INDETERMINACY OF EQUILIBRIUM IN THE OVERLAPPING GENERATIONS MODEL: A SURVEY

Michael Woodford

Columbia University

May 1984

Preliminary; Comments welcome

INDETERMINACY OF EQUILIBRIUM IN THE OVERLAPPING GENERATIONS MODEL: A SURVEY

by

Michael Woodford*

One of the most important properties of competitive equilibrium, in the familiar model with a finite number of agents and a finite number of commodities, is that under generically valid conditions, equilibrium will be locally unique. This property is necessary in order for it to be possible to compute the comparative statics properties of such equilibria. It is also relied upon in arguments for the desirability of the competitive system that make use of the second welfare theorem. When one argues that any efficient allocation (and hence, the state of affairs that maximizes one's social welfare function) can be supported as a competitive equilibrium for an appropriate initial distribution of endowments, and from this that there would be no need for interference with the working of the market if the assumptions of the Walrasian model were true and lump-sum taxation were possible, one assumes implicitly that a correct distribution of initial claims to resources would suffice to pick out the desired equilibrium allocation. For this it is necessary that the competitive equilibrium associated with that particular distribution of endowments be at least locally unique.

It is therefore of no small importance to the theory of competitive economies that when the number of agents and the number of commodities both become infinite, this local uniqueness property need no longer hold. In particular, in the case of an infinite-horizon dynamic competitive economy with overlapping generations of finite-lived agents, there may be an uncountably infinite set of competitive equilibria arbitrarily near (in any of the familiar topologies used in treatments of such models) a given equilibrium. This indeterminacy does not result from any missing markets, and the examples characterized by indeterminacy are robust in the sense that small pertubations of agents' preferences or endowments or of the production technologies available will not resolve the indeterminacy.

It follows that preferences, endowments, and technology alone may not suffice to determine the allocation of resources in a dynamic economy, even when perfectly competitive markets exist for all goods. There may be an independent role for the beliefs of agents in the determination of economic outcomes, even if one restricts one's attention to equilibria in which the expectations of all agents are correct. This means, for example, that "speculative bubbles" in asset markets need not indicate either irrational behavior or incorrect expectations on the part of any traders. (The problem, however, is not simply one of asset valuation. Equilibrium may be indeterminate even in the absence of infinite-lived assets, and even in the case of economies in which all goods are perishable.)

It also follows that there may be a role for active government policy (other than to obtain a correct initial distribution of endowments), even in the case of competitive economies for which the first and second welfare theorems hold. Even if endowments are such that the desired allocation can be supported as a competitive equilibrium, equilibrium may be indeterminate in the absence of active policy, so that market forces cannot be relied upon to produce the desired result. Furthermore, active stabilization policies may exist which render the desired allocation a locally unique competitive equilibrium.

The plan of the survey is as follows. In Section I, a series of simple examples are presented, which are intended to demonstrate that the problem of indeterminate equilibrium is not simply an artifact associated with the overlapping generations model of money. These examples show that there is no general connection between indeterminacy of equilibrium and Pareto inefficiency; that equilibrium may be indeterminate in models without valued fiat money; and that equilibrium may be indeterminate in models in which various kinds of non-monetary assets exist. In Section II, the problem of indeterminate perfect foresight equilibrium is considered more generally, and existing results on the general conditions necessary for indeterminacy to be possible are summarized. The three important general results are that the dimension of indeterminacy can be as large as, but no larger than, the number of goods traded per period; that a sufficient number of the agents alive at each point in time must be finite-lived in order for indeterminacy to be possible; and that income effects must be sufficiently important, in the response of the consumption demands of the finite lived agents to price variations, in order for a Pareto optimal equilibrium to be indeterminate. In Section III, the relation between indeterminacy and the existence of equilibrium cycles is discussed, and in Section IV it is argued that there is a close relationship between indeterminary of perfect foresight equilibrium and the existence of stationary rational expectations equilibria in which "sunspots matter". Section V discusses possible responses to the problem of indeterminate equilibrium, and considers the possibility of stabilization policy to render equilibrium determinate.

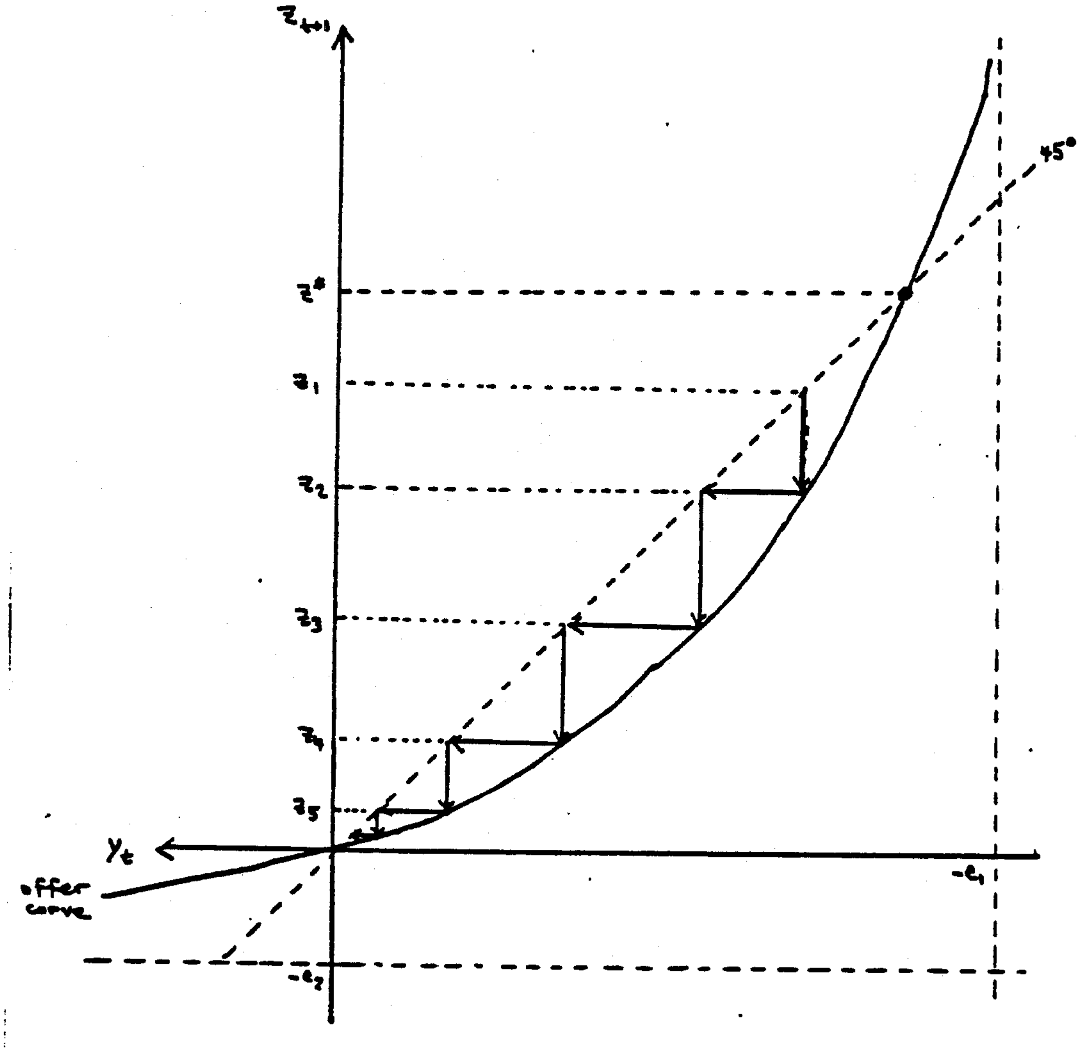
I. Indeterminacy of Perfect Foresight Equilibrium: Examples

In this section, a series of simple examples is presented. These illustrate the possible indeterminacy of perfect foresight equilibrium in the overlapping generations model. They will also give some indication of the variety of models in which this problem may occur, as well as showing that it does not occur in the case of all overlapping generations economies.

Example 1: This example is one analyzed by Gale [1973], what he calls the "Samuelson case". In this economy, there is one perishable consumption good per period. All agents live for two periods, and an equal number belong to each generation; the agents who live in periods t and $t+1$ will be called generation t (for $t = 1,2,\ldots$). All agents have identical preferences with respect to consumption in the two periods of life, and identical lifetime patterns of endowment of the consumption good (e_1 in the first period of life, e_2 in the second). There is no production, but intergenerational exchange of the consumption good is possible using fiat money. In period one, in addition to the young members of generation one, there exists a group of agents who live only in period one, referred to as generation zero; these agents hold a certain quantity $M > 0$ of fiat money at the beginning of period one. The stock of fiat money in circulation is never altered. The money is spent each period by the old in that period, for some of the endowment of the current young, who acquire money in order to spend it the following period. Fiat money is the only asset in this economy, and so provides the only means by which agents can save.

The perfect foresight equilibria of the model may be displayed using a diagram like Figure 1. (This method of exposition, which is also used for the examples to follow, is taken from Cass, Okuno, and Zilcha [1979].) Let y_t be the excess demand of the young for the consumption good in period t, and z_t be the excess demand of the old in period t. Let us assume that y_t and z_{t+1} are both continuous functions of the relative price p_t/p_{t+1}. (Here p_t is the money price of the consumption good in period t, so that p_t/p_{t+1} is the gross rate of return on savings for the members of generation t.) The way these excess demands vary with p_t/p_{t+1} can be represented by an offer curve in the y_t-z_{t+1} plane, shown in Figure 1. The offer curve passes through the origin, and remains always in the first and third quadrants. (The part of the offer curve with $y_t > 0$, $z_{t+1} < 0$ corresponds to demands that would only be made if it were possible to borrow, which has not been allowed for here; but that part of the offer curve is not used in the constructions below.) The offer curve also remains above the line $z_{t+1} = -e_2$, and to the left of the line $y_t = -e_1$, since no relative price can induce the agents to supply more than their

Figure 1



endowment in either period. Let us further assume that the offer curve passes through the origin with a slope less than 45°. This assumption, which distinguishes Gale's "Samuelson case" from his "classical case", is necessary and sufficient for the existence of an equilibrium with positive valued fiat money. As a final assumption, let first and second period consumption be gross substitutes in the excess demands of all agents, so that z_{t+1} is an increasing function, and y_t is a decreasing function, of p_t/p_{t+1}.

A perfect foresight equilibrium is then a sequence $\{z_t\}$, for $t = 1,2,\ldots$, with $z_t \geq 0$ for all t, such that for each t, $(-z_t, z_{t+1})$ is a point on the offer curve. In each period, $z_t = M/p_t$, so that there is a one-to-one correspondence between sequences of prices and sequences $\{z_t\}$. The requirement that $z_t \geq 0$ is simply the requirement that the value of money be non-negative in all periods; the second half of the definition simply states that at equilibrium prices, one will have $y_t + z_t = 0$ in each period--total excess demand is zero. Perfect foresight is assumed, insofar as it is assumed that the supply decision of the young is made in each period with a correct understanding of the value of money in the following period.

In a case like that shown in Figure 1, it is apparent that there are two possible steady state equilibria, i.e., perfect foresight equilibria in which z_t is constant for all t. These are the two intersections of the offer curve with the 45° line. One is at the origin; in this equilibrium, $y_t = z_t = 0$ in all periods. Money is never valued, and all agents consume exactly their endowments. In the other steady state, $-y_t = z_t = z^* > 0$ in all periods. This is the monetary steady state. Samuelson [1958] compared these two steady states, noting that the non-monetary steady state does not achieve a Pareto optimal allocation of resources, and is Pareto dominated by the monetary steady state; from this comparison he concluded that there is an important function served by valued fiat money in such a model.

Gale's analysis, however, showed that there are many other perfect foresight equilibria for this economy. One such is illustrated in Figure 1. Beginning with a positive value for money somewhat less than obtains in the monetary steady state $(z_1 < z^*)$, it is possible to construct successive values of z_t so that $(-z_t, z_{t+1})$ is always a point on the offer curve. Since it is possible to continue this construction forever, with $z_t > 0$ in each case, one has found another perfect foresight equilibrium. In this equilibrium money is valued, but its value decreases over time, and asymptotically approaches zero $(z_t \to 0$ as $t \to \infty)$. The rate of price inflation increases over time, asymptotically approaching π^*, where $1/(1 + \pi^*)$ is the slope of the offer curve through the origin. As this asymptotic limit is approached, the allocation of resources approaches that of the non-monetary steady state (autarchy). The existence of "hyperinflationary" monetary equilibria of this kind indicates that inflationary expectations may have a self-fulfilling character.

The point of interest to us here is that the point z_1 chosen in Figure 1 is arbitrary. For any choice of z_1 between zero and z^*, a similar construction is possible; hence to each such choice of z_1

(alternatively, choice of the initial price level p_1) there corresponds a perfect foresight equilibrium. There is thus an uncountably infinite set of distinct equilibria. Furthermore, for any $\varepsilon > 0$, there is an uncountably infinite set of equilibria in which $|z_t| < \varepsilon$ for all t, i.e., of equilibria which are uniformly close to the non-monetary steady state. We therefore say that the non-monetary steady state is an <u>indeterminate</u> equilibrium. The monetary steady state, on the other hand, is <u>determinate</u> in the terminology of this paper, because there are no other equilibria arbitrarily close to it, in the sense that $|z_t - z^*| < \varepsilon$ for all t, for arbitrarily small ε.[1/] There exist topologies under which one could say that there do exist other equilibria "arbitrarily close" to the monetary steady state. For example, if one used the product topology on the set of sequences $\{z_t\}$ satisfying the bounds $0 < z_t < z^*$, then any neighborhood of the monetary steady state contains an uncountable number of other equilibria. For open sets in the product topology place bounds on only a finite number of the z_t's, so that what happens to a sequence asymptotically does not prevent it from being considered "close" to the monetary steady state, as long as all the early elements in the sequence are close to z^*.

There are, however, advantages to the terminology used here. For one thing, it makes the determinacy or indeterminacy of a given equilibrium a local property of the perfect foresight dynamics near that equilibrium. One needs only to check whether other perfect foresight trajectories which start near the equilibrium converge to it asymptotically, or diverge from it, to settle whether or not it is determinate in the proposed sense. This will allow purely local methods to be used in determining general conditions for determinacy of steady states in Section II. Of course, the multiplicity of perfect foresight equilibria possessed by a given model is not changed by any such choice of definition. However, when it is possible to demonstrate the existence of indeterminacy in the sense proposed here, one has surely discovered a multiplicity of a very disturbing sort; the possible existence of even larger multiplicities neglected in our consideration of indeterminacy in the sensed proposed here would only mean that the problem is worse.[2] Furthermore, there is something of particular interest about the type of indeterminacy which involves a continuum of equilibria all converging asymptotically to the same steady state. It will be suggested in Section IV that it is in exactly these cases that stationary "sunspot equilibria" exist near the steady state. Such multiplicities of stationary equilibria are arguably of more interest than non-stationary equilibria of the sort displayed above, on the ground that perfect foresight or rational expectations may only be plausibly assumed in a stationary equilibrium.

The large multiplicity of equilibria in this example has been known for some time. However, many have noticed that the indeterminacy of equilibrium in this example is closely related to certain very special, and rather unsatisfactory, aspects of the model. For example, the "hyperinflationary" equilibria exist because the economy in question possesses pure fiat money, which is valued only as an asset. Plainly, a family of equilibria of this sort would not exist in a dynamic model without fiat money, or in a model in which money is needed for other reasons. Thus Brock and Scheinkman [1980] propose that indeterminacy will not exist if money must be held to pay taxes, or if it is required by a transactions technology.

