varphi(x))$, where $\varphi(\cdot)$ is a known convex function.

28. This intuition does not depend upon the constraints being private information. To see this, consider the variant in which all bidders face the same cost-of-spending function (i.e. $C(x, w) = C(x)$ for all w, where $C''(\cdot) > 0$). Fix the auction form (first-price or second-price) and let $p(v)$ and $c(v)$ denote the equilibrium probability of winning and the expected cost of the payment, respectively, for a buyer with valuation v. The buyer's expected utility is $U(v) \equiv vp(v) - c(v) = \max_{\tilde{v}} vp(\tilde{v}) - c(\tilde{v})$. By the Envelope Theorem, $U(v) = \int_{\underline{v}}^{v} p(z)dz$ for all v. Since $p(v) = F(v)^{N-1}$ in both auction forms, a given bidder's expected utility is the same in the two auction forms. This implies that the seller compensates a bidder for the additional cost that he bears due to the randomness of the winner's payment in the second-price auction, so the first-price auction yields higher expected revenue.

29. Our working paper version (Che and Gale (1995)) shows that all-pay auctions revenue-dominate first-price auctions in the absolute budget limit case.

offered can also shed light on many important issues that have long challenged practitioners but have only recently received academic attention. The following is a short list.

(*i*) *The size of sales units*

Sellers often have some control over the value of the object for sale. In mineral rights auctions, for example, the length of a lease and the area covered by it can both be varied to change its value. Similarly, the value of a license to operate in a market depends critically on how many licenses are sold in the same market. Sales of more licenses will lead to a more competitive market structure and may lead to a smaller combined value of the licenses sold. Scale and network economies, which are often present in government auctions, favour increasing the size of the sales unit. The presence of financial constraints favors the opposite. Limiting the size and value of auctioned objects can alleviate the financial constraints that bidders face and increase bidding competition. The recent debate over the size of the market areas in the PCS auctions reflects such a trade-off.

(*ii*) *Joint bidding and mergers*

The size issue is intimately related to the issues of joint bidding and mergers of potential bidders. Absent budget constraints, joint bidding and mergers of potential bidders simply reduce the number of potential bidders, which may lower the revenue. In the presence of financial constraints, however, these arrangements can increase bidding competition by easing the constraints.

(*iii*) *Sale of corporate control, design of bankruptcy procedures*

Financial constraints are important in the market for corporate control. Shleifer and Vishny (1992) suggest that corporate control may not accrue to the party with the highest cash-flow potential, because of capital market imperfections. Our model has the same feature. Moreover, our analysis suggests that the precise sales format matters. This perspective can shed further light on how to improve bankruptcy procedures, specifically when it may be desirable to hold an immediate auction,[30] and which rule should be used to allocate the bankrupt assets.

APPENDIX

Proof of Lemma 1

The proof has three parts. The first part shows that any symmetric equilibrium bid function must take the form $B_f(w, v) = \min\{w, b_f(v)\}$ for some continuous, strictly increasing function $b_f(\cdot)$. The second part establishes existence of a unique bid function satisfying (6) that is strictly increasing. The third part shows that the bid function constitutes an equilibrium. Here, we present the third part. The first and second parts are contained in Che and Gale (1995) and are also available from the authors upon request.

We show that the bidding strategy described in Lemma 1 is an equilibrium bidding strategy. Consider an arbitrary bidder with $(w, V) \geqq (r_f, r_f)$. If all others employ $B_f(\cdot, \cdot)$, the bidder chooses $b \in [r_f, w]$ to maximize

$$U_f(b; w, v) \equiv [v-b]F_f(b_f^{-1}(b))^{N-1}.$$

(The inverse of $b_f(\cdot)$ is well defined since the bid function is strictly increasing.) The Lagrangean associated with the problem is

$$L(b, \lambda, \mu) \equiv U_f(b; w, v) + \lambda(w-b) + \mu(b-r_f),$$

30. See Aghion, Hart, and Moore (1994) and Baird (1986).

where λ and μ are multipliers associated with the two constraints. Observe that

$$\frac{\partial U_f(b; w, v)}{\partial b} = -F_f(b_f^{-1}(b))^{N-1} + [v-b]\frac{dF_f(b_f^{-1}(b))^{N-1}}{db}$$

$$= [v - b_f^{-1}(b)]\frac{dF_f(b_f^{-1}(b))^{N-1}}{db}$$

$$\begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \quad \text{if} \quad b \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} b_f(v).$$

