PROJECT DESCRIPTION

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1. Introduction

The applicant's research interests span the fields of dispersive partial differential equations, microlocal analysis and harmonic analysis. In this section, current and completed research projects are discussed. Most of this work will focus on the behavior of solutions to the nonlinear Schrödinger equation (NLS). There will be some work on the linear theory of pseudodifferential operators (PDO's), including low regularity well-posedness and Strichartz estimates. The applicant is also interested in billiard problems.

1.1. **Stable perturbations of solitons for saturated NLS in Three Dimensions.** As seen in [13], a nonlinear Schrödinger equation with a saturated nonlinearity is of the form:

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

for

$$\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{1 + s^{\frac{q-p}{2}}},$$

where $1 + \frac{4}{n-2} > q > 1 + \frac{4}{n} > p > 1$ for $n \geq 3$ and $\infty > q > 1 + \frac{4}{n} > p > 1$ for $n < 3$. Soliton solutions of such an equation are solutions of the form $e^{i\lambda^2 t} Q_\lambda(x)$, where Q_λ is the unique, radial, decreasing solution to

$$-\Delta Q_\lambda + \lambda^2 Q_\lambda - \beta(|Q_\lambda|^2) Q_\lambda = 0.$$

Saturated nonlinearities are those that behave like supercritical nonlinearities for small values and subcritical nonlinearities for large values. See [20] and [25] for discussions of such nonlinearities. Numerics show that theise nonlinearities have a soliton of minimal L^2 mass. By [7], it is known that such a soliton is unstable under general perturbations. The goal of this project is to use the methods of [2] and [12] in order to construct a stable class of perturbations for this minimal mass soliton, say Q. Numerical experiments examining the nature of perturbations of saturated nonlinearity solitons will also be carried out to suggest behavioral properties of the solutions. The key methods involved are a detailed discussion of the spectral theory for the Hamiltonian linearized about Q, as well as development of a general distorted Fourier basis theory in order to push forward the existence argument.

1.2. **Scattering for q-NLS across delta potentials.** In [9], the applicant, Justin Holmer and Maciej Zworski explore the scattering matrix that occurs when a soliton for cubic-NLS interacts with a delta function potential. Specifically,

$$\begin{cases} iu_t + \Delta u + \frac{1}{2}|u|^2 u + q\delta_0(x)u = 0 \\ u(0,x) = e^{ivx} Q(x - x_0) \end{cases}$$

is solved as $v \to \infty$, where Q is the ground state soliton solution and $q \neq 0$. This will be referred to as the q-NLS equation. The Strichartz estimates still hold for such an equation using the linear scattering theory as developed in [26]. Then, one is able to prove dynamical results modulo errors of size $O(|v|^{-\alpha})$ for $\alpha > 0$ as defined in the result. The solution is close to either nonlinear evolution of the data away from the soliton or close to the linear evolution near the soliton. One is able to describe modulo small errors the profile of the solution after interaction with the delta function. Then, using inverse scattering, one shows that in long times this solution resolves into a combination of solitons and dispersion. A numerical companion examining the values of v for which these asymptotics begin to appear and a study of $q < 0$ can be found in [10].

1.3. **Integrability along the flow suggests local in time well-posedness for Schrödinger-type PDO's.** In collaboration with Jason Metcalfe and Daniel Tataru [15], it is shown that under the assumption

$$\int_0^1 |\partial_x^\alpha \partial_\xi^\beta a(x^s, \xi^s)|^{1+\epsilon}\, ds < C,\ |\alpha| + |\beta| \geq 2,$$

where ϵ is not necessarily small, x^s, ξ^s is the Hamilton flow and A is the operator affiliated with the symbol a,

$$\begin{aligned} (i\partial_t + A(x,D))u &= f, \\ u(0) &= u_0, \end{aligned}$$

is well-posed locally in time. In fact, if $u_0 \in \mathcal{S}$, then $u(x,t) \in \mathcal{S}$ for all $0 \leq t \leq 1$, where $\mathcal{S}$ is the Schwartz class of functions. This result uses a phase space transform approach as described in [28] and developed further in [11]. This will have applications for the local well-posedness results for quasilinear Schrödinger equations. Other results in this direction include [8] and the references contained within.