Similarly, the "hyperinflationary" equilibria are all characterized by a fall in the real rate of return to some negative level (how far negative depends upon the slope of the offer curve through the origin in Figure 1). It is therefore evident that if a storage technology existed (for each unit of consumption good stored, one has s units the following period, where $1/(1 + \pi^*) < s < 1$), or some other production technology that would place a sufficiently high lower bound on the equilibrium real rate of return, equilibria tending to autarchy would not exist. (However, there can still exist a continuum of equilibria in the case of a storage technology. Total saving is constant over time in these equilibria, but the share of saving which consists of money balances decreases over time, approaching zero asymptotically, while the share of saving which consists of storage increases to compensate for the decline in the value of money.) This has led some to suppose that indeterminacy of equilibrium is not possible in production economies or in the presence of durable goods.

It is also evident that the indeterminacy of the non-monetary steady state in this example depends upon the Pareto inefficiency of that steady state (and likewise of the "hyperinflationary" equilibria converging to it). If one assumes that the offer curve passes through the origin with a slope greater than 45° (Gale's "classical case"), the non-monetary steady state becomes Pareto optimal--and it ceases to be indeterminate. Some have therefore concluded that the problem of indeterminacy is intimately bound up with the other well-known pathology of overlapping generations economies, the failure of the first welfare theorem. Since it is known that the inefficient equilibria can be ruled out by various modifications of the overlapping generations model, such as adding even a small amount of non-depreciating "land" or even a very small number of infinite-horizon maximizers[3/], which modifications arguably make the model more realistic, it might be supposed that these modifications would also rule out indeterminacy of equilibrium.

These are good reasons not to be very interested in the "hyperinflationary" equilibria in Example 1, and other reasons will be offered below. But none of these arguments suffice to dispose of the problem of indeterminate equilibria. In fact, equilibrium may be indeterminate in models without fiat money; monetary equilibrium may be indeterminate even when real balances remain bounded away from zero; equilibrium may be indeterminate in models with production or with storage; and equilibrium may be indeterminate in models with land or infinite lived agents. All of these propositions will be established by the examples to follow.

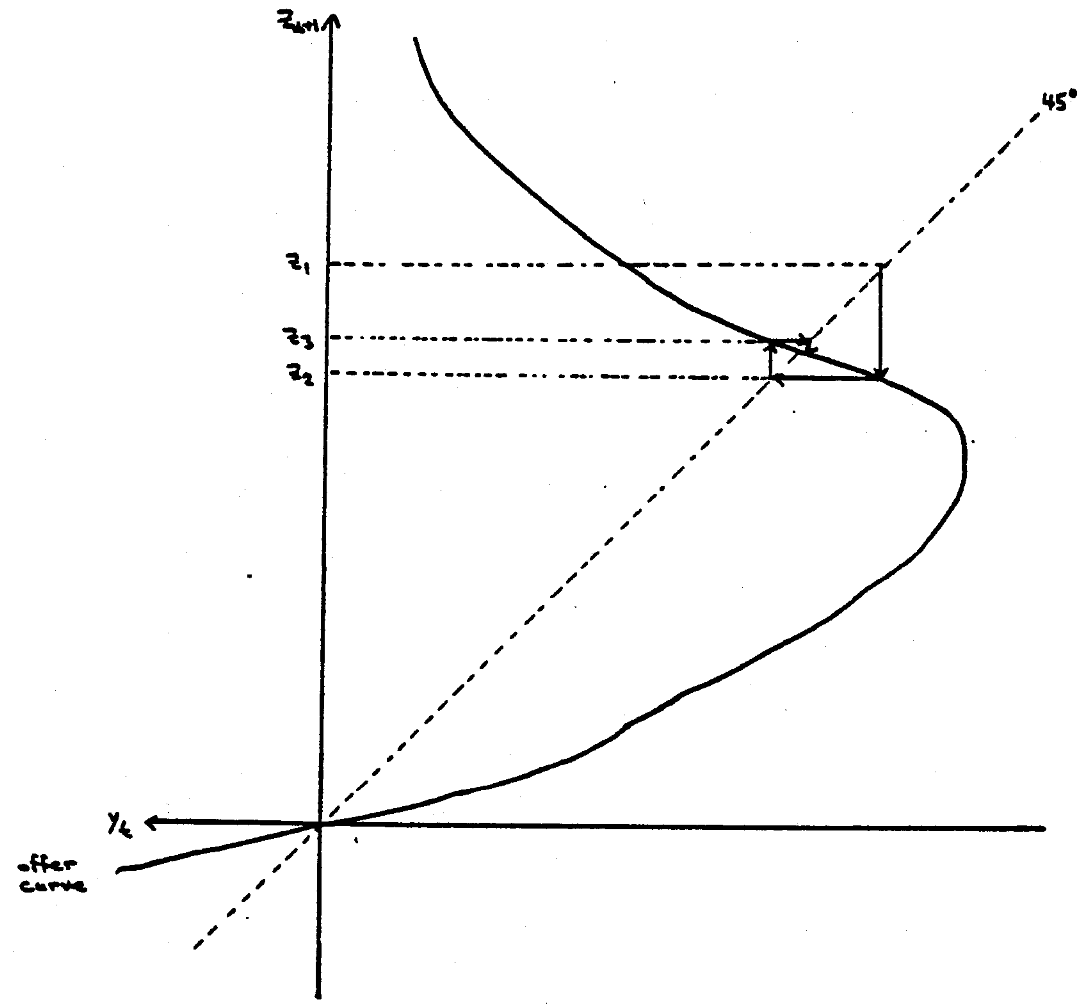
Example 2: This example was first discussed by Cass, Okuno, and Zilcha [1979]; the equilibria of this model are discussed in great detail in Grandmont [1983b]. The economy is identical to that of Example 1, except that the assumption of gross substitutes is dropped.

If one continues to assume that both first and second period consumption are normal goods and that all agents have identical preferences, then z_{t+1} must be an increasing function of p_t/p_{t+1} in the first quadrant, but y_t need not be a monotonically decreasing function in that quadrant. Hence one may have a backward-bending offer curve of the sort shown in Figure 2. Let us assume further that the offer curve is so sharply backward-bending that at the point where it crosses the 45° line (the monetary steady state), its slope is between zero and minus one.

Perfect foresight equilibria are defined as before. As shown in Figure 2, if the offer curve bends back sharply enough, the monetary steady state is indeterminate. That is, there will exist an entire interval of values for z_1 (including the monetary steady state value, z^*) which may be continued into perfect foresight equilibria that converge to z^* in the long run.

This example answers some of the objections to Example 1. For example, the indeterminacy of monetary equilibrium has nothing to do with money losing its value asymptotically; in all of these equilibria, real balances remain bounded away from zero. Hence the Brock-Scheinkman suggestion, that a certain amount of real balances be required each period for payment of taxes (which are then given back as transfers), would not rule out the indeterminacy in this case. Nor would the existence of storage, or some other arbitrage possibility placing a lower bound on the real rate of return, rule out the existence of a continuum of equilibria converging to the monetary steady state, as long as that lower bound satisfies $s < 1$. And the set of equilibria

Figure 2



converging to the monetary steady state are all Pareto optimal, as is the steady state itself. Hence this example shows that there is no general connection between indeterminacy and inefficiency. (It will also be seen that Pareto optimal non-monetary equilibria can be indeterminate. See Example 7 in Section II.) This should suggest that the sorts of modifications which suffice to rule out inefficient competitive equilibria will not rule out indeterminacy. The next pair of examples shows that this is the case.

Example 3: Let us suppose that in a stationary overlapping generations exchange economy of the sort considered in Example 1 there exists a certain amount of "land". By "land" we mean an asset that yields a constant stream of the consumption good, forever.[4/] Let us suppose that this asset is initially owned by the members of generation zero, and let $a > 0$ be the yield of the land each period. The members of generation zero consume the yield of the land in period one, and then sell the land to the young members of generation one for part of their first period endowment. The members of generation one hold the land as a means of saving; they consume the yield in period two, then sell the land to the young members of generation two. Thus the land passes from each generation to the next, exactly as did the fiat money in Examples 1 and 2. There is assumed to be no fiat money in this economy. In fact, if the members of generation zero also held a stock of fiat money, there would exist no perfect foresight equilibria in which the money was valued.[5/]

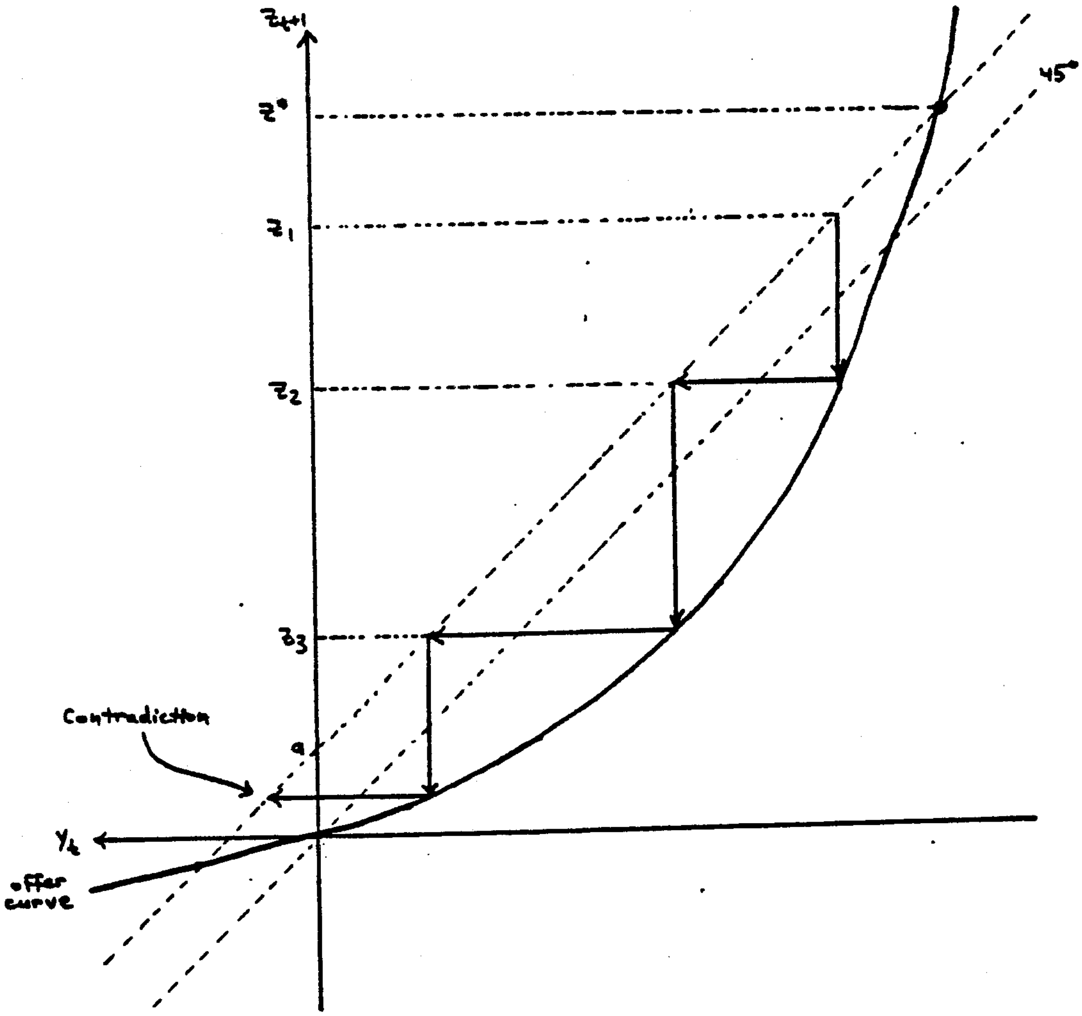
A perfect foresight equilibrium for this economy is a sequence

$\{z_t\}$ of excess demands by the old in each period, satisfying $z_t > a$ for all t, and such that for each t, $(a - z_t, z_{t+1})$ is a point on the offer curve. This is because the goods market equilibrium condition in period t is $y_t + z_t = a$. In each period, $-y_t$ will be the value of the land at the end of the period, so the requirement $z_t > a$ is just a requirement that the land always trade at a non-negative price. Hence the equilibria of this economy can be constructed using the offer curve diagram as above, only the 45° line must be replaced by a vertical translation (of distance a) of the 45° line, as shown in Figure 3.

Steady state equilibria of this economy will be points of intersection of the offer curve and the translated 45° line, with $z > a$. In Figure 3, there is only one such intersection, the point labeled z^*. It is also easily verified that this steady state is the only equilibrium. If one were to start with a value $z_1 < z^*$, as was possible in Example 1, one finds that the sequence $\{z_t\}$ necessary for $(a - z_t, z_{t+1})$ to be always a point on the offer curve leads in a finite number of steps to a value of z_t less than a, as shown in Figure 3.

Thus if one perturbs the economy in Example 1 by adding even a tiny amount of land, the entire uncountably infinite set of inefficient equilibria disappear, while there continues to be an equilibrium allocation close to the monetary steady state of Example 1. (In the new steady state, the title to land plays the role of fiat money in Example 1.) This result gives one a reason to consider the "hyperinflationary" equilibria in Example 1 not to be a robust property of the model.

Figure 3

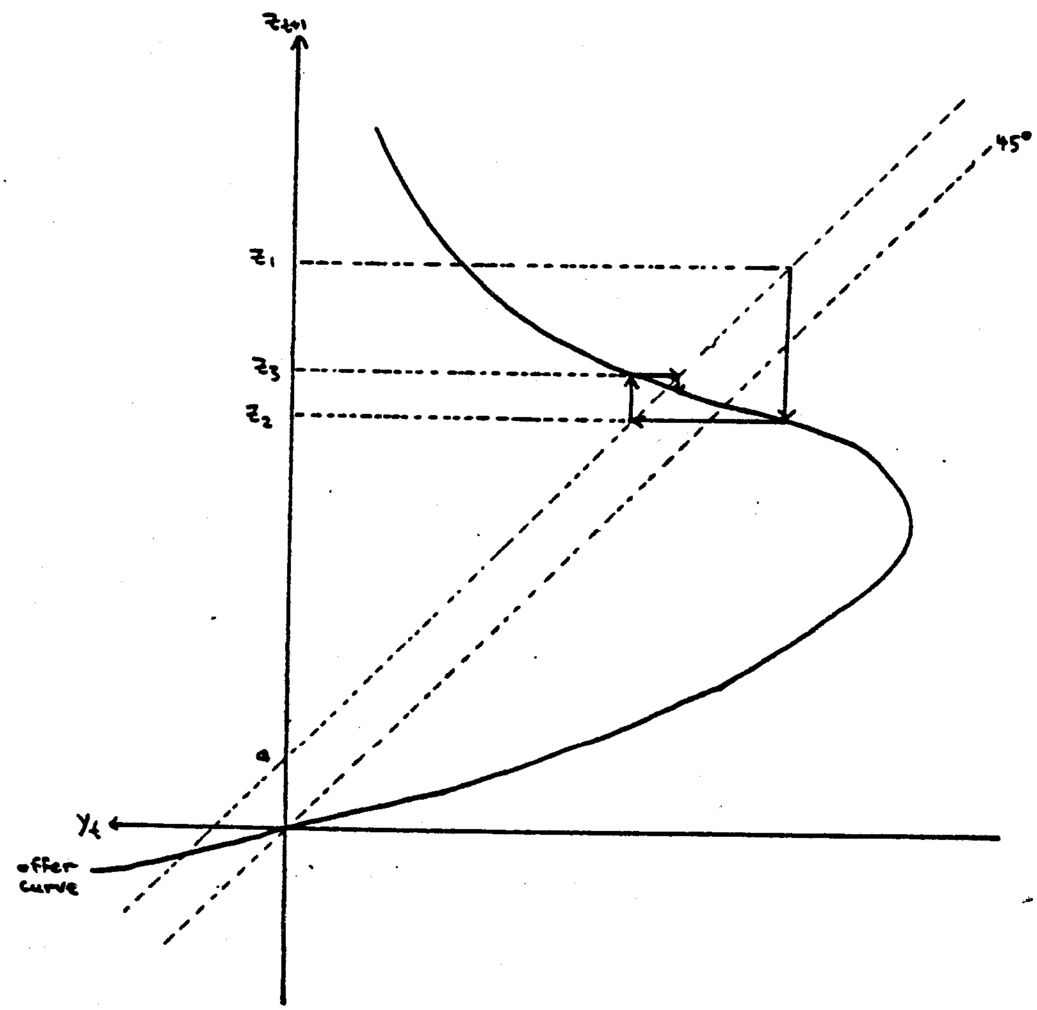


Indeed, none of the inefficient equilibria of stationary overlapping generations models are robust, under this criterion.[6] However, the next example shows that indeterminacy is still possible in the presence of land.

