10/6/05
6101

SOLUTIONS: HW 4

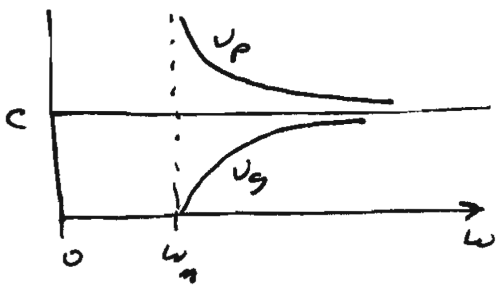
#1) p. 4.3

$$\omega^2 = \omega_m^2 + c^2 k^2$$

$$v_p = \frac{\omega}{k} = c\,\frac{\omega}{ck} = \frac{c}{\sqrt{1 - \omega_m^2/\omega^2}}$$

$$v_g = \frac{\partial \omega}{\partial k}, \quad \text{BUT} \quad 2\omega \frac{\partial \omega}{\partial k} = 2 c^2 k$$

$$\text{SO} \quad v_g = c^2 k/\omega = c^2/v_p = c\sqrt{1 - \omega_m^2/\omega^2}$$



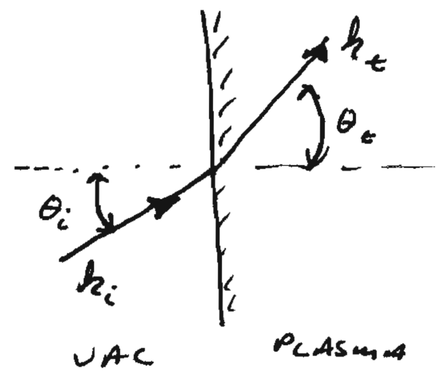
THIS IS JUST LIKE E.M. PLASMA WAVES, WHEN $\omega_m^2 = \omega_p^2$

#2) p. 4.4

SNELL'S LAW:

$$n_i \sin\theta_i = n_t \sin\theta_t$$

WHEN $\theta_t = \pi/2$, WE HAVE "TOTAL REFLECTION". THUS, THE CRITICAL ANGLE IS



$$\sin\theta_c = n_t/n_i . \quad n_i = 1 . \quad n_t = \frac{ck}{\omega} = \sqrt{1 - \omega_p^2/\omega^2}$$

$$\text{SO} \qquad \sin\theta_c = \sqrt{1 - \omega_p^2/\omega^2}$$

#3) p. 4.5 FOR HARMONIC PERTURBATIONS, $E(t) \sim \mathrm{Re}\{E e^{-i\omega t}\}$

$$\text{SO...} \quad -i m_s \omega v_s + \nu_s m_s v_s = e_s (E + v_s \times B_0)$$

$$\underbrace{-i\omega\, m_s\left(1 + \frac{i\nu_s}{\omega}\right)}\, v_s = e_s (E + v_s \times B_0)$$

$$m_s^* = m_s (1 + i\nu_s/\omega)$$

#3) CONTINUED

b) FOR FIELD-FREE E.M. WAVES

$$k^2 c^2 = \omega^2 - \omega_p^2$$

SO
$$\frac{kc}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}\left(1 + i\frac{\nu_s}{\omega}\right)^{-1}}$$

IF $\omega_p^2 \ll \omega^2$ AND $\nu_s \ll \omega$, THEN

$$\frac{kc}{\omega} \approx 1 - \frac{1}{2}\left(\frac{\omega_p^2}{\omega^2}\right)\left(1 - i\frac{\nu_s}{\omega} + \dots\right) + \dots$$
$$\simeq 1 - \frac{1}{2}\left(\frac{\omega_p^2}{\omega^2}\right) + i\,\frac{1}{2}\,\frac{\omega_p^2 \nu_s}{\omega^3} + \dots$$

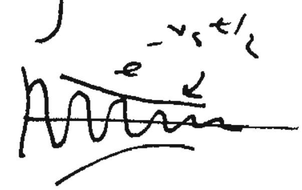
c) FOR LONGITUDINAL WAVES ...

$$-i\omega m_s\left(1 + i\nu_s/\omega\right)V_s = e_s E$$

SO
$$\omega^2 = \omega_p^2\left(1 + \frac{i\nu_s}{\omega}\right)^{-1}$$

AND
$$\omega \approx \omega_p\left(1 - \frac{1}{2}\frac{i\nu_s}{\omega} + \dots\right)$$
$$\simeq \omega_p - \frac{i}{2}\nu_s$$

