RESEARCH STATEMENT

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My research interests span the fields of dispersive partial differential equations, microlocal analysis and harmonic analysis. In the first section, we discuss current and completed research projects. Most of this work will focus on the behavior of solutions to the nonlinear Schrödinger equation (NLS). There will be some work on the linear theory of pseudodifferential operators (PDO's), including work on billiard problems, low regularity well-posedness and Strichartz estimates. The second section contains ideas for future research.

1. Current Research

In this section, we present a brief discussion of several research projects involving dispersive partial differential equations.

1.1. **Stable perturbations of solitons for saturated NLS equations in three dimensions.** As seen in [13], a nonlinear Schrödinger equation with a saturated nonlinearity is of the form:

$$\begin{aligned} iu_t + \Delta u + \beta(|u|^2)u &= 0 \\ u(0,x) &= u_0(x), \end{aligned}$$

for

$$\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{1 + s^{\frac{q-p}{2}}},$$

where $1 + \frac{4}{n-2} > q > 1 + \frac{4}{n} > p > 1$ for $n \geq 3$ and $\infty > q > 1 + \frac{n}{2} > p > 1$ for $n < 3$. Soliton solutions of such an equation are solutions of the form $e^{i\lambda^2 t} Q_\lambda(x)$, where Q is the unique, radial, decreasing solution to

$$-\Delta Q_\lambda + \lambda^2 Q_\lambda - \beta(|Q_\lambda|^2) Q_\lambda = 0.$$

Saturated nonlinearities are those that behave like supercritical nonlinearities for small values and subcritical nonlinearities for large values. See [20] and [25] for discussions of such nonlinearities. Numerics show that these nonlinearities have a soliton of minimal L^2 mass. By [7], we know that such a soliton is unstable under general perturbations. The goal of this project is to use the methods of [2] and [12] in order to construct a stable class of perturbations for this minimal mass soliton, say Q. Numerical experiments examining the nature of perturbations of saturated nonlinearity solitons will also be carried out to suggest behavioral properties of the solutions. The key methods involved are a detailed look, both analytically and numerically, at the spectral theory of the Hamiltonian linearized about Q, as well as development of a general distorted Fourier basis theory in order to push forward the existence argument. This is work done in my Ph.D. thesis directed by Daniel Tataru.

1.2. **Scattering for q-NLS across delta potentials.** With Justin Holmer and Maciej Zworski, we explore the scattering matrix that occurs when a soliton for cubic-NLS interacts with a delta function potential. Specifically, we solve

$$\begin{aligned} iu_t + \Delta u + \frac{1}{2}|u|^2 u + q\delta_0(x)u &= 0 \\ u(0,x) &= e^{ivx}Q(x-x_0) \end{aligned}$$

as $v \to \infty$, where Q is the ground state soliton solution and $q > 0$. This will be referred to as the q-NLS equation. We find that the Strichartz estimates still hold for such an equation using the linear scattering theory as developed in [26]. Then, we are able to prove closeness modulo errors of size $O(|v|^{-1})$ that the solution is close to either nonlinear evolution of the data away from the soliton and close to the linear evolution of the soliton near the soliton. Not only that, we are able to describe, modulo errors again of size $O(|v|^{-1})$, the profile of the solution after interaction with the delta function. Then, we are able to use inverse scattering to show that in long times this solution resolves into a combination of solitons and dispersion again with small errors inversely related to the size of $|v|$. This work is contained in [9] and a numerical companion examining the values of v for which these asymptotics begin to appear and a study of $q < 0$ can be found in [10].

1.3. **Integrability along the flow suggests local in time well-posedness for Schrödinger-type PDO's.** In collaboration with Jason Metcalfe and Daniel Tataru, we show that under the assumption

$$\int_0^1 |\partial_x^\alpha \partial_\xi^\beta a(x^s,\xi^s)| \, ds < \epsilon, \ |\alpha| + |\beta| \geq 2,$$

where ϵ is small, x^s, ξ^s is the Hamilton flow and a is a PDO, we have

$$\begin{aligned} (\partial_t + A(x,D))u &= f, \\ u(0) &= u_0, \end{aligned}$$

is well-posed locally in time. In fact, if $u_0 \in \mathcal{S}$, then $u(x,t) \in \mathcal{S}$ for all $0 \leq t \leq 1$, where $\mathcal{S}$ is the Schwartz class of functions. This result uses a phase space transform approach as described in [28] an developed further in [11]. This result will then be applied in order to attempt to develop dispersive estimates for such operators. Also, we believe this will have applications for the local well-posedness results for quasilinear Schrödinger equations. Other results in this direction include [8] and the references contained within.