Example 4: In this example, there exists "land" yielding a units of the consumption good each period, as in Example 3, but agents' preferences are as in Example 2. Perfect foresight equilibrium is defined as in Example 3. As shown in Figure 4, there will in this case exist an uncountably infinite set of perfect foresight equilibria, all converging asymptotically to the unique steady state. Hence indeterminacy is possible even when land exists. This example also shows that indeterminacy is possible in a non-monetary economy. (However, this result may still suggest that indeterminacy is intrinsically related to the problem of pricing a non-depreciating asset. For a demonstration that indeterminacy is possible in a non-monetary economy when all goods are perishable, see Example 7 in Section II.)

This result should lead the reader to suspect that the mere presence of a small number of infinite lived agents need not rule out indeterminacy either. For Example 4 may be interpreted as an economy in which there exists an infinite lived agent. Suppose that instead of there existing "land" initially in the possession of generation zero, there exists an infinite lived agent with an endowment of a each period. Suppose further that the preferences of the infinite lived agent are such that he desires consumption only in period one. Then the infinite lived consumer will consume his endowment in period one, and

Figure 4



sell the rights to his entire future endowment stream to the young of generation one. These agents consume the infinite lived agent's endowment in period two, and sell the residual rights to the young of generation two. Thus the rights to the endowment stream of the infinite lived agent are traded in exactly the same manner as the land in the above interpretation. The definition of perfect foresight equilibrium will be the same, except that now z_1 will represent, not the excess demand of the members of generation zero, but instead the period one consumption of the infinite lived agent. And thus, as shown in Figure 4, perfect foresight equilibrium is indeterminate.

Note that the addition of an infinite lived agent, even with a very small endowment, has a far from trivial effect on the set of equilibria. All inefficient equilibria are ruled out (assuming that the endowment of the infinite lived agent is stationary), for the real rate of return cannot be persistently non-positive in the long run if the budget set of the infinite lived agent is to be defined. But this does not affect the possibility of indeterminacy. Once this is seen, it should be evident that even if the infinite lived agent desires consumption in all periods, as long as he discounts future consumption at a sufficiently high rate the perfect foresight dynamics will be close to those of Example 4, and hence equilibrium will be indeterminate. (For a demonstration that indeterminacy is possible even when the rate of time preference of the infinite lived agent is very low, and when the consumption of the infinite lived agent in equilibrium remains bounded away from zero forever, see Example 8 in Section II.)

Example 5: This example is due to Sargent [1984].[7] Suppose that agents, commodities, preferences and endowments are as in Example 1. Suppose that in addition there exists a government that is able to finance its expenditures (within limits) by printing new fiat money. Suppose that the government's policy is to consume a quantity $g > 0$ each period of the consumption good; the amount of new fiat money issued is whatever amount is necessary to purchase this quantity of consumption in each period.

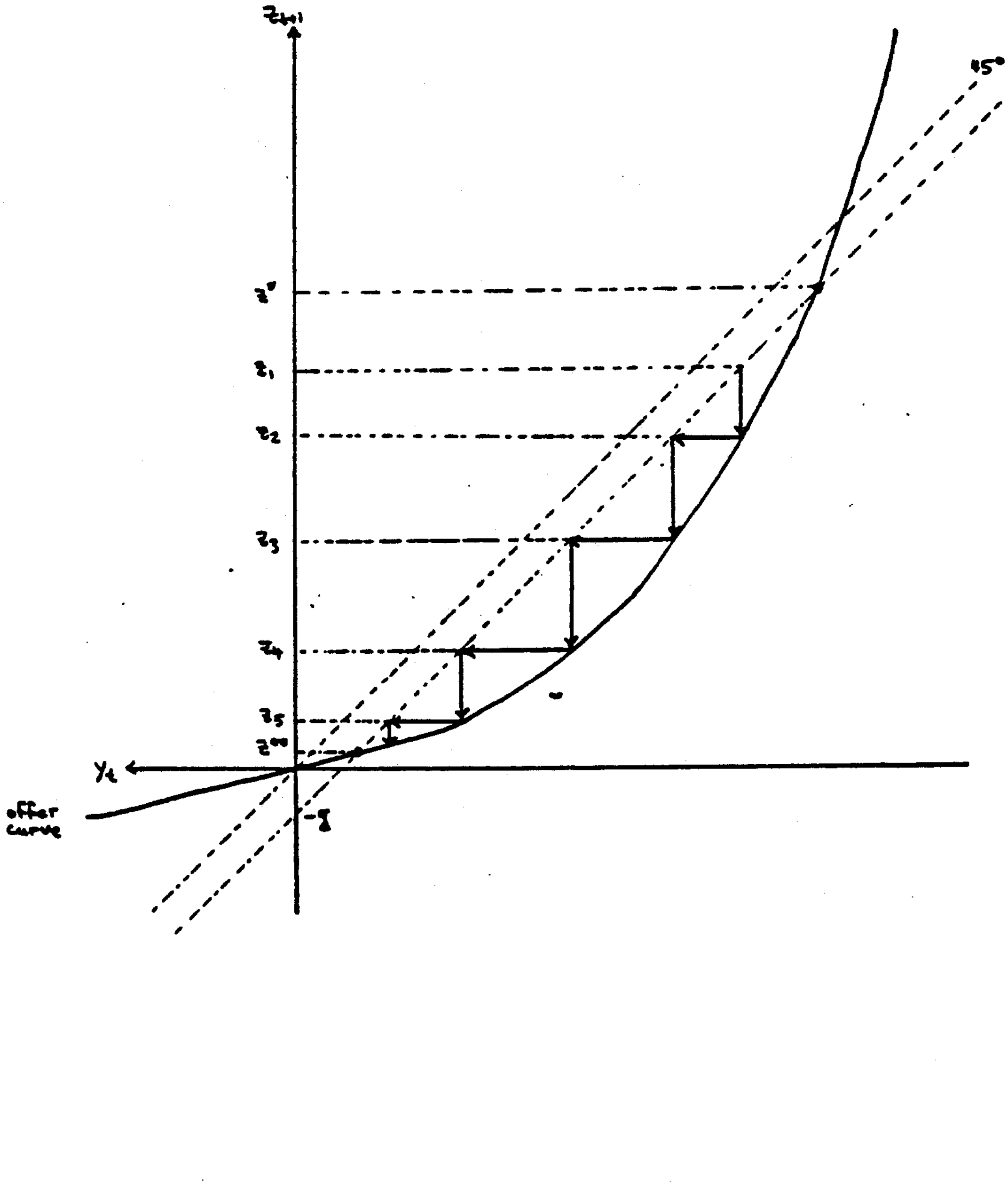
This economy is very similar to that considered in Example 3. For Example 3 may be reinterpreted in the following manner. Instead of the land being owned at all times by a finite lived agent, and held at the beginning of period one by the members of generation zero, let the land be owned for all time by the government, and let the members of generation zero hold a quantity M of fiat money at the beginning of period one. Let the government's policy be to sell the yield of the land each period for money, which it then retires from circulation. The equilibrium allocation is then exactly as in Example 3, with the fiat money functioning in the same way as the title to land, since in the present interpretation a given fraction of the existing fiat money gives one a claim to that fraction of the current and future yield of the government land.

This reinterpretation has the advantage that one immediately sees that a constant rate of government consumption financed by new money creation is like Example 3 with a negative quantity of land. Thus a perfect foresight equilibrium is a sequence $\{z_t\}$ with $z_t > 0$ for all t, such that for each t, $(-g - z_t, z_{t+1})$ is a point on the offer curve. The equilibria can be examined by means of the same diagram as before, but now the 45° line must be translated downward by a distance g, as shown in Figure 5.

Steady states are intersections of the offer curve with this translated 45° line, with $z > 0$. For a sufficiently small level of g, there will be two steady states, labeled z^* and z^{**} in Figure 5. These correspond to the two rates of money creation that yield the same level of government revenue, a situation familiar from the literature on seignorage revenue. (See, e.g., Bailey [1956], Calvo and Peel [1983].) The steady state with the lower level of real balances (z^{**}) corresponds to the higher rate of money growth and hence of inflation. There also exist an uncountably infinite set of non-stationary perfect foresight equilibria of this economy. For any z_1 between z^{**} and z^*, it is possible to construct an equilibrium, as shown in Figure 5. Note that for all $z_1 < z^*$, z_t must asymptotically approach z^{**}. Thus the higher-inflation steady state is indeterminate.[8/]

This example shows that it is not necessary that income effects be very strong compared to substitution effects, as in Example 2, in order for an indeterminate monetary equilibrium to occur.[9/] The conditions on preferences necessary for the indeterminacy to occur in this example are quite weak--if any monetary equilibrium is possible, then for sufficiently small $g > 0$ there must exist an indeterminate monetary steady state.

Figure 5



Example 6: This example is a variant of one presented by Geanakoplos and Polemarchakis [1983]. Thus far we have not considered any economies with production. It might be wondered in particular if the possibility of capital accumulation restricts the possible perfect foresight dynamics in some way that could guarantee determinacy. Furthermore, in all the examples of indeterminacy presented thus far, agents have had to price some nondepreciating asset (money or land). The present example shows that indeterminacy is also possible when the only asset consists of capital goods that are used up entirely in the following period's production.

Because a non-trivial model with production must involve more than one good per period, the simple diagrams used thus far cannot be extended to this case. Hence very special functional forms are chosen for the utility function and production function, which allow the complete set of perfect foresight equilibria to be solved explicitly. It might be wondered, then, if the results for this example do not represent merely a degenerate case. The considerations advanced in Section II, however, show that similar results would be obtained for any other smooth functions close to these ones, at least as far as the question of local indeterminacy is concerned.

Suppose that agents live two periods, work in the first period of life, and consume in the second. Let the utility of a member of generation t be

$$u(n_t, c_{t+1}) = \frac{c_{t+1}^{1-\gamma}}{1-\gamma} - n_t$$

where n_t is his labor supply, c_{t+1} his consumption, and $\gamma > 0$. The single consumption good is produced using a Cobb-Douglas production technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}$$

where k_t is the capital stock in period t and $0 < \alpha < 1$. Capital goods are produced from consumption goods set aside as investment the period before; one unit of consumption good foregone in period t produces one unit of capital for use in production in period $t+1$. There is no fiat money;[10] investment in the production of capital is the only means agents have of saving.

It follows from the Cobb-Douglas technology that the competitive rewards to factors in each period will be in the proportions

$$R_t k_t = \frac{\alpha}{1 - \alpha} w_t n_t$$

where R_t is the rental rate on capital and w_t is the wage, both in units of the consumption good. Furthermore, in each period the entire wage bill will be spent on investment, so that

$$k_{t+1} = w_t n_t$$

It follows then that

(1.1) $$k_{t+1} = \frac{1 - \alpha}{\alpha} R_t k_t$$

Since each generation's consumption is always given by

$$c_{t+1} = n_t w_t R_{t+1}$$

the first order condition for optimal labor supply is

$$(1.2) \qquad R_{t+1}^{1-\gamma} = n_t^{\gamma} w_t^{\gamma-1} = k_{t+1}^{\gamma} w_t^{-1}$$

The factor-price frontier for the Cobb-Douglas technology is

$$w_t = (1 - \alpha)\left(\frac{\alpha}{R_t}\right)^{\frac{\alpha}{1-\alpha}}$$

Substituting this and (1.1) into (1.2) yields

$$(1.3) \qquad R_{t+1}^{1-\gamma} = (1 - \alpha)^{\gamma-1}\left(\frac{R_t}{\alpha}\right)^{\gamma+\frac{\alpha}{1-\alpha}} k_t^{\gamma}$$

A perfect foresight equilibrium is then a sequence $\{k_t, R_t\}$, for $t = 1,2,\ldots$, with $k_t > 0$ and $R_t > 0$ for all t, with k_1 given as an initial condition, and such that (1.1) and (1.3) hold for all t. The above derivation shows that these conditions are necessary for a competitive equilibrium. It is easily seen that they are sufficient as well, since from any $\{k_t, R_t\}$ sequences it is possible to construct unique values for n_t, w_t, y_t, c_t consistent with equilibrium and then to verify that all markets are indeed in equilibrium.

The unique steady state for this economy is given by

$$R^* = \frac{\alpha}{1 - \alpha}$$

$$k^* = \alpha^{\frac{1-\gamma}{\gamma}}(1 - \alpha)^{1+\frac{\alpha}{\gamma(1-\alpha)}}$$

It is an equilibrium, of course, only if the initial capital stock is consistent with it, i.e., only if $k_1 = k^*$. Let us suppose that this is the case. Is the steady state the only equilibrium? No. In fact, it is clear that for any value of $R_1 > 0$ that one might choose, one can use (1.1) and (1.3) to calculate k_2 and R_2, and they will satisfy $k_2 > 0$, $R_2 > 0$. Hence the process may be repeated indefinitely, and thus there corresponds a distinct perfect foresight equilibrium to each possible value $R_1 > 0$. And this is not only true when $k_1 = k^*$. For any $k_1 > 0$, there is a distinct equilibrium corresponding to each possible $R_1 > 0$. Hence there always exists an uncountably infinite set of equilibria.

If, however, these equilibria all diverge from one another, none of them are indeterminate in the sense defined above. In order to determine the asymptotic behavior of the equilibria, a coordinate transformation is useful. Let

$$\tilde{k}_t \equiv \log k_t - \log k^*$$

$$\tilde{R}_t \equiv \log R_t - \log R^*$$

Then (1.1) and (1.3) become

$$(1.4) \qquad \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{R}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\gamma}{1-\gamma} & \frac{\gamma}{1-\gamma} + \frac{\alpha}{1-\alpha}\frac{1}{1-\gamma} \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{R}_t \end{bmatrix}$$

The asymptotic behavior of the equilibria obviously depends upon the eigenvalues of the matrix in (1.4). One eigenvalue is always between zero and one. For $0 < \gamma < 1$, the other eigenvalue is greater than 1; for $1 < \gamma < 2/(1 - \alpha)$, it is less than -1; and for $\gamma > 2/(1 - \alpha)$, it is between zero and -1. (Singular cases will not be discussed.) Hence, when $\gamma > 2/(1 - \alpha)$, the steady state is indeterminate, in the sense defined above. When $\gamma < 2/(1 - \alpha)$, the steady state is exactly determinate, in the sense that for any given initial capital stock k_1, there is a unique initial rental R_1 consistent with a perfect foresight equilibrium that converges asymptotically to the steady state. This steady state thus possesses the "saddle point" property often discussed in the rational expectations macroeconomics literature. Note, however, that this unique initial rental R_1 is not demanded by the assumption of perfect foresight equilibrium. There is still a perfect foresight equilibrium for each possible initial rental rate $R_1 > 0$, even when $\gamma < 2/(1 - \alpha)$. It is true that the divergent equilibrium paths eventually involve extreme behavior--arbitrarily large values of labor supply per capita, for example--but they are perfect foresight equilibria nonetheless.