The second equality follows from (6). Clearly, $\mu = 0$ since $b_f(v) \geqq r_f$. If $b_f(v) \leqq w$, the solution is $b = b_f(v)$. If $b_f(v) > w$, then $\lambda > 0$, and complementary slackness implies that $b = w$. ||

Proof of Proposition 2

In an arbitrary auction M in $\mathcal{F}$, a revenue-maximizing seller will choose a reserve price so that $\hat{v}_M$ maximizes (5). In first- and second-price auctions, a reserve price of r entails $\hat{v}_f = \hat{v}_s = r$. The unconstrained types' equilibrium bids are independent of the reserve price in a second-price auction, but they depend on the reserve price in a first-price auction. With a slight abuse of notation, we write $b_f(v; r)$ to denote the equilibrium bid of an unconstrained type with $v \geqq r$, given a reserve price r, in the first-price auction (i.e. the function $b_f(\cdot\,; r)$ satisfies (6) for $r_f = r$.) It is useful to establish first the following result.

Fact 1. $b_f(v; r) \geqq b_f(v; r')$ for $v \geqq r > r'$; *i.e.*, $b_f(v; \cdot)$ is nondecreasing.

Proof. Fix r and $r' < r$. By (6), $b_f(r; r) = r > b_f(r; r')$. Since $b_f(v; r)$ is continuous in $v \geqq r$, by Lemma 1, it then follows that $b_f(v'; r) > b_f(v'; r')$ for $v' > r$, but sufficiently close to r. By continuity, it now suffices to show that, whenever $b_f(v; r) = b_f(v; r')$ for some $v > r$, $b_f(\tilde{v}; r) = b_f(\tilde{v}; r')$ for all $\tilde{v} > v$. The latter fact holds since, whenever $b_f(v; r) = b_f(v; r') \equiv b$,

$$b_f'(v; r) = \frac{(N-1)(v-b)G(b, v)^{N-2}G_2(b, v)}{G(b, v)^{N-1} - (N-1)(v-b)G(b, v)^{N-2}G_1(b, v)} = b_f'(v; r'). \qquad ||$$

Using Fact 1, we are now ready to prove the first statement of Proposition 2. Note first that $F_f(r) = G(b_f(r), r) = G(r, r) = F_s(r)$ for a reserve price r. Now, let $R_f(r)$ and $R_s(r)$ denote the expected revenue given a reserve price r in the first- and second-price auctions, respectively. Differentiating (5) with respect to r, for $M = f$ and s, we obtain

$$R_f'(r) = N\{1 - G(r, r) - r[G_1(r, r) + G_2(r, r)]\}G(r, r)^{N-1} - \int_r^{\bar{v}} G_1^{(2)}(b_f(v; r), v)\frac{\partial b_f(v; r)}{\partial r}dv$$

$$\leqq N\{1 - G(r, r) - r[G_1(r, r) + G_2(r, r)]\}G(r, r)^{N-1}$$

$$= R_s'(r),$$

where $G^{(2)}(b_f(\cdot\,; r), \cdot)$ is the distribution of the second-order statistic of N random variables generated by $G(b_f(\cdot\,; r), \cdot)$ and $G_1^{(2)}(b_f(v; r), v)$ denotes its derivative with respect to the first argument. The above inequality holds since $G_1^{(2)}(b_f(v; r), v) \geqq 0$, and $\partial b_f(v; r)/\partial r \geqq 0$ for almost every $v \geqq r$, by Fact 1. The inequality implies that whenever it strictly pays to lower the reserve price in a second-price auction, it also does so in a first-price auction. It then follows that, given a revenue-maximizing reserve price in a second-price auction, a weakly lower reserve price maximizes expected revenue in a first-price auction. With a (weakly) lower reserve price, the rankings of Proposition 1 clearly follow: The revenue ranking follows by Proposition 1 and the seller's revealed preference, and the social surplus rankings follow from Theorem 1 for $\hat{v}_f = r_f^* \leqq r_s^* = \hat{v}_s$. ||

Proof of Lemma 2

Clearly, P1 is satisfied by definition of a second-price auction. The other properties are shown to hold in order.