1.4. **Local smoothing and Strichartz estimates on asymptotically flat manifolds without a nontrapping condition.** With Jason Metcalfe and Daniel Tataru [16], the connection between long-time Strichartz estimates and local smoothing estimates for Schrödinger equations with C^2, asymptotically flat coefficients is studied. A second order elliptic operator,

$$A(t,x,D) = D_i a^{ij}(t,x) D_j,$$

is fixed, where $a^{ij} \in C^2$ is symmetric in i, j and satisfies

$$a^{ij}\xi_i\xi_j \geq \delta_0 |\xi|^2, \quad \delta_0 > 0, \quad \text{for any } \xi.$$

Then, the Schrödinger evolution

$$Pu = (D_t + A(t,x,D))u = f, \quad u(0) = u_0 \tag{1.1}$$

is considered. Using the outgoing parametrix construction of Tataru [28], the proof of Strichartz estimates is reduced to proving certain frequency-localized local smoothing estimates. Then, for any asymptotically flat coefficients, including those with trapping, these smoothing estimates are proved outside of a bounded region, V, modulo a lower order error term on the bounded set. As a result, one immediately obtains local-in-time Strichartz estimates, without any loss of regularity, in the exterior of V. Moreover, it is shown that in order to obtain global-in-time exterior estimates, it suffices to prove good smoothing estimates over this bounded set.

1.5. **Scattering thresholds for NLS equations in one dimension.** This project with Justin Holmer is meant to explore the dynamics at work in soliton resolution in one dimension. In subcritical monomial NLS, the behaviors of the solution with initial data of the form $\alpha Q(x)$ are studied, where Q is the ground state soliton and $\alpha < 1$. As seen in [9], by inverse scattering methods, this problem is well understood in cubic-NLS. However, it is important to understand why the resolution is much less clear in other subcritical NLS equations. To this end, a numerical study of the scattering threshold is conducted for various monomial nonlinearities and saturated nonlinearities. Analytically, the applicant will search for asymptotic descriptions for the behavior of the solution with this initial data.

2. Research objectives, methods, and significance

In this section, the applicant discusses future projects and their possible significance. Such studies include stability theory of solitons, describing the spectrum of linearized Hamiltonians, scattering thresholds for nonlinear equations, and well-posedness results for general classes of nonlinearities. As a main focus, the applicant will pursue results for the nonlinear Schrödinger equation, but will also study nonlinear wave equations and relativity. The methods required will include, but are not limited to microlocal/semiclassical analysis, phase space transforms, functional analysis, Sobolev estimates, numerical analysis, variational methods and spectral theory. The equations of interest have broad applications to the study of Bose-Einstein condensates, fiber optic communications, water waves and other physical applications.

2.1. **Soliton interaction dynamics, stable perturbations of minimal mass solitons in other dimensions, and further development of numerical spectral theory results.** Building upon the work in [14], the applicant will study the dynamics of solitons interacting both with perturbations and with other solitons. Especially in the case of minimal mass solitons, it is very much open what a collision at relatively low velocity would do to the structure of the solution. It could either lead to decay or there might be stability. Either way, it is interesting to study these phenomenological effects using similar numerical and analytical techniques as developed above, particularly in one-dimension. The applicant will continue to develop the theory presented in [14] for other dimensions. Currently the spectral theoretical results and iteration arguments rely on some special conditions in three dimensions, but could be generalized to other dimensions. The numerics determining whether or not there are small eigenvalues contained in the continuous spectrum of the linearized Hamiltonian could also be improved substantially. The applicant will explore the use of Evans functions as discussed in [19] in order to improve the efficiency and accuracy of these numerical calculations.