This example illustrates several points. It shows that indeterminacy is possible in a production economy. It shows that determinacy has no necessary connection with efficiency. For the steady state of the economy is Pareto optimal if and only if $\alpha > 1/2$ (i.e., $R^* > 1$), while it is determinate if and only if $\gamma < 2/(1 - \alpha)$. Thus one can have a Pareto optimal steady state that is indeterminate, or an inefficient steady state that is determinate, or any of the other combinations of these attributes. Finally, it shows that indeterminate steady states, such as mainly have been discussed thus far and will be our exclusive concern in the following section, are not the only kind of large multiplicities of perfect foresight equilibria that may occur. Even when none of the steady states of a given economy are indeterminate, there may exist an uncountable infinity of perfect foresight equilibria, which may well be considered a problem for equilibrium analysis. However, the difficulties posed by an indeterminate steady state are of particular importance; it is shown in Section IV, for the example just discussed, that stationary "sunspot equilibria" exist exactly in the case that $\gamma > 2/(1 - \alpha)$.

II. Indeterminacy of Perfect Foresight Equilibrium: General Results

Let us now consider the problem of indeterminate perfect foresight equilibrium more generally. It is worth recalling first why competitive equilibrium is generically locally unique, in a Walrasian model with a finite number of agents and commodities, and smooth excess demand functions. In such a model, a competitive equilibrium is a price vector p^* satisfying $Z(p^*) = 0$, where $Z(p)$ is the vector of excess demand functions. (For the sake of simplicity, we consider a pure exchange economy.) If there are n commodities, then there are $n - 1$ independent equilibrium conditions (one is implied by the others, because of Walras' Law), for $n - 1$ relative prices (scalar multiplication of an equilibrium price vector gives one another equilibrium price vector, because of the homogeneity of the excess demand functions). If the regularity condition

$$\text{(2.1)} \qquad \text{Det } [DZ(p^*)] \neq 0$$

is satisfied, then there exists an open neighborhood of p^* in which no other equilibria exist, by the inverse function theorem. Intuitively, if one were to consider a small perturbation of one of the $n - 1$ relative prices, there would then be $n - 1$ independent equations for the necessary perturbations of the remaining $n - 2$ relative prices, which, for a small enough perturbation, certainly cannot all be satisfied. An economy such that (2.1) holds for all equilibria is called a "regular economy"; the property is known to be generic in the class of smooth excess demand functions.

The above argument plainly depends upon the existence of a finite number of commodites. (One cannot say that $n - 1$ equations is in general too many to be satisfied by $n - 2$ relative prices, unless n is finite.) However, a similar argument can be made, even with an infinite number of commodities, in the case that there are only a finite number of agents. As long as there are only a finite number of agents, the familiar proof of the first welfare theorem (involving addition of the budget constraints of the agents) is still possible. Then all competitive equilibria belong to the set of Pareto optimal allocations, which set can be parameterized by the $H - 1$ relative weights on the various agents' utility functions in the social welfare function (where H is the number of agents). The conditions for a competitive equilibrium can then be written as a set of $H - 1$ independent equations for the $H - 1$ relative weights, and the inverse function theorem can again be used to guarantee local uniqueness. (See Kehoe and

Levine [1982] for this argument in more detail.)

If, however, there is both an infinite number of commodities, and an infinite number of agents, as in any economy with an overlapping generations structure, no argument of this sort is possible. Instead one has the situation detailed in the following section.

1. The general n-good stationary exchange economy

The treatment to follow is based upon the work of Kehoe and Levine [1982, 1983a, 1983b]. Consider an overlapping generations exchange economy with n goods per period, and in which all agents live for two consecutive periods. (The assumption of two period lives is not restrictive, as long as the number of goods per period is arbitrary, for the reasons discussed by Balasko, Cass and Shell [1980].) Let the economy be stationary, in the sense that the same distribution of preferences and endowments is repeated in each generation. Let generation t, which trades and consumes in periods t and $t+1$, have an aggregate excess demand vector $y(p_t, p_{t+1})$ for the n goods in period t, and an aggregate excess demand vector $z(p_t, p_{t+1})$ for the n goods in period $t+1$, where p_t is the vector of consumption goods prices in period t. The excess demand functions are assumed to be smooth, homogeneous of degree zero in (p_t, p_{t+1}), and to satisfy Walras' Law:

$$p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) \equiv 0$$

Certain boundary conditions are also assumed, which have no bearing upon the analysis of determinacy with which we are here concerned. These functions describe the consumption choices of generations $t = 1,2,\ldots$. The members of generation zero consume only in period one, and have an aggregate excess demand vector $z_o(p_1)$ for the n goods in period one. In the case of a non-monetary economy, the excess demands of generation zero satisfy $p_1'z_0(p_1) \equiv 0$; but in the case of a monetary economy, the identity is instead $p_1'z_0(p_1) \equiv M$, where M is the total nominal money stock in the hands of the members of generation zero at the beginning of period one. A perfect foresight equilibrium is then a sequence $\{p_t\}$ of non-negative prices, for $t = 1,2,\ldots$, that satisfies the sequence of market clearing conditions

$$z_0(p_1) + y(p_1,p_2) = 0 \tag{2.2.1}$$

$$z(p_1,p_2) + y(p_2,p_3) = 0 \tag{2.2.2}$$

$$\cdots$$

$$z(p_{t-1},p_t) + y(p_t,p_{t+1}) = 0 \tag{2.2.t}$$

$$\cdots$$

It is evident that in the case of a sequence of equilibrium conditions like this, equilibrium need not be locally unique. Suppose that $\{p_t^*\}$ is an equilibrium. The question is whether, for $p_1 \neq p_1^*$ sufficiently close to p_1^*, there can exist a sequence of prices, beginning with p_1 and remaining always close to the $\{p_t^*\}$ sequence, that is also an equilibrium. Simply counting equations and unknowns does not indicate impossibility. Equations (2.2.1) give n conditions which should determine the prices p_2 consistent with p_1, equations (2.2.2) then give n conditions which should determine the prices p_3 consistent with p_1 and p_2, and so on. Whether prices p_1 are consistent with perfect foresight equilibrium depends upon whether or not the infinite sequence of equilibrium conditions can be solved in this manner. Examples 1 and 2 of Section I show that in some cases it is possible.

As noted in Section I, we are especially interested in the determinacy of perfect foresight equilibrium near a steady state equilibrium. A steady state is an equilibrium of the form $p_t = \beta^{t-1}p$, where p is some positive vector and $\beta > 0$ is a scalar factor by which prices are inflated or deflated as time progresses. The possibility of a scalar factor $\beta \neq 1$ is allowed because of the homogeneity of the excess demand functions; even when $\beta \neq 1$, the allocation of resources will be the same in all periods. Thus a steady state is a pair (p, β) which satisfy

$$z(p, \beta p) + y(p, \beta p) = 0 \tag{2.3}$$

It follows immediately from Walras's Law that any steady state will satisfy

$$(1 - \beta)p'z(p, \beta p) = 0$$

Therefore either $\beta = 1$, or $p'z(p, \beta p) = 0$, or both. It can be shown that, for a generic smooth economy, both equalities do not hold for any one steady state. (See Kehoe and Levine [1983b].) Hence we can treat separately steady states of two sorts: those in which $\beta = 1$, and $p'z(p,\beta p) \neq 0$, and those in which $\beta \neq 1$, and $p'z(p,\beta p) = 0$. Now $p'z(p,\beta p) = -p'y(p,\beta p)$ is just aggregate saving in each period in the steady state; it is clear that in a monetary equilibrium, $p_t'z(p_{t-1},p_t) = M$ in each period, while in a non-monetary equilibrium, $p_t'z(p_{t-1},p_t) = 0$ in each period. So the former steady states are <u>monetary</u> steady states, while the latter are <u>non-monetary</u> steady states. Only a steady state with $p'z(p,\beta p) > 0$ represents a steady state with valued fiat money, of course; but economic interpretations might be given for solutions to (2.3) with $p'z(p,\beta p) < 0$ as well (an infinite lived intermediary exists that allows net borrowing each period, of a constant nominal magnitude). Kehoe and Levine [1983b] prove that at least one monetary and at least one non-monetary steady state exist under standard assumptions on preferences; the monetary steady state may, however, be one with $p'z(p,\beta p) < 0$.

Suppose that (p,β) satisfies (2.3), and the initial conditions are consistent with this equilibrium, i.e., $z_o(p) = z(p,\beta p)$. Then $p_t = \beta^{t-1}p$, for $t = 1,2,\ldots$, is one perfect foresight equilibrium. We wish to know whether, for an arbitrarily small neighborhood N of p, there exist any other equilibria $\{p_t\}$ such that $\beta^{1-t}p_t \in N$ for all t. Assuming that

$$\text{Det } [D_2y(p,\beta p)] \neq 0 \tag{2.4}$$

where D_2y is the matrix of derivatives of $y(p_1,p_2)$ with respect to p_2, then the implicit function theorem guarantees, for each p_1

sufficiently close to p, the existence of a unique p_2 such that $\beta^{-1}p_2$ is near p and (2.2.1) is satisfied. The same regularity condition (2.4) suffices for application of the implicit function theorem again, to show the existence of a unique $p_3 = \beta^2\phi(\beta^{-1}p_2,p_1)$ such that for every $(\beta^{-1}p_2,p_1)$ sufficiently close to (p,p), ϕ is close to p and (2.2.2) is satisfied. Repeated application of the implicit function theorem in this manner will allow the construction of an entire sequence of equilibrium prices, beginning with any p_1 sufficiently close to p, if (2.4) holds, and the map ϕ has the property that at each stage in the construction, $\beta^{-t}p_{t+1} = \phi(\beta^{1-t}p_t,\beta^{2-t}p_{t-1})$ is sufficiently close to p for $(\beta^{-t}p_{t+1},\beta^{1-t}p_t)$ to be within the domain of the map ϕ (and the neighborhood $N \times N$) as well. In other words, if the sequence $\{\beta^{1-t}p_t\}$ constructed does not diverge from p, it is possible to continue the construction using the map ϕ, and a perfect foresight equilibrium exists corresponding to every vector p_1 sufficiently close to p. (Even if all $p_1 \neq p$ resulted in sequences that diverge from p, it might nonetheless be possible to continue the construction of a perfect foresight equilibrium, as in Example 6 for the case $\gamma < 2/(1 - \alpha)$. But as noted before, we are principally interested in perfect foresight equilibria that remain uniformly close to the steady state.)

Kehoe and Levine [1983b] show that the regularity condition (2.4) holds for generic smooth economies; hence the only question is whether the map ϕ results in the construction of convergent or divergent sequences $\{\beta^{1-t}p_t\}$. The answer to this depends upon the eigenvalues of the matrix[12/]

$$G = \begin{bmatrix} D_1\phi & D_2\phi \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -D_2y^{-1}(\beta^{-1}D_1y + D_2z) & -D_2y^{-1}(\beta^{-1}D_1z) \\ I & 0 \end{bmatrix}$$

which is just the derivative of the map which gives $(\beta^{-2}p_3, \beta^{-1}p_2)$ as a function of $(\beta^{-1}p_2, p_1)$. Here D_1z, D_2z represent the derivatives of $z(p_1,p_2)$ with respect to p_1, p_2 respectively, and likewise for the derivatives of $y(p_1,p_2)$; all derivatives are evaluated at $(p,\beta p)$. It follows from the stable manifold theorem for diffeomorphisms (Irwin [1980], Theorem 6.17) that if G has n^s eigenvalues inside the unit circle, there is an n^s-dimensional local stable manifold. This is a set of values of $(\beta^{-1}p_2, p_1)$, including (p,p), such that repeated application of the map ϕ to one of these initial values results in a sequence of prices such that $(\beta^{-t}p_{t+1}, \beta^{1-t}p_t)$ belongs to the stable manifold for all t, and converges to (p,p) asymptotically. If $n^s = 2n$, the stable manifold must include a neighborhood of (p,p), so that this is a sufficient condition for the construction described above to work for all $(\beta^{-1}p_2, p_1)$ sufficiently close to (p,p), and hence for all p_1 sufficiently close to p. In such a case, we say that the <u>dimension of indeterminacy</u> is n. The set of p_1 values near p consistent with perfect foresight equilibrium converging asymptotically to the steady state has the structure of a local n-dimensional manifold;

i.e., there are n independent possible directions of variation of p_1, all consistent with extension to a convergent perfect foresight equilibrium. However, even if $n^s < 2n$, the steady state may be indeterminate. For it may be possible to continue the construction indefinitely for <u>some</u> $p_1 \neq p$ arbitrarily close to p, though not for all p_1 arbitrarily close to p. All initial values $(\beta^{-1}p_2, p_1)$ that belong to the stable manifold and are consistent with the initial conditions (2.2.1) correspond to convergent equilibria. The set of such initial values will thus be the intersection of the stable manifold with the manifold defined by (2.2.1), and will be itself a manifold; the dimension of this manifold we call the dimension of indeterminacy. We turn now to the question of the constraints placed by economic theory on the dimension of the stable manifold.

The homogeneity degree zero of the demand functions implies that ϕ is homogeneous degree one. Therefore

$$G \begin{matrix} p \\ p \end{matrix} = \begin{matrix} p \\ p \end{matrix}$$

and one eigenvalue of G must be exactly one. Kehoe and Levine prove that, generically, no other eigenvalues have modulus exactly equal to one. Thus G has n^s stable eigenvalues, one eigenvalue exactly equal to one, and $2n - n^s - 1$ unstable eigenvalues. The homogeneity of (2.2.t) implies that for any $(\beta^{-1}p_2, p_1)$ belonging to the stable manifold, $(\alpha\beta^{-1}p_2, \alpha p_1)$ will represent initial values which, under repeated application of the map ϕ, extend to a perfect foresight equilibrium converging to $(\alpha p, \alpha p)$, for any positive scalar α. There is therefore an $n^s + 1$ dimensional local manifold of initial values $(\beta^{-1}p_2, p_1)$ which do not diverge from (p,p); all initial values near (p,p) not on this manifold diverge from (p,p) eventually. Note also that $(\alpha p, \alpha p)$, for some positive scalar α, represents the same steady state allocation as (p,p). It represents simply an alternative normalization of prices in the case of a non-monetary steady state, or a different nominal money stock in the case of a monetary steady state.