1.4. **Local smoothing and Strichartz estimates on asymptotically flat manifolds without a nontrapping condition.** With Jason Metcalfe and Daniel Tataru, we study the connection between long-time Strichartz estimates and local smoothing estimates for Schrödinger equations with C^2, asymptotically flat coefficients. We fix a second order elliptic operator

$$A(t,x,D) = D_i a^{ij}(t,x) D_j,$$

where $a^{ij} \in C^2$ is symmetric in i, j and satisfies

$$a^{ij}\xi_i\xi_j \geq \delta_0|\xi|^2, \quad \delta_0 > 0, \quad \text{for any } \xi.$$

We then consider the Schrödinger evolution

$$Pu = (D_t + A(t,x,D))u = f, \quad u(0) = u_0. \tag{1.1}$$

Using the outgoing parametrix construction of Tataru [28], we reduce the proof of Strichartz estimates to proving certain frequency-localized local smoothing estimates. Then, for any asymptotically flat coefficients, including those with trapping, we prove these smoothing estimates outside of a bounded region modulo a lower order error term on the bounded set. As a result, we immediately obtain local-in-time Strichartz estimates, without any loss of regularity, in the exterior of the said bounded set. Moreover, we show that in order to obtain global-in-time exterior estimates, it suffices to prove good smoothing estimates over this bounded set.

1.5. **Scattering thresholds for NLS equations in one dimension.** This project with Justin Holmer is meant to explore the dynamics at work in soliton resolution in one dimension. In subcritical monomial NLS, we study the behaviors initial data that are $\alpha Q(x)$, where Q is the ground state soliton and $\alpha < 1$. As seen in [9], by inverse scattering methods, this problem is well understood in cubic-NLS. However, we wish to understand why the resolution is much less clear in other subcritical NLS equations. To this end, we numerically study the scattering threshold in the critical norm for various monomial nonlinearities and saturated nonlinearities. Analytically, we search for asymptotic descriptions for the behavior of the solution with this initial data.

1.6. **Eigenfunctions for partially rectangular billiards.** In this note [13], we further develop the methods of Burq-Zworski [5] to study eigenfunctions for billiards which have rectangular components: these include the Bunimovich billiard, the Sinai billiard, and the recently popular pseudointegrable billiards [3]. The results are an application of a "black box" point of view as presented in [4] by the same authors.

By a partially rectangular billiard, we mean a connected planar domain, Ω, with a piecewise smooth boundary, which contains a rectangle, $R \subset \Omega$, such that if we decompose the boundary of R, into pairs of parallel segments, $\partial R = \Gamma_1 \cup \Gamma_2$, then $\Gamma_i \subset \partial\Omega$, for at least one i.

We show that for such billiards, the eigenfunctions of the Dirichlet, Neumann, or periodic Laplacian cannot concentrate in closed sets in the interior of the rectangular part. A combination of this elementary result with the now standard, but highly non-elementary, propagation results of Melrose-Sjöstrand [17] and Bardos-Lebeau-Rauch [1], can give further improvements – see [4],[5].

The original work presented in this work proves further non-concentration results, away from the obstacle in the Sinai billiard. The key extension is to actually prove concentration in regions of certain pseudointegrable billiards provided there are certain classical trajectories.

2. Future Research

In this section, we discuss future projects and their possible significance. Such studies include stability theory of solitons, describing the spectrum of linearized Hamiltonians, scattering thresholds for nonlinear equations, and well-posedness results for general classes of nonlinearities. As a main focus, I will pursue results for the nonlinear Schrödinger equation, but will also study nonlinear wave equations and relativity. The methods required will include, but are not limited to microlocal/semiclassical analysis, phase space transforms, functional analysis, Sobolev estimates, numerical analysis, variational methods and spectral theory. The equations of interest have broad applications to the study of Bose-Einstein condensates, fiber optic communications, water waves and other physical applications.