2.2. **Dispersion for initial data below the minimal L^2 mass for saturated nonlinearities that lack scale invariance.** As shown in [25], for the L^2 critical NLS equation, there is a minimal mass M, for the initial data below which only linear dispersion occurs. This is proved by relating the ideal constant for a Gagliardo-Nirenberg inequality to the mass of the soliton. For the saturated nonlinearity, numerically there is a minimal mass soliton. Both analytically describing the shape of the soliton curve in general situations and proving dispersion below the minimal mass become incredibly difficult without scaling. The applicant will continue to work in this direction building off of the work in [23] and [22] in order to anaytically prove scattering below this minimal L^2 mass. Scattering results that do not depend upon scaling are very rare and would go a long way towards understanding soliton resolution.

2.3. **A class of initial data for which scattering occurs in subcritical monomial NLS equations and the applications to soliton resolution.** As an extension of [14], with Sarah Raynor and Doug Wright, the applicant will work to prove scattering for a certain class of solutions for quadratic-NLS in three dimensions. Then, the authors will attempt extend this to have soliton resolution in this equation when the soliton is perturbed by a similar class of functions. This is an extension of [27] by Terence Tao, who proves a partial result towards soliton resolution under the assumption that you have a globally well-behaved solution for a super-critical NLS problem.

2.4. **Integrability along the flow suggests local in time well-posedness for wave-type PDO's.** This, in conjunction with Matt Blair, Jason Metcalfe and Daniel Tataru will be an extension of the current results in [15], where it is shown that for Schrödinger-like PDO's one can build a solution in general with minimal regularity requirements on the symbol of the operator. However, for wave-like PDO's, the phase space transform presented in [28] is used to build solutions on various frequency scales, then patch them all together in a meaningful sense.

2.5. **Asymptotic stability of solitons for NLS on manifolds depending upon the geometry.** This is a project conceived in a conversation with Maciej Zworski. The applicant will discuss this project with Andrew Hassell who has a very good understanding of the interaction between PDE and the geometry of the manifolds on which they are solved. The general idea is that with enough curvature, solitons of small energy and mass should be spread out enough to be quite self-interacting. If this self-interaction can cause dispersion, then standard results for asymptotic stability of solitons might be possible without defining the nonlinearity in order to avoid these small solitons. While the applicant knows of no references in this direction, for general asymptotic stability results, see [6].

3. Enhancement of career development

This fellowship would have a substantial impact on the applicant's career. Through contact with Professor Michael Weinstein, the applicant will become further aware of the applications associated to the studies of dispersive partial differential equations. As a result, the applicant can address a broader scope of problems of physical interest. In addition, Professor Weinstein is an expert in modulational stability and asymptotic stability, an area of research the applicant will pursue diligently as a post-doctoral fellow. Therefore, such a collaboration would be extremely beneficial. Finally, with the funding provided by such an award, the applicant would have more time to establish interesting research results and continue to develop deeper ties in the mathematical community through travel, communication and collaboration. Travel funding will be especially improtant since the author has met several of his collaborators and began several projects during conferences.

4. Justification of sponsoring scientist and host institution

Professor Weinstein at Columbia University was chosen by the applicant as a sponsoring scientist due to his expertise in the field of nonlinear dispersive equations. He has developed many of the mathematical methods for this subject and, having worked for Bell Labs, also understands the important problems to work on for physical applications. His results are known both for their clarity and depth, as displayed in [30]. Professor Weinstein is very active in both applied and pure mathematical research projects in the direction of the applicant's doctoral work. Hence, there are several possibilities for future projects and many similar interests that will lead to a productive working relationship over the course of the fellowship period.

The Applied Mathematics and Physics Department at Columbia University provides an excellent location for research. There are many excellent faculty there, as well as in the Mathematics department. As an active research university, there will also be several visitors to interact with on a regular basis. Plus, the campus is within travelling distance of the Courant Institute, Rutgers, and Princeton, all of whom have very active research going on in the field of Partial Differential Equations. In particular, the applicant hopes to interact with professors whose publications have influenced his previous work, such as Sergiu Klainerman, Igor Rodnianski, Percy Deift, Jalal Shatah, and Avy Soffer.

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