Another property of the map ϕ is that along all perfect foresight equilibrium paths, $p'_{t+1}z(p_t, p_{t+1})$ is the same in all periods; it is equal to the nominal money stock, as noted previously. This implies the existence of a left eigenvector for G: $f'G = (1/\beta)f'$, where f' is the derivative of the functional $F(\beta^{-t}p_{t+1}, \beta^{1-t}p_t) \equiv \beta^{-t}p'_{t+1}z(p_t, p_{t+1})$. Hence one eigenvalue of G is always $1/\beta$. Kehoe and Levine show that this proposition, and the existence of an eigenvalue equal to one, exhaust the restrictions on the eigenvalues of G that follow from the assumption that the excess demands derive from utility maximization. (The proof relies upon the Sonnenschein-Debreu-Mantel result that essentially any smooth function satisfying Walras' Law and the homogeneity requirement can be derived as an excess demand function for some finite set of agents with well-behaved preferences.) Hence the only propositions about the eigenvalues of G that hold generally are:

- In the case of a non-monetary steady state, one eigenvalue is 1 and another is $1/\beta$.

- In the case of a Pareto optimal non-monetary steady state (i.e., one with $\beta < 1$)[13/], $0 \leq n^s \leq 2n - 2$.
- In the case of an inefficient non-monetary steady state ($\beta > 1$), $1 \leq n^s \leq 2n - 1$.
- In the case of a monetary steady state, one eigenvalue is 1, so that $0 \leq n^s \leq 2n - 1$.

Kehoe and Levine [1983b] show that open sets of smooth economies exist for which n^s takes any of the values consistent with the above inequalities.

It must next be determined how many of the initial values belonging to the non-divergent manifold are consistent with the period one equilibrium conditions (2.2.1). Consider first the case of a monetary steady state. All initial values consistent with (2.2.1) must be such that $F(\beta^{-1}p_2,p_1) = \beta^{-1}M$; other initial values would correspond to nominal money stocks other than the quantity M in existence at the beginning of period one. Thus all initial values consistent with (2.2.1) must correspond to price sequences that converge to (p,p), rather than to $(\alpha p,\alpha p)$ for some $\alpha \neq 1$; therefore they must belong to the n^s-dimensional stable manifold. Equation (2.2.1) places $n - 1$ additional restrictions upon the initial values, besides the one stated above, which hold for all points of the stable manifold. Since there need be, in general, no relations of dependence between these initial conditions and the equations that define the stable manifold, the dimension of the local manifold of initial values corresponding to perfect foresight equilibria will be $\max(0,n^s-n+1)$. Since n^s can take any value in the range $0 < n^s < 2n - 1$, the dimension of indeterminacy can be as large as n. If $n^s = n - 1$, the steady state is exactly determinate. This means not only that perfect foresight equilibrium is locally unique, for the initial conditions consistent with the steady state equilibrium, but that a unique equilibrium converging to the steady state will continue to exist under perturbations of the initial conditions. (An instance of exact determinacy is Example 6 in the case that $\gamma < 2/(1 - \alpha)$; for each value of k_1, not just $k_1 = k^*$, there exists a unique equilibrium converging to the steady state.) If $n^s < n - 1$, the steady state is unstable, in the sense that for almost all small perturbations of the initial conditions there ceases to exist any equilibrium converging to the steady state.

Consider next the case of a Pareto optimal $(\beta < 1)$ non-monetary steady state. In this case (2.2.1) is homogeneous degree zero in $(\beta^{-1}p_2,p_1)$, as are the conditions defining the non-divergent manifold. Thus for every $(\beta^{-1}p_2,p_1)$ that corresponds to an equilibrium converging to (p,p), $(\alpha\beta^{-1}p_2,\alpha p_1)$ will correspond to an equilibrium converging to $(\alpha p,\alpha p)$, for any $\alpha > 0$. These will be the same equilibria in all respects except the normalization of prices. Therefore we may choose, as a price level normalization, to consider only perfect foresight equilibria converging to (p,p). Then we again restrict our attention to the n^s-dimensional stable manifold. One of the n restrictions implied by (2.2.1) is that $F(\beta^{-1}p_2,p_1) = 0$. However, this condition holds for all points of the stable manifold. This is because the tangent space of the stable manifold at the steady state is a linear space spanned by a set of right eigenvectors v such that $f'v = 0$. (Note that $f'v = 0$ for all right eigenvectors except the eigenvector whose eigenvalue is $1/\beta$.[14] The latter eigenvector is not part of the subspace in question, because $1/\beta > 1$.) In addition to this, (2.2.1) implies $n - 1$ further restrictions, which will in general be independent of the conditions that define the stable manifold. Again the dimension of indeterminacy (neglecting price level renormalizations) is $\max(0,n^s-n+1)$. But since in this case n^s only takes values in the range $0 \leq n^s \leq 2n - 2$, the maximum possible dimension of indeterminacy is $n - 1$. Note that this implies that a Pareto optimal non-monetary steady state cannot be indeterminate, in the case of a one-good model.

Consider finally the case of an inefficient $(\beta > 1)$ non-monetary steady state. As above, one may restrict one's attention to the n^s-dimensional stable manifold, and equation (2.2.1) constitutes $n - 1$ additional restrictions, besides the requirement that $F(\beta^{-1}p_2,p_1) = 0$. But in this case one of the stable eigenvalues of G is $1/\beta$. The n^s-dimensional tangent space to the stable manifold at the steady state can be decomposed into an $n^s - 1$ dimensional subspace, spanned by the $n^s - 1$ stable right eigenvectors of G other than the eigenvector corresponding to the eigenvalue $1/\beta$, and a one-dimensional subspace spanned by the right eigenvector whose eigenvalue is $1/\beta$. Let V be the invariant submanifold (under the map ϕ) whose tangent space of the steady state is the $n^s - 1$ dimensional subspace just defined. Then the only points in the stable manifold consistent with (2.2.1) will have to be points of the submanifold V, since (2.2.1) implies $F(\beta^{-1}p_2, p_1) = 0$; other points of the stable manifold, with $F(\beta^{-1}p_2, p_1) \neq 0$, imply the existence of valued fiat money. Equation (2.2.1) places $n - 1$ further restrictions upon the points of V, so that the dimension of indeterminacy of non-monetary equilibrium is in this case $\max(0, n^s - n)$. Since n^s may take any value in the range $1 < n^s < 2n - 1$, the dimension of indeterminacy of non-monetary equilibrium can be as large as $n - 1$.

The above considerations assume that in the case of a non-monetary steady state, the initial conditions (2.2.1) involve excess demands $z_0(p_1)$ satisfying $p_1' z_0(p_1) \equiv 0$, i.e., there is no valued fiat money. But one might also consider initial conditions in which $p_1' z_0(p_1) = M > 0$. That is, one might assume that the members of generation zero start out with a positive quantity of fiat money, and ask if there exist equilibria which nonetheless approach the non-monetary steady state as the money loses its value. Example 1 of the previous section shows that this is possible. It is clear, however, that this could only happen in the case of an inefficient $(\beta > 1)$ non-monetary steady state. This is because $p_{t+1}' z(p_t, p_{t+1})$ remains constant over time, so that $F(\beta^{-t}p_{t+1}, \beta^{1-t}p_t)$ must explode at a rate $1/\beta > 1$ in a steady state with $\beta < 1$. But if $\beta > 1$ in the steady state, $F(\beta^{-t}p_{t+1}, \beta^{1-t}p_t)$ must shrink at a rate $1/\beta < 1$, so that it is possible for F to approach zero as the equilibrium approaches the steady state, even though $\beta^t F = M > 0$ for all time. If we assume initial conditions of this sort, then the equilibrium conditions are no longer homogeneous in prices, so that we must consider the entire non-divergent manifold rather than simply the stable manifold. Equation (2.2.1) then constitutes n restrictions, so that one obtains in this case a dimension of indeterminacy of $\max(0,n^s-n+1)$, of which one dimension (if one or more exist) will correspond to monetary equilibria that converge asymptotically to the non-monetary steady state.

To summarize, Kehoe and Levine show that in the case of a stationary exchange economy with n goods per period:

- a manifold of dimension up to n of monetary equilibria may converge to a monetary steady state;
- a manifold of dimension up to $n-1$ of non-monetary equilibria may converge to a Pareto optimal non-monetary steady state;
- a manifold of dimension up to n of equilibria may converge to an inefficient non-monetary steady state; if such a steady state has dimension of indeterminacy $d > 1$, then a $d-1$ dimensional submanifold of the convergent equilibria are non-monetary equilibria, while the rest are monetary equilibria in which money asymptotically loses its value.

These results show to what extent the results of Gale [1973] generalize to an n-good economy. They show that the one-good case is very special: in that case one never has local non-uniqueness of non-monetary equilibrium, and a non-monetary steady state is indeterminate if and only if it is inefficient. But for $n > 1$, it is possible to have a continuum of non-monetary equilibria converging to a non-monetary steady state, whether it is efficient or not. In particular, when $n > 1$, it is possible to have a Pareto optimal non-monetary steady state that is nonetheless indeterminate. This is illustrated by the following example.

Example 7: (From Kehoe and Levine [1983a].) Let there be one good per period, but let agents live three periods. As shown by Balasko, Cass and Shell [1980], this is equivalent to a model with two goods per period and two-period lives; hence it is possible for equilibrium to be indeterminate in the absence of fiat money. Let all agents in each generation have preferences described by a constant elasticity of substitution utility function $u(c_1,c_2,c_3) = 2c_1^{-4} + 2c_2^{-4} + c_3^{-4}$, where c_1, c_2, and c_3 are consumption in the first, second, and third periods of life respectively. Let the endowment pattern be 3 units of the good in the first period of life, 15 in the second, and 2 in the third. The non-monetary steady states of this model are not autarchic. Agents borrow in the first period of life from middle-aged members of the generation before their own, repaying in the second period (when their creditors are old); they lend in the second period of life to young members of the generation after their own, and are repaid in the third period (when their debtors are middle-aged). It is easily verified that this economy has three non-monetary steady states, corresponding to $\beta = .04$, $\beta = .93$, and $\beta = 53.81$, and one monetary steady state $(\beta = 1)$. The $\beta = .93$ non-monetary steady state is Pareto optimal, since $\beta < 1$. Evaluated at this steady state, the eigenvalues of the matrix G are 1, $1/\beta$, and a complex pair with modulus less than one. It follows that $n^s = 2$, and there is one dimension of indeterminacy.

2. Land and Infinite Lived Agents

A similar analysis is possible in the case that land exists, as is shown by Muller and Woodford [1983, 1984]. Suppose that land exists yielding a vector a of perishable consumption goods each period. Let p_t be the vector of consumption goods prices in period t, and q_t the vector of land prices; q_t is an n-vector, one price for each type of land, on the supposition that the claims to flows of the different consumption goods may be separately traded. Then a competitive equilibrium is a sequence $\{p_t, q_t\}$, for $t = 1,2,\ldots$, such that

(2.5.1) $$z_0(q_1, p_1) + y(p_1, p_2) = a$$

(2.5.2) $$z(p_1, p_2) + y(p_2, p_3) = a$$

$$\cdots$$

(2.5.t) $$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = a$$

$$\cdots$$

(2.6) $$q_t = p_t + q_{t+1} \quad , \quad t = 1,2,\ldots$$

Note that q_1 enters as an argument of the excess demand function of generation zero, because the members of generation zero own the land at the beginning of period one. Walras' Law for the generation zero demands is in this case $p_1' z_0(q_1, p_1) \equiv q_1' a$. Land prices are not arguments of the excess demand functions of any other generations, as members of subsequent generations are neither net suppliers nor net consumers of land; the land they purchase in one period they sell the next.

A steady state of this economy is an equilibrium of the form $p_t = \beta^{t-1}p$, $q_t = \beta^{t-1}q$, for some positive vectors p and q and some positive scalar β. Thus a steady state is a triple (p,q,β) satisfying

(2.7) $$z(p,\beta p) + y(p,\beta p) = a$$

(2.8) $$q(1 - \beta) = p$$

For both p and q to be positive, it is plainly necessary that $\beta < 1$. So a steady state exists if and only if there exist (p,β) satisfying (2.7) with $\beta < 1$. Muller and Woodford [1983] prove the existence of a steady state under standard assumptions on preferences.

The determinacy or indeterminacy of a steady state with land depends upon the dimension of its stable manifold, i.e., the set of $(\beta^{-1}p_2,p_1,q_1)$ that may be extended to a perfect foresight equilibrium $(\beta^{-t}p_{t+1},\beta^{1-t}p_t,\beta^{1-t}q_t)$ converging to (p,p,q). Since (2.5.t) does not involve land prices at all, the question of which $(\beta^{-1}p_2,p_1)$ may be extended to a perfect foresight equilibrium $(\beta^{-t}p_{t+1},\beta^{1-t}p_t)$ converging to (p,p) can be answered without reference to the land prices. And for any $\{p_t\}$ sequence with that property, there is a unique $\{q_t\}$ sequence consistent with it, since (2.6) implies

$$q_t = \sum_{s=t}^{\infty} p_s$$

Hence the dimension of the stable manifold is equal to n^s, the number of eigenvalues within the unit circle possessed by the matrix G, defined as before. As before, the equilibrium conditions are homogeneous in prices, so that one eigenvalue of G is one, and the non-divergent manifold is of dimension $n^s + 1$. The initial conditions (2.5.1) are also homogeneous degree zero in $(\beta^{-1}p_2, p_1, q_1)$, so that for any $\{p_t, q_t\}$ constituting a perfect foresight equilibrium, $\{\alpha p_t, \alpha q_t\}$ constitutes one as well, for any positive α. Since $\{\alpha p_t, \alpha q_t\}$ represents the same allocation as $\{p_t, q_t\}$, the additional dimension of the non-convergent manifold may be suppressed by a price level normalization choice. Initial conditions (2.5.1) provide only $n - 1$ additional restrictions upon the values of $(\beta^{-1}p_2, p_1, q_1)$, since any $(\beta^{-1}p_2, p_1, q_1)$ belonging to the stable manifold must satisfy

$$p_1'[z_0(q_1, p_1) + y(p_1, p_2) - a] = (q_1' - p_1')a + p_1'y(p_1, p_2)$$

$$= q_2'a - p_2'z(p_1, p_2) = 0$$

(Equation (2.5.t) implies that $q_t'a - p_t'z(p_{t-1}, p_t)$ has the same value in all periods $t \geq 2$, and since $q'a - p'z(p, \beta p) = 0$ in a steady state, $q_2'a - p_2'z(p_1, p_2) = 0$ for any perfect foresight equilibrium converging to a steady state.[15/]) Hence, in the generic case, the dimension of indeterminacy will be $\max(0, n^s - n + 1)$.

Equilibrium conditions (2.2.t) are just a special case of (2.5.t), with $a = 0$. Hence for a small enough, the eigenvalues of G evaluated at the steady state with land will be close to the eigenvalues of G evaluated at the corresponding steady state of the economy without land. Hence the range of possible values of n^s must be as great as in the case of Pareto optimal non-monetary steady states without land or monetary steady states. So the range is $0 < n^s < 2n - 1$; it follows that the degree of indeterminacy may be as large as n.

Examples of indeterminate perfect foresight equilibrium with land can be constructed by adding a sufficiently small amount of land to an exchange economy with either a Pareto optimal non-monetary steady state, or a monetary steady state with positively valued fiat money. (A steady state with land cannot be obtained through a small perturbation of a non-monetary steady state with $\beta > 1$, since any small perturbation of such a steady state must still have $\beta > 1$; nor through a small perturbation of a monetary steady state with $p'z(p,\beta p) < 0$, since any small perturbation of such a steady state must still have a negative supply of savings.) An example of the latter possibility is the way Example 4 was constructed by adding land to Example 2. An example of the former possibility would be the economy of Example 7, with a sufficiently small amount of land added.