(i) P2: We constructed a dominant strategy in the text, and we show its continuity here. For any (w, v), the equilibrium bid $b = B_s(w, v) \leqq \delta(w)$ is the highest $\tilde{b}$ satisfying $v \geqq C(\tilde{b}, w)$. We show that, for any interval

(b', b'') containing b, the equilibrium strategy for a type (w', v') is also contained in that interval, if (w', v') is sufficiently close to (w, v). Fix any $b' < b$ and consider $\varepsilon > 0$ such that $b' + \varepsilon < b$. Since $b' + \varepsilon < \delta(w)$, Assumption 2 implies that $C(b' + \varepsilon, w')$ goes to $C(b' + \varepsilon, w)$ as w' goes to w. So, $B_s(w', v') \geqq b' + \varepsilon > b'$ for (w', v') sufficiently close to (w, v). Now fix any $b'' > b$. If $b < \delta(w)$, then $v = C(b, w)$ (since $C(x, w)$ is continuous in x for $x \in (0, \delta(w))$, by convexity), so $v'' < C(b'', w'')$ for (w'', v'') sufficiently close to (w, v). Now, suppose that $b = \delta(w)$. Assumption 2 implies that $\delta(\cdot)$ is continuous, so $b'' > \delta(w'')$ for w'' sufficiently close to w, which implies that $v'' < C(b'', w'') = \infty$ for (w'', v'') sufficiently close to (w, v). In sum, $B_s(w'', v'') < b''$ for (w'', v'') sufficiently close to (w, v). Combining the two arguments, we conclude that $B_s(\cdot, \cdot)$ is continuous.

(ii) P3: Weak monotonicity follows easily from Assumptions 1, 3, and 4, which imply that $C(x, w)$ is nondecreasing in $(x, -w)$. To show strict monotonicity, for any (w, v), again let $b = B_s(w, v) \leqq \delta(w)$ be the highest $\tilde{b}$ satisfying $v \geqq C(\tilde{b}, w)$. Consider any $(w', v') \gg (w, v)$. By Assumption 3, for any $\varepsilon > 0$, there exists $b' > b$ such that

$$C(b', w') - C(b, w') < (M_w + \varepsilon)(b' - b).$$

It then follows that there exists $b'' > b$ such that $v' \geqq C(b'', w')$, which implies that $B_s(w', v') > B_s(w, v)$.

(iii) P4: Given P3 and Assumption 4, a type-$(\bar{w}, \bar{v})$ bidder makes the highest bid, $\bar{v}$, and is unconstrained. Similarly, a type-$(\bar{w}, r_s)$ bidder makes the lowest bid, r_s, and is unconstrained. Given the continuity of $B_s(\cdot, \cdot)$, P4 then holds. ||

Proof of Lemma 3

Given a continuous Bayes–Nash equilibrium, weak monotonicity holds since the payment increases with the bid in the sense of first-order stochastic dominance (see footnote 16). The equilibrium bid must increase strictly when both w and v rise (strict monotonicity), or else there would exist a rectangle of types that make the same bid, which is inconsistent with equilibrium since raising the bid slightly is preferable. Thus, P3 holds. Given the continuity and monotonicity of bids, P4 also holds since bidders with $\bar{w}$ make both the maximum and minimum equilibrium bids, and they are unconstrained, by Assumption 4.

We now show that $p(\cdot)$ is strictly increasing and differentiable in the interior of the range of the equilibrium bids. Strict monotonicity is immediate, since no bid can be made with positive probability. If $p(\cdot)$ is increasing, it is differentiable almost everywhere in the range. Pick an arbitrary b in the interior of the range of the equilibrium bids and let $p'(b^+)$ and $p'(b^-)$ denote the right-hand and left-hand derivatives at b, respectively. (The right-hand and left-hand limits of $p(\cdot)$ at b are both equal to $p(b)$, since $p(\cdot)$ is continuous). Since b is in the range of the equilibrium bids, $b \in [r_f, \bar{v})$. Furthermore, there exists a type $(\bar{w}, v)$, $v \geqq r_f$, that bids b (by P4). By Assumption 4, $C(b, \bar{w}) = b$ for $b \leqq \bar{v}$, so

$$p'(b^+)(v - b) - p(b) \leqq 0, \tag{A1}$$

$$p'(b^-)(v - b) - p(b) \geqq 0. \tag{A2}$$

Suppose that (A1) holds with a strict inequality. Then, there is an open rectangle of types sufficiently close to $(\bar{w}, v^*)$, with $v' > v$, who also prefer to bid b. It follows that the bid b is submitted with positive probability, which cannot occur in equilibrium since it would then be better to bid slightly higher than b. Therefore, (A1) must hold with equality. But the symmetric argument, (A2) holds with equality. We conclude that $p'(b^+) = p'(b^-)$. ||

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