SO ..
$$E(t) = \mathrm{Re}\left\{E_0 e^{-i\omega_p t} e^{-\nu_s t/2}\right\}$$
$$= E_0 \cos(\omega_p t)\, e^{-\nu_s t/2}$$



#4) P. 4.7

$$\mathrm{Det}\begin{bmatrix} S - n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{bmatrix}$$

$$\mathrm{Det} = \left(S - n^2\cos^2\theta\right)\left(S - n^2\right)\left(P - n^2\sin^2\theta\right) - n^4\sin^2\theta\cos^2\theta\left(S - n^2\right) - D^2\left(P - n^2\sin^2\theta\right)$$
$$= \left(S - n^2\right)\left[SP - n^2\left\{P\cos^2\theta + S\sin^2\theta\right\} + \cancel{n^4\sin^2\theta\cos^2\theta} - \cancel{n^4\sin^2\theta\cos^2\theta}\right] - D^2\left(P - n^2\sin^2\theta\right)$$
$$= n^4\left(P\cos^2\theta + S\sin^2\theta\right) - n^2\left[SP + S\left(P\cos^2\theta + S\sin^2\theta\right) - D^2\sin^2\theta\right] + S^2P - D^2P$$

#4) (CONT.)

So $\det = A n^4 - B n^2 + C$

$$A = S\sin^2\theta + P\cos^2\theta$$
$$B = SP(1+\cos^2\theta) + S^2\sin^2\theta - D^2\sin^2\theta$$
$$C = P(S^2 - D^2) = PRL$$

But $S^2 - D^2 = RL$ (see P. 4.6)

And $B = SP(1+\cos^2\theta) + RL\sin^2\theta$ QED

#5 a)

$$\tau = \int \frac{ds}{v_g} \qquad v_g = \frac{\partial\omega}{\partial k} \qquad \omega^2 = k^2c^2 + \omega_p^2$$
$$\frac{\partial\omega}{\partial k} = \frac{c^2 k}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

$\therefore$ IF $\frac{\omega_p^2}{\omega^2} \ll 1$, THEN

$$\tau = \int \frac{ds}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-\frac{1}{2}} \simeq \int \frac{ds}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2} + \cdots\right)$$
$$\simeq \int \frac{ds}{c} + \frac{1}{2c\omega^2}\int ds\, \omega_p^2 \qquad \omega_p^2 = \frac{e^2 n}{m_0 \epsilon_0}$$
$$\simeq \int \frac{ds}{c} + \frac{e^2}{2c\, m_0 \epsilon_0 \omega^2}\int ds\, n$$

b) COEFFICIENT, $C = \dfrac{e^2}{2c\, m_0 \epsilon_0 (2\pi)^2} = \dfrac{c}{2\pi}\Bigg(\underbrace{\dfrac{e^2}{4\pi\epsilon_0 m_0 c^2}}_{\substack{\text{CLASSICAL RADIUS} \\ \text{OF ELECTRON} \\ 2.8\times10^{-15}\,m}}\Bigg)$

c) $$\tau_{100m} - \tau_{200m} = 2 = D\left[\frac{1}{(10^8)^2} - \frac{1}{4(10^8)^2}\right] = \frac{3}{4}\frac{D}{10^{16}}$$

$$\therefore \quad D = \frac{8}{3}\times10^{16}$$

#5) CONT.

D) IF $n = 0.03\ cm^{-3} = 3 \times 10^{4}\ m^{-3}$, THEN

$$D = \frac{c}{2\pi}\, 2.8 \times 10^{-15}\, L\ 3 \times 10^{4} = \frac{8}{3} \times 10^{16}$$

$$\therefore\ L = \frac{8}{3} \times 10^{16}\ \frac{2\pi}{c}\ \frac{1}{3 \times 10^{4}}\ \frac{1}{2.8 \times 10^{-15}}$$

$$= 6.6 \times 10^{18}\ m$$

BUT 1 PARSEC $= 3 \times 10^{16}\ m$ SO $L = 222$ PARSEC

E) MAGNETIC FIELDS, REFRACTION, AND SCATTERING MAKE THIS MEASUREMENT DIFFICULT.