Muller and Woodford also show how a similar analysis is possible in the case of economies with an arbitrary number of smooth constant return to scale production technologies which allow the transfer of resources from one period to the next. Since the results are essentially identical to those in the case of an exchange economy,[16/] this extension will not be treated here. Of somewhat more interest is an extension of the analysis to allow for infinite lived as well as finite lived agents. We will consider here only the case of a single infinite lived agent with a stationary endowment, who consumes a constant non-zero vector of goods in the steady state under consideration. (As noted in Section I, the case of a steady state in which the infinite lived agent does not consume, because his rate of time discount is higher than the real rate of return in the steady state, is essentially the same as a steady state with land.) Complications arising from the existence of several infinite lived agents, with either the same or different rates of time discount, are treated by Muller and Woodford.

Let the infinite lived agent's preferences be stationary and additively separable over time; i.e., let him maximize a utility function of the form

$$V = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

where c_t is his consumption vector in period t, $0 < \beta < 1$ is a discount factor, and $u(c)$ is a monotone, strictly concave, twice continuously differentiable function satisfying standard boundary conditions. Let the infinite lived agent have a positive endowment vector a each period. His lifetime budget constraint is then

$$\sum_{t=1}^{\infty} p_t' c_t < \sum_{t=1}^{\infty} p_t' a + p_1' w$$

where w represents the vector of goods owed the infinite lived agent by the members of generation zero, at the beginning of period one. (Negative elements of w represent debts of the infinite lived agent at the beginning of period one. One supposes that period one does not represent the first period of life for either the infinite lived agent or the members of generation zero; since the finite lived agents will generally borrow from or lend to the infinite lived agent in an equilibrium, it is reasonable to suppose that some such debts will already exist at the beginning of period one. It is assumed that debts are specified in terms of a vector of goods, rather than a nominal aggregate, so as to eliminate the possibility of a trivial indeterminacy having to do with the arbitrary real value of initial debts specified in nominal terms.)

Let us assume that a steady state exists in which the infinite lived agent consumes a constant positive vector of goods c each period.[17/] In such a steady state, equilibrium prices must be of the form $p_t = \beta^{t-1}p$, for some positive vector p, where β is the discount factor of the infinite lived agent. Furthermore, the vector p must be such that

$$Du(c) = \lambda p \tag{2.9}$$

for some positive scalar λ. Finally, in each period, supply of the good must equal demand:

$$z(p,\beta p) + y(p,\beta p) + c = a \tag{2.10}$$

Thus a steady state is a triple (p,c,λ) such that (2.9) and (2.10) are satisfied. (Note that λ may be chosen arbitrarily; different choices of λ simply amount to alternative price level normalizations. We do not suppress the λ, however, for the sake of the treatment below of equilibrium outside of a steady state.)

It might be thought that some additional restriction is necessary, relating the value of steady state consumption by the infinite lived agent to the size of his endowment. This is not so. By the time the economy has settled into a steady state, the infinite lived agent may have accumulated debts that he rolls over every period (borrowing from the young in each period to repay the old); or he may have accumulated wealth in addition to his endowment (claims against the old in each period, part of the proceeds of which he lends the young, against whom he then has claims at the beginning of the following period). In fact, in a generic steady state $p'c \neq p'a$. When this is true, the infinite lived agent must hold net claims against the current old, at the beginning of each period, of nominal value $\beta^{t-1}p'w$, where

$$(1 - \beta)p'w = p'c - p'a \tag{2.11}$$

Thus, in order for the steady state to be an equilibrium from period one onward, rather than being only the asymptotic limit of an equilibrium, the infinite lived agent must start out with a vector of claims w satisfying (2.11) at the beginning of period one.

Strict concavity of u implies that $D^2u(c)$ is negative definite, and hence that $\mathrm{Det}\ D^2u(c) \neq 0$. It follows that (2.9) can be inverted in a neighborhood of the steady state, to yield a unique smooth function $c = s(\lambda p)$ for λp in some neighborhood of the steady state value. Now in any perfect foresight equilibrium, the consumption demand of the infinite lived agent in each period will satisfy $Du(c_t) = \lambda_t p_t$, for some λ_t. It follows that in any perfect foresight equilibrium sufficiently close to the steady state, $c_t = s(\lambda_t p_t)$ each period. Furthermore, the Lagrange multipliers in consecutive periods must satisfy $\lambda_t = \beta\lambda_{t+1}$. Hence a perfect foresight equilibrium is a sequence $\{p_t, \lambda_t\}$, with $\lambda_t p_t$ always in the domain for which $s(\lambda p)$ is defined, satisfying

(2.12.1) $$z_0(p_1) + y(p_1,p_2) + s(\lambda_1 p_1) = a$$

(2.12.2) $$z(p_1,p_2) + y(p_2,p_3) + s(\lambda_2 p_2) = a$$

$$\cdots$$

(2.12.t) $$z(p_{t-1},p_t) + y(p_t,p_{t+1}) + s(\lambda_t p_t) = a$$

$$\cdots$$

(2.13) $$\lambda_{t+1} = \beta^{-1}\lambda_t \quad , \quad t = 1,2,\ldots \quad .$$

Note that this system of equations is homogeneous degree zero in $\{p_t, \lambda_t^{-1}\}$, so that either one price or one Lagrange multiplier may be chosen arbitrarily as a price level normalization.

Equilibrium conditions (2.12) and (2.13) embody all of the first order conditions for optimal consumption on the part of the infinite lived agent, but it might be supposed that an additional condition is necessary, representing the budget constraint of the infinite lived agent. This is not true when there is only one infinite level agent. In any solution to (2.12) and (2.13), the budget constraints of all other agents are satisfied; hence the budget constraint of the infinite lived agent must be satisfied as well. Hence (2.12) and (2.13) are the complete set of equilibrium conditions.[18/]

Assuming that the vector of initial claims w satisfies (2.11), the steady state $p_t = \beta^{t-1}p$, $\lambda_t = \beta^{1-t}\lambda$, is one solution to (2.12) and (2.13). Are there other solutions arbitrarily close to this one? The answer depends upon the dimension of the stable manifold of the steady state, i.e., the set of $(\beta^{-1}p_2, p_1, \lambda_1)$ that may be extended to a perfect foresight equilibrium converging to the steady state. It is apparent from (2.13) that $\beta^{t-1}\lambda_t$ does not converge to λ unless $\lambda_1 = \lambda$. Hence we need only look for sequences $\{p_t\}$ that satisfy (2.12) with $\lambda_t = \beta^{1-t}\lambda$. The dimension of the stable manifold is thus equal to n^s, the number of eigenvalues within the unit circle possessed by the matrix G, defined as before, except that the expression $(\beta^{-1}D_1y + D_2z)$ in the upper left block of the previous definition must be replaced by $(\beta^{-1}D_1y + D_2z + \beta^{-1}A)$, where the matrix A is defined by

$$A \equiv \lambda Ds(\lambda p) = \lambda[D^2u(c)]^{-1}$$

Note that G need no longer have an eigenvalue equal to one when this term is added; although equations (2.12) and (2.13) are homogeneous degree zero in $\{p_t, \lambda_t^{-1}\}$, equations (2.12) with $\lambda_t = \beta^{1-t}\lambda$ are not homogeneous in $\{p_t\}$. Because of this, n^s may vary over the range $0 < n^s < 2n$. Initial conditions (2.12.1) impose n independent additional constraints, so that the dimension of indeterminacy is $\max(0, n^s - n)$. It follows that there may be as many as n dimensions of indeterminacy.

Examples of indeterminate steady states with infinite lived agents can be constructed by a perturbation method like that discussed in the case of land. Starting with a stationary overlapping generations economy with an indeterminate steady state which is either non-monetary and Pareto optimal or monetary with positive valued fiat money, one adds a small positive vector of land a. The resulting economy has an indeterminate steady state (p^*,β^*), with $\beta^* < 1$, as discussed above. If one replaces the land by an infinite lived agent with a stationary endowment a and discount factor $\beta = \beta^*$, the resulting economy has a steady state in which the consumption of the infinite lived agent is zero (i.e., (2.10) is satisfied with $c = 0$). Now consider the smooth function $c(p,\beta)$, defined as that vector c which solves (2.10). We know that $c(p^*,\beta^*) = 0$. Assuming a generically valid transversality condition, there exist points (p,β) arbitrarily close to (p^*,β^*) where the sign pattern of $c(p,\beta)$ takes on each of the 2^n possible values. Let (p,β) be values very close to (p^*,β^*) such that all components of $c(p,\beta)$ are positive. Then it is possible to choose a utility function $u(c)$ such that $Du(c(p,\beta))$ is a scalar multiple of p. Now let the infinite lived agent have utility function

$$V = \sum_{t=0}^{\infty} \beta^{t-1} u(c_t)$$

where β is the value near β^* just chosen, and $u(c)$ is the function just chosen. The resulting economy has a steady state in which the infinite lived agent consumes $c(p,\beta) >> 0$. The matrix G for this economy will be close to the matrix G for the original overlapping generations economy, and so its eigenvalues will be close as well. However, one is no longer an eigenvalue, after the infinite lived consumer is added; there will still be a real eigenvalue near one, but it may be greater or less than one. Hence the addition of the infinite lived agent either increases n^s by one, or leaves it unchanged. Since with the addition of the infinite lived agent, the dimension of indeterminacy becomes $n^s - n$, rather than $n^s - n + 1$, it follows that the addition of the infinite lived agent either leaves the dimension of indeterminacy unchanged, or decreases it by one. (The condition that determines which occurs is discussed in Muller and Woodford [1984].)

It follows from the above analysis that it is possible to have an indeterminate Pareto optimal non-monetary steady state with only one perishable good per period and no assets other than the IOU's of agents, if an infinite lived agent is added to the model. The following example shows that this is the case.

Example 8: Let the preferences of the finite lived agents be as in Example 6, but suppose that there is no capital; labor power is directly converted into the consumption good. This is then equivalent to a one good model (the labor endowment being effectively just an endowment of the consumption good) with utility function $u(y,z) = y + z^{1-\gamma}/(1-\gamma)$, for $y < 0$, $z > 0$. Let there also be an infinite lived agent with utility function $V = \sum_{t=1}^{\infty} \beta^{t-1} \log c_t$, with $0 < \beta < 1$, and with a stationary endowment $a > 0$.

The excess demand functions for the finite lived agents are

$$y(p_t, p_{t+1}) = -\left(\frac{p_t}{p_{t+1}}\right)^{\frac{1-\gamma}{\gamma}}$$

$$z(p_t, p_{t+1}) = \left(\frac{p_t}{p_{t+1}}\right)^{\frac{1}{\gamma}}$$

In order for there to be a steady state in which the infinite lived agent consumes, p_{t+1}/p_t must equal β, and from (2.10) the steady state consumption of the infinite lived agent must be

$$c = \beta^{\frac{\gamma-1}{\gamma}} - \beta^{-\frac{1}{\gamma}} + a$$

This quantity is only positive if

$$a > \beta^{-\frac{1}{\gamma}}(1 - \beta) \tag{2.14}$$

If (2.14) is satisfied, such a steady state exists. Note that in the steady state $c < a$ each period. This is because the infinite lived agent maintains a constant level of debt in the steady state, upon which he must pay interest at the rate $(1/\beta) - 1$ each period. Since there is no outside money in this economy, all saving by the young of each finite lived generation is achieved by holding the debt of the infinite lived agent. Each period, then, the young lend $-y = \beta^{(\gamma-1)/\gamma}$ to the infinite lived agent, who must repay them $1/\beta$ times this amount, or $\beta^{-1/\gamma}$ in the following period. Because the infinite lived agent's repayments each period, $\beta^{-1/\gamma}$, are greater than his new borrowing, $\beta^{(\gamma-1)/\gamma}$, his consumption c is less than a. In this model, then, the steady state level of claims against the current old with which the infinite lived agent begins each period is $w = -\beta^{-1/\gamma}$, in accordance with (2.11).

Writing $R_{t+1} \equiv p_t/p_{t+1}$ for the gross real rate of interest, the first order conditions for the consumption demands of the infinite lived agent are

$$c_{t+1} = \beta R_{t+1} c_t \tag{2.15}$$

The market clearing relations for periods $t \geq 2$ are

$$R_t^{\frac{1}{\gamma}} - R_{t+1}^{\frac{1-\gamma}{\gamma}} + c_t = a \tag{2.16}$$

The initial condition (market clearing in period one) is

$$-w - R_2^{\frac{1-\gamma}{\gamma}} + c_1 = a \tag{2.17}$$

since $z_0 = -w$. A perfect foresight equilibrium is then a sequence $\{c_t\}$ for $t = 1,2,\ldots,$ and a sequence $\{R_t\}$ for $t = 2,3,\ldots,$ satisfying (2.15), (2.16), and (2.17). If $w = -\beta^{-1/\gamma}$ in (2.17), the steady state is one solution.

Linearizing (2.15) and (2.16) around the steady state yields

$$\begin{bmatrix} R_{t+1} - \frac{1}{\beta} \\ c_{t+1} - c \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\gamma}\frac{1}{\beta} & \frac{\gamma}{1-\gamma}\beta^{(1-2\gamma)/\gamma} \\ \frac{c}{1-\gamma} & 1 + \frac{\gamma c}{1-\gamma}\beta^{(1-\gamma)/\gamma} \end{bmatrix} \begin{bmatrix} R_t - \frac{1}{\beta} \\ c_t - c \end{bmatrix}$$

The eigenvalues of this matrix are the solutions to

$$(2.18)\quad \lambda^2 - [1 + \frac{1}{1-\gamma}\frac{1}{\beta} + \frac{\gamma c}{1-\gamma}\beta^{(1-\gamma)/\gamma}]\lambda + \frac{1}{1-\gamma}\frac{1}{\beta}$$
$$= \lambda^2 - [\frac{1}{1-\gamma} + \frac{1}{\beta} + \frac{\gamma a}{1-\gamma}\beta^{(1-\gamma)/\gamma}]\lambda + \frac{1}{1-\gamma}\frac{1}{\beta} = 0$$

In the case that $\gamma < 1$, one root lies between zero and one, while the other is greater than one. So the stable manifold is of dimension one, the initial condition (2.17) imposes one restriction, and the steady state is exactly determinate. On the other hand, in the case that $\gamma > 1$, one root lies between zero and one, while the other root lies between zero and -1 if

$$(2.19)\quad a < (1 + \beta)\beta^{-\frac{1}{\gamma}}(\frac{\gamma - 2}{\gamma})$$

but is less than -1 if the inequality in (2.19) is reversed. Hence the steady state is indeterminate if and only if (2.19) holds. Note that the interval for a established by (2.14) and (2.19) is nonempty if and only if

$$(2.20)\quad \gamma > \frac{1 + \beta}{\beta}$$

Thus no matter what the value of β is, a value of γ greater than two is always necessary for existence of an indeterminate steady state in which the infinite lived agent consumes.

3. Conditions Guaranteeing Determinacy

Muller and Woodford [1984] establish some additional general conditions for the possibility of indeterminacy, in addition to the bounds on the dimension of indeterminacy given by the number of goods per period. One of these is that the consumption demands of infinite lived agents cannot play too large a role in the determination of equilibrium, if indeterminacy is to be possible. It has already been mentioned that indeterminacy is impossible if the economy consists only of a finite number of infinite lived consumers. But it can also be shown for a hybrid model (i.e., both infinite lived agents and overlapping generations of finite lived agents exist), that as the endowment of the infinite lived agents is increased (or the population of infinite lived agents with a given endowment per capita is increased), for a given population of finite lived agents in each generation, a point is eventually reached at which no steady state can any longer be indeterminate. This is illustrated in Example 8. As the parameter a is increased in that example (i.e., either the endowment, and hence the steady state consumption level of a single infinite lived agent is increased, or, equivalently, the population of infinite lived agents is increased), eventually the bound (2.19) is reached, beyond which the steady state becomes determinate.