2.1. **Soliton interaction dynamics.** Building upon the work in my thesis, I will study the dynamics of solitons interacting both with perturbations and also with other solitons. Especially in the case of minimal mass solitons, it is very much open what a collision at relatively low velocity would do to the structure of the solution. It could lead to decay which represents annihilation of some sort, or there might be stability. Either way, it will be interesting to study these phenomenological effects, particularly in one-dimension.

2.2. **Stable perturbations of minimal mass solitons in other dimensions and further development of numerical spectral theory results.** I will to continue to develop the theory presented in my thesis for other dimensions. Currently the spectral theoretical results and iteration arguments rely on some special conditions in three dimensions, but could be generalized to other dimensions. Also, the numerics determining whether or not there are small eigenvalues contained in the continuous spectrum of the linearized Hamiltonian can be improved substantially. Currently, they rely on very strict asymptotic analysis. I will explore the use of Evans functions in order to improve the efficiency and accuracy of these numerical calculations.

2.3. **Dispersion for initial data below the minimal L^2 mass for saturated nonlinearities that lack scale invariance.** As proven by Weinstein and shown in [25], for the L^2 critical NLS equation, there is a minimal mass for the initial data. Below this minimal mass, only dispersion occurs and there are no solitons. This is proved by relating the ideal constant for a Gagliardo-Nirenberg inequality to the mass of the soliton. For the saturated nonlinearity, numerically we see there is a minimal mass soliton. Both analytically describing the shape of the soliton curve in general situations and proving dispersion below the minimal mass become incredibly difficult without scaling. In conjunction with Justin Holmer, we will continue to work in this direction building off of the work in [23] and [22] in order to anaytically prove scattering below this minimal L^2 mass. Scattering results that do not depend upon scaling are very rare and would go a long way towards understanding soliton resolution.

2.4. **A class of initial data for which scattering occurs in subcritical monomial NLS equations and the applications to soliton resolution.** As an extension of my thesis work, with Sarah Raynor and Doug Wright, we will prove scattering for a certain class of solutions for quadratic-NLS in three dimensions. Then, we will extend this to have soliton resolution in this equation when we perturb by a similar class of functions. This is similar to the work of Tao [27], who proves a partial result towards soliton resolution under the assumption that you have a globally well-behaved solution for a super-critical NLS problem.

2.5. **Integrability along the flow suggests local in time well-posedness for wave-type PDO's.** This, in conjunction with Matt Blair would be an extension of the current results with Jason Metcalfe and Daniel Tataru. For Schrodinger-like PDO's, we can build a solution in general. However, for wave-like PDO's, we will have to use the phase space transform method developed by Tataru [28] and presented in [15] to build solutions on various frequency scales, then patch them all together in a meaningful sense. A similar assumption about integrability along the flow will then lead to local well-posedness results there. Hopefully, similar applications to nonlinear wave equations would exist.

2.6. **Asymptotic stability of solitons for NLS on manifolds depending upon the geometry.** This is a project conceived in a conversation with Maciej Zworski. I will discuss this project with Andrew Hassell who has a very good understanding of the interaction between PDE's and the geometry of the manifolds on which they are solved. The general idea is that with enough curvature, solitons of small energy and mass should be spread out enough to be quite self-interacting. If this self-interaction can cause dispersion, then standard results for asymptotic stability of solitons would be possible without defining the nonlinearity in order to avoid these small solitons. While I know of no references in this direction, for general asymptotic stability results, see [6].

2.7. **Sharp resolution of partially rectangular billiard problem.** As a counterpart to my earlier result on partially rectangular billiards, I will resolve the issue as to whether or not there is concentration in certain regions without the need of these classical trajectories. This could mean constructing a set of eigenvalues that do not have such classical orbits and determining that concentration is impossible in the interested regions. Also, it is possible from the work of [18] that we could overcome the reflection principle arguments inherent in the previous work on the subject. Though I am unclear which direction this project will go, I will be very interested in resolving the issue.

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