This is true for the general n-good model as well, because the eigenvalues of G are the $2n$ solutions to

$$\text{(2.21)} \qquad \text{Det}[D_2y\lambda^2 + (\beta^{-1}D_1y + D_2z + \beta^{-1}A)\lambda + \beta^{-1}D_1z] = 0$$

In the fully general model, A is a sum, over the infinite lived agents who consume in the steady state, of individual terms $\lambda^h[D^2u^h(c^h)]$ for each agent h. Thus as the number of infinite lived agents is made larger, A comes to be a larger and larger negative definite matrix. Suppose that all the eigenvalues of A are of order N, where N is a large number (some measure of the importance of the infinite lived agents in the economy). Then (2.21) will have n solutions of order N (and hence outside the unit circle), approximately equal to the eigenvalues of $-\beta^{-1}D_2y^{-1}A$, and n solutions of order $1/N$ (and hence inside the unit circle), approximately equal to the eigenvalues of $-D_1zA^{-1}$. Hence the steady state must be exactly determinate.[19]

Note that it is large elements in the matrix A that rule out indeterminacy rather than a large population of infinite lived agents or large consumption by them as such. What is important is the extent to which the aggregate consumption demand of the infinite lived agents responds to price changes. Even if the infinite lived agents consume a sizeable fraction of the total product, if their consumption demands are sufficiently inelastic their presence will not rule out indeterminacy of equilibrium. For example, suppose that in Example 8, the utility function of the infinite lived agent were $V = \sum_{t=1}^{\infty} \beta^{t-1}(c_t^{1-\delta}/(1-\delta))$, for some $\delta > 1$. Then $A = -c/\delta$, and (2.18) becomes instead

$$\lambda^2 - [1 + \frac{1}{1-\gamma}\frac{1}{\beta} - \frac{1}{\delta}\frac{\gamma c}{1-\gamma}\beta^{\frac{1-\gamma}{\gamma}}]\lambda + \frac{1}{1-\gamma}\frac{1}{\beta} = 0$$

Following the same argument as before, one finds that the steady state is indeterminate whenever $\gamma > 1$ and

$$c < 2\delta\beta^{-\frac{1}{\gamma}}[\frac{\beta(\gamma-1)-1}{\gamma}]$$

But c can be made arbitrarily large, and still satisfy this inequality, as long as δ is made proportionately large. A large

δ, of course, means a very small elasticity of substitution of consumption between periods on the part of the infinite lived agent.

Another very general conclusion is that there must always be significant income effects in the responses of the consumption demands of the finite lived agents to price changes, if indeterminacy is to be possible in the case of a Pareto optimal steady state. This also follows from (2.21). Suppose one suppressed all income effects from the derivatives of the excess demand functions y and z. Then

$$\begin{bmatrix} D_1y & D_2y \\ D_1z & D_2z \end{bmatrix}$$

would be a matrix of Slutsky substitution terms only, and hence would be symmetric. But A is symmetric as well, from which it follows that if λ solves (2.21), $1/(\beta\lambda)$ solves it as well. Thus half the eigenvalues of G would be of modulus less than $\beta^{-1/2}$. In the case of a Pareto optimal steady state $(\beta < 1)$, this would imply that no more than n eigenvalues have modulus less than or equal to one. Thus in the absence of infinite lived agents, so that one must be one of the eigenvalues, $n^s \leq n - 1$, and indeterminacy is impossible. On the other hand, in the case that infinite lived agents exist that consume in the steady state, the bound becomes $n^s \leq n$, but indeterminacy is still impossible. Thus in the case of an autarchic steady state (each agent consumes exactly his endowment), where there are no income effect terms, it is not possible that the steady state is both Pareto optimal and indeterminate.

An autarchic steady state is a very special case, of course, once one allows more than one good per period, introduces production, or introduces an infinite lived agent. But because of the continuity of the eigenvalues of G as functions of the demand derivatives, it will also be the case that in any steady state at which the income effect terms are small compared to the substitution effect terms, half the eigenvalues are of modulus less than $\beta^{-1/2}$, so that indeterminacy is again incompatible with Pareto optimality. Hence examples characterized by indeterminate Pareto optimal steady states must always involve significant income effects.

It would be desirable to be able to express quantitatively how large the income effects must be for indeterminate Pareto optimal equilibrium to be possible. In the examples with Pareto optimal steady states presented in this paper, excess demands satisfying the "gross substitutes" condition (excess demand for each good decreasing in its own price, increasing in all other prices) suffice to rule out the possibility of indeterminacy. Thus in Examples 2 and 4, gross substitutes would mean an offer curve which is not backward bending; this would prevent the existence of the indeterminate steady state. In Examples 6 and 8, gross substitutes would mean $\gamma < 1$, but $\gamma > 2$ is always required in those examples for indeterminacy. However, gross substitutes does not seem to be a sufficient condition for determinacy more generally. Calvo [1978] presents an example of an overlapping generations model with two production technologies (producing consumption goods and capital goods respectively) and two factors of production (capital and inelastically supplied labor) used in both sectors. Consumers demand only the single consumption good each period. Calvo shows that it is possible for such a model to possess an indeterminate Pareto optimal steady state, even when agents' preferences over consumption in the two periods of life satisfy the gross substitutes condition.

David Levine has suggested the following sufficient condition for indeterminacy. Let the bound of a matrix M be defined as

$$\|M\| \equiv \sup_{\|x\|=1} \|Mx\|$$

where $\|x\|$ denotes the Euclidean norm of vector x. Then

$$\|[\beta^{-1}D_1y + D_2z + \beta^{-1}A]^{-1}\| \; (\|D_2y\| + \|\beta^{-1}D_1z\|) < 1 \tag{2.22}$$

is a sufficient condition for indeterminacy. For consider the possibility of a sequence of vectors $\{x_t\}$, $t \geq 0$, satisfying $x_o = 0$, $x_t \to 0$ as $t \to \infty$, and

$$D_2yx_{t+2} + (\beta^{-1}D_1y + D_2z + \beta^{-1}A)x_{t+1} + \beta^{-1}D_1zx_t = 0$$

for $t \geq 0$. It follows from $x_o = 0$ and (2.22) that $\|x_1\| < \|x_2\| < \|x_3\| < \ldots$, so that $x_t \to 0$ as $t \to \infty$ is possible only if $x_t = 0$ for all t. Hence the stable manifold must be of dimension no greater than n. Next consider the possibility of a sequence of vectors $\{x_t\}$, $t \leq 0$, satisfying $x_o = 0$, $x_t \to 0$ as $t \to -\infty$. This too is possible only if $x_t = 0$ for all t; hence the stable manifold must be of dimension no smaller than n. It follows that the stable manifold is of dimension n, and equilibrium is exactly determinate.

The Levine criterion provides a quantitative measure of how large the effects of the demands of the infinite lived agents must be in order to insure determinacy of equilibrium: the terms contributed by A must be large enough to make the bound of $[\beta^{-1}D_1y + D_2z + \beta^{-1}A]^{-1}$ less than a certain quantity. But it provides no quantitative measure of how large income effects must be in order for indeterminacy to be possible, at least none that is easily interpreted.

In summary, the present section has shown that the economies for which equilibrium is indeterminate are not isolated cases. First, the analysis of Kehoe and Levine shows that the property of having an indeterminate steady state depends only upon certain inequalities being satisfied by the derivatives of the excess demand functions. Hence examples of economies with indeterminate steady states are robust in the sense that small changes in preferences or endowments will not affect the result. Second, the analysis of Muller and Woodford shows that many such examples can even be perturbed by adding land or infinite lived agents to the economy without changing the dimension of the indeterminacy. This makes it clear that the problem does not arise only in the case of models that are extremely special in some respect.

On the other hand, there are assumptions about the world that, if one were willing to grant them, would suffice to rule out the indeterminacy problem. Two have been discussed in particular. If one believed that a sufficient number of agents behaved like infinite lived agents (say, for the reasons given by Barro) for the responses of these agents' demands to constitute a large part of the response of aggregate demand to price changes, then one could forget about indeterminacy. Alternatively, if one believed that the preferences of finite lived agents were such that substitution effects were always much stronger than income effects, and one were interested only in Pareto optimal equilibria, one would again be able to forget about indeterminacy. The latter assumption in particular is one that many economists are willing to make in other contexts, in order to insure the validity of conclusions drawn from simple parables. Ideally, of course, one would want to be able to empirically test the validity of such assumptions.

III. Indeterminacy and Equilibrium Cycles

The work of Jean-Michel Grandmont [1983b] has stimulated renewed interest in the existence of equilibrium cycles for certain overlapping generations economies. Grandmont shows that for a broad class of one-good overlapping generations exchange economies of the sort considered in Example 2 of Section I, there exists a detailed set of necessary relationships between the existence and determinacy of periodic equilibria of various periods. For example, in a special class of models that he considers in detail, a 2-period monetary equilibrium cycle exists if and only if the monetary steady state is indeterminate, a 4-period monetary equilibrium exists if and only if there exists an indeterminate 2-period cycle, and so on.[20/] It might be wondered, therefore, if the connection observed in that case between indeterminacy and the existence of cycles holds more generally.

It is easily shown that in the case of a one-good overlapping generations exchange economy of the sort considered in Example 2 of Section I, indeterminacy of the monetary steady state is a sufficient condition for the existence of a 2-period equilibrium cycle. The following proof is given by Azariadis and Guesnerie [1984]. Figure 6a shows the offer curve of Figure 2, and superimposed upon the same diagram, the reflection of that offer curve about the 45° line. Any intersection of the two curves represents an equilibrium that repeats itself every two periods. Intersections on the 45° line represent steady states; intersections off the 45° line represent 2-period cycles. (Intersections of the latter sort come in pairs, since each such intersection must have a reflection that is a point of intersection as well. The two points in the pair represent the two phases of a single 2-period cycle.) It follows from the geometry of the figure that if the slope of the offer curve through the monetary steady state is between +1 and -1 (the case of indeterminacy), the curves must intersect somewhere off the 45° line.

On the other hand, the above proof allows one to see that indeterminacy of the monetary steady state is not a necessary condition for the existence of a 2-period cycle. Figure 6b shows how it is possible to have 2-period cycles without an indeterminate steady state. (The sort of offer curve displayed in Figure 6b does not satisfy Grandmont's assumption of a "negative Schwarzian derivative.") Furthermore, once one moves beyond the confines of this particular type of economy, indeterminacy need not be sufficient for existence of an

Figure 6a

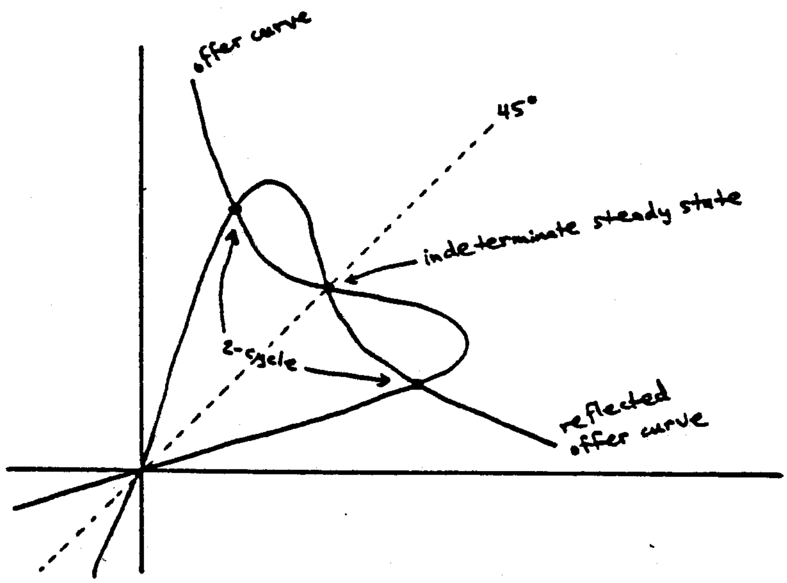
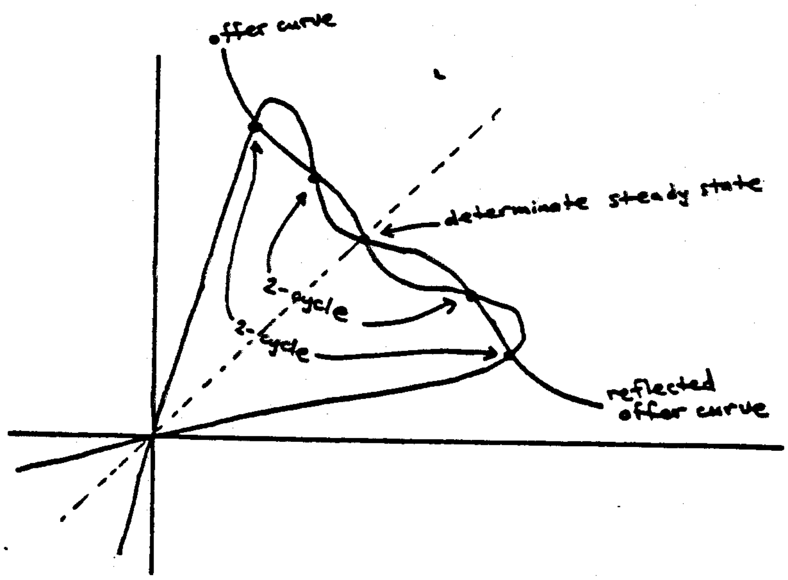


Figure 6b



equilibrium cycle. Example 5 of Section I, for example, possesses an indeterminate monetary steady state, but no equilibrium cycles. (The requirement that $(-g - z_t, z_{t+1})$ be a point on the offer curve makes z_{t+1} a monotonically increasing function of z_t, so that the only possible kind of stationary equilibrium is a steady state.)

Outside the class of one-good overlapping generations models, it is even clearer that there is no general relationship between indeterminacy and the existence of cycles. Benhabib and Nishimura [1984] demonstrate the possibility of a 2-period equilibrium cycle in an optimal growth model; but equilibrium is unique in that model. A comparison may be instructive between the way equilibrium cycles arise in their model and the way they arise in the model considered by Grandmont. In both papers, a one-parameter family of economies is considered. For low values of the parameters, the steady state equilibrium is exactly determinate and there is no equilibrium cycle; but at a critical parameter value, a "flip bifurcation" occurs (see Guckenheimer and Holmes [1983], Chap. 3), and for parameter values beyond this point a 2-period cycle exists in addition to the steady state. The point at which the flip bifurcation occurs is always the parameter value at which one of the eigenvalues controlling the convergence or divergence of equilibrium paths near the steady state changes stability by passing through the value -1, and this will necessarily change the dimension of the stable manifold of the steady state by one. But in Grandmont's case, the eigenvalue passes from less than -1 to greater than -1 (unstable to stable), so that the previously determinate steady state becomes indeterminate, while in the Benhabib-Nishimura example, the eigenvalue passes from greater than -1 to less than -1 (stable to unstable), so that the previously exactly determinate steady state becomes unstable.

This indicates why, despite the generality of the flip bifurcation theorem (it holds for mappings of an arbitrary number of dimensions), one cannot expect there to be any general connection between indeterminacy and cycles. Even if the only way that 2-period cycles could arise under a continuous deformation of one economy into another were via a flip bifurcation (this is not so--two cycles may be created at once in a "saddle-node bifurcation"), it would follow only that the emergence of a 2-period cycle would always be associated with a change in the dimension of the stable manifold of the steady state. In a general multiple-good model, this could mean a transition from three dimensions of indeterminacy to only two, or vice versa, or from exact determinacy to instability, or many other things. The connection between indeterminacy and cycles that appears in the special case studied by Grandmont is a consequence, then, of the very special geometry of that case.

Similar conclusions are reached in the case of continuous-time dynamic models. There too one finds a connection between indeterminacy and cycles in a special, low-dimensional case. Suppose that the perfect foresight equilibrium conditions describe a dynamical system with one state variable and one "jump variable" (control variable or price), so that one quantity is given by the history of the economy up to the present, while the other is determined by expectations about the future. In such a case, exact determinacy requires that a stationary equilibrium have a one-dimensional stable manifold (the "saddle point" property). It is easily shown in such a case that if an equilibrium cycle exists, perfect foresight equilibrium is indeterminate. An example of this is the Diamond [1982] search model of aggregate demand fluctuations. In this model the single state variable is the fraction of the population with inventories to trade (e), and the single control variable is the reservation production cost (c^*) which determines which production opportunities will be accepted by the agents without inventories. Diamond shows that perfect foresight equilibrium implies certain dynamics in the $e - c^*$ phase plane. Diamond and Fudenberg [1983] show that this model may possess an equilibrium cycle. It follows that in that case perfect foresight equilibrium is indeterminate. For either the perfect foresight dynamics near the cycle spiral toward it, or they spiral away from it. In the first case, the cycle is indeterminate (see Figure 7a). That is, for a given initial value $e(o)$, there exists an entire interval of $c^*(o)$ values consistent with perfect foresight equilibrium that converges asymptotically to the cycle. In the second case, there must be some limit set in the interior of the cycle that is indeterminate (such as the indeterminate steady state shown in Figure 7b, or another equilibrium cycle).

This result, however, only applies in the case of a system with a single state variable and a single control. With additional dimensions it is possible to have a cycle without indeterminacy. Thus in the

Figure 7a

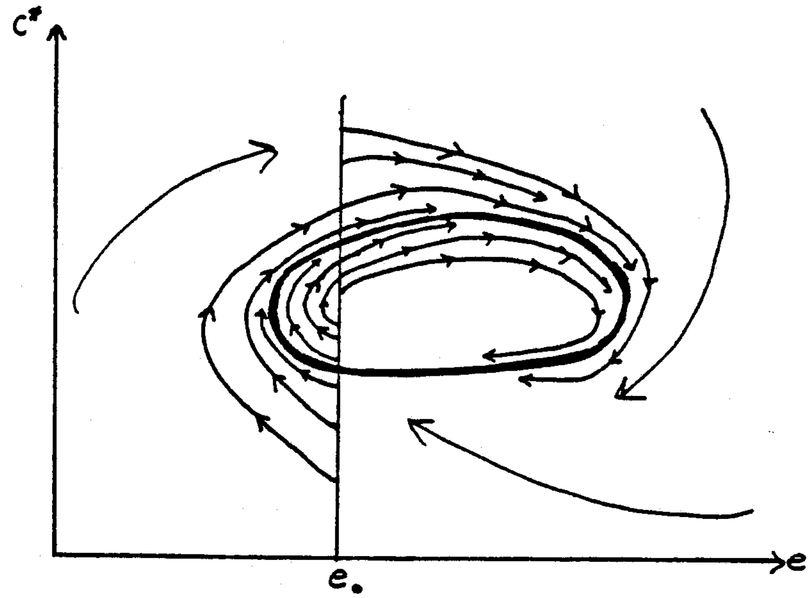
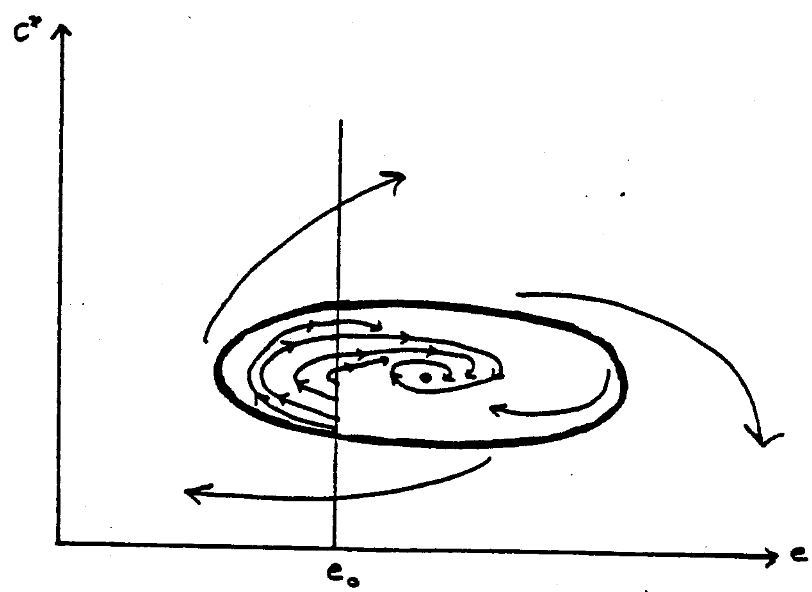


Figure 7b



optimal growth example of Benhabib and Nishimura [1979], with two capital stocks and two prices, there exists a continuous time competitive equilibrium cycle, but perfect foresight equilibrium is unique.

IV. Indeterminacy and Sunspot Equilibria

Some readers may not find the indeterminacy of perfect foresight equilibrium demonstrated above a serious problem. For, they might argue, perfect foresight equilibrium as a solution concept only makes sense in the case of stationary equilibria, in which case one may expect a rational agent to have learned what to expect. And in the generic case, the stationary perfect foresight equilibria are locally isolated[21]; indeterminacy only appears if one counts the non-stationary equilibria as well. Others may agree that perfect foresight equilibrium is the correct equilibrium notion, even outside of a stationary equilibrium, but may feel that the indeterminacy is not troubling as long as all of the possible equilibrium paths are Pareto optimal. And there are various ways to insure this, such as assuming the existence of a small amount of land or a single infinite lived consumer. Others may regard indeterminacy as not greatly disturbing as long as the role that self-fulfilling expectations can play in determining allocations is a purely transient one.[22] Still others may hold that indeterminacy of a steady state is a desirable state of affairs, calling it "stability" and regarding it as a guarantee that the economy will eventually settle into a steady state equilibrium and return to it in the event of transient disturbances.

All of these grounds for complacency are unsettled by the observation that indeterminacy of perfect foresight equilibrium brings with it the possibility of stationary "sunspot equilibria." By this is meant stationary rational expectations equilibria in which prices and allocations are affected by random variables which are observed by agents but do not affect preferences, endowments or technology. It is reasonable to consider these stochastic equilibria, if they exist, to be additional equilibria of the non-stochastic model. And these equilibria cannot be rejected on the ground that rational expectations are only plausible in the case of a stationary equilibrium, for the sunspot equilibria to be displayed below are all stationary. There are a very large number of them; in the examples below, there are an uncountably infinite number of such stationary equilibria that remain uniformly close to the non-stochastic steady state equilibrium. Furthermore, all of these equilibria are inefficient in an expected utility sense, even if the non-stochastic steady state is Pareto optimal, since they involve unnecessary randomization of the equilibrium allocation. (See Balasko [1983] for a general argument to this effect.) And finally, the existence of these equilibria shows that an indeterminate steady state does not mean that the economy eventually reaches the steady state allocation in every rational expectations equilibrium.

The connection asserted here between indeterminacy of perfect foresight equilibrium and the existence of stationary sunspot equilibria can be illustrated in the case of the examples of indeterminacy given in Section I. Example I, however, provides an exception to the general result. It is possible to show the existence of a large number of non-stationary sunspot equilibria for such an economy. Early examples were given by Shell [1977] and Azariadis [1981], and Peck [1984] has shown how to construct a large number of such equilibria for the general one-good model, assuming that a monetary equilibrium exists. But these equilibria are all transient, in the sense that the sunspot variable ceases to affect allocations asymptotically. This results from the "hyperinflationary" character of the non-stationary monetary equilibria of this model. The situation is different in the case either of an indeterminate monetary steady state, or of an indeterminate non-monetary steady state with a multiplicity of non-monetary equilibria converging to it; in these cases, it is possible that sunspot variables could produce equilibrium fluctuations around the steady state that are stationary. This provides another reason to consider the "hyperinflationary" indeterminacy of Example 1 less important than the kinds of indeterminacy illustrated by the other examples.

Example 2 revisited: Azariadis [1981] showed, for the case of an additively separable utility function, that stationary sunspot equilibria can exist for an economy like that of Example 2, and that indeterminacy of the monetary steady state is a sufficient condition for the existence of such equilibria.[23/] The analysis of this example has been further extended by Azaradis and Guesnerie [1982, 1984] and Spear [1983], who show, among other things, that the result does not depend upon additive separability. The presentation here is a generalization of that of Azariadis [1981], since this approach is most easily generalized to allow treatment of Examples 4 and 5 as well.

Let the utility function of the two period lived agents be $u(y,z)$, where y and z are consumption in excess of endowment in the first and second periods of life respectively; u is defined on the set $\{y > -e_1, z > -e_2\}$. Let u be smooth, monotone, strictly concave, and satisfy boundary conditions sufficient to insure that optimal consumption demands are never completely specialized. Let u_1 and u_2 denote partial derivatives of u with respect to its first and second arguments respectively. Azariadis shows the existence of stationary rational expectations equilibria in which $\{z_t\}$ follows a two-state stationary Markov process. (It follows that the price level p_t follows such a process as well, since $z_t = M/p_t$ in all periods, but it is convenient to consider the dynamics of the z_t variable, as before.) Let the two values of z_t be (z_a,z_b) and let

$$\text{Prob}(z_{t+1} = z_a | z_t = z_a) = \pi_a$$

$$\text{Prob}(z_{t+1} = z_b | z_t = z_b) = \pi_b$$

The question is under what circumstances there exist (π_a,π_b,z_a,z_b) with $0 < \pi_a < 1$, $0 < \pi_b < 1$, $z_a > 0$, $z_b > 0$, $z_a \neq z_b$, such that this process represents a rational expectations equilibrium.

It is clear that (π_a,π_b,z_a,z_b) constitutes an equilibrium if and only if the following pair of first order conditions are satisfied:

$$\text{(4.1)} \quad \begin{aligned} &\pi_a z_a u_1(-z_a, z_a) + (1 - \pi_a) z_a u_1(-z_a, z_b) = \\ &\pi_a z_a u_2(-z_a, z_a) + (1 - \pi_a) z_b u_2(-z_a, z_b) \end{aligned}$$

$$\text{(4.2)} \quad \begin{aligned} &\pi_b z_b u_1(-z_b, z_b) + (1 - \pi_b) z_b u_1(-z_b, z_a) = \\ &\pi_b z_b u_2(-z_b, z_b) + (1 - \pi_b) z_a u_2(-z_b, z_a) \end{aligned}$$

(The first order conditions suffice to characterize optimal choice because of the smoothness and strict concavity of u and the boundary conditions.) For any z_a, z_b $(0, e_1)$, it is possible to calculate the values of π_a, π_b that would satisfy (4.1) and (4.2):

$$\text{(4.3)} \qquad \frac{\pi_a}{1 - \pi_a} = \frac{z_b u_2(-z_a, z_b) - z_a u_1(-z_a, z_b)}{z_a u_1(-z_a, z_a) - z_a u_2(-z_a, z_a)}$$

$$\text{(4.4)} \qquad \frac{\pi_b}{1 - \pi_b} = \frac{z_a u_2(-z_b, z_a) - z_b u_1(-z_b, z_a)}{z_b u_1(-z_b, z_b) - z_b u_2(-z_b, z_b)}$$

assuming that the numerator and denominator are not both zero in the case of either (4.3) or (4.4); in such a case the probability is indeterminate. If (4.3) and (4.4) indicate $0 < \pi_a < 1$, $0 < \pi_b < 1$, one has found a stationary sunspot equilibrium. One's task is therefore to find quantities z_a, $z_b \in (0, e_1)$ such that the numerator of (4.3) has the same sign as the denominator, and likewise for (4.4).

The moneraty steady state is given by z^* such that

$$u_1(-z^*, z^*) = u_2(-z^*, z^*)$$

One solution to (4.1) and (4.2) is therefore $z_a = z_b = z^*$, for arbitrary (π_a, π_b). Now consider $z_a = z^* + m\varepsilon$, $z_b = z^* + n\varepsilon$, for m

and n nonzero. Substituting these into (4.3), and taking the limit as $\varepsilon \to 0$ yields

$$(4.5) \qquad \frac{\pi_a}{1 - \pi_a} \to \frac{\frac{n}{m} - S}{S - 1}$$

where

$$(4.6) \qquad S \equiv -\frac{dz}{dy} = \frac{u_1 + z^*(u_{21} - u_{11})}{u_2 + z^*(u_{22} - u_{12})}$$

is the slope of the offer curve through the steady state, and the derivatives in (4.6) are all evaluated at $(-z^*,z^*)$. Similarly, taking the limit of (4.4) yields

$$(4.7) \qquad \frac{\pi_b}{1 - \pi_b} \to \frac{\frac{m}{n} - S}{S - 1}$$

One has $0 < \pi_a < 1$, $0 < \pi_b < 1$ in the limit if and only if both expressions (4.5) and (4.7) are positive. It is possible to choose m and n so that both expressions are positive if and only if $-1 < S < 1$, i.e., exactly in the case that the monetary steady state is indeterminate.

The sunspot equilibria that are guaranteed to exist by this argument are close to the steady state allocation, in the sense that z_a and z_b are both near z^*. And it is clear from the above that for any neighborhood of z^*, there exists a two dimensional manifold of choices of (z_a,z_b) both within that neighborhood that correspond to two-state Markov process stationary equilibria. (It is only necessary that m/n lie within a certain interval, and for each m/n in that interval, there will be an interval of $m\varepsilon$ values close enough to zero so that expressions (4.5) and (4.7) have the same sign as they do for $\varepsilon \to 0$.) Nor do two-state Markov process equilibria exhaust the set of stationary sunspot equilibria. The equilibrium conditions for a Markov process with a larger number of states can be written in a form like equations (4.1) and (4.2), and the two-state equilibria will be particular solutions to those equations. Then, by elementary continuity arguments, solutions to those equations will also exist that are not reducible to a two-state process. Hence an uncountable number of n-state Markov process equilibria will exist, for all $n > 2$.[24/] The degree of indeterminacy is thus very severe, even when one restricts one's attention to stationary equilibria.

While indeterminacy of the monetary steady state is a sufficient condition for the existence of stationary sunspot equilibria, it is not necessary. In the example depicted in Figure 6b, the monetary steady state is determinate, yet two-state Markov process equilibria exist. Let (z_1^*, z_2^*) be the two phases of one of the 2-period deterministic cycles for the economy depicted in Figure 6b. Then one solution to (4.1) and (4.2) is $z_a = z_1^*$, $z_b = z_2^*$, $\pi_a = \pi_b = 0$. A simple continuity argument then guarantees the existence of solutions to (4.1) and (4.2) with $\pi_a, \pi_b > 0$, for π_a and π_b sufficiently small. (See Azariadis and Guesnerie [1984].) There will therefore exist an uncountable number of stationary sunspot equilibria near each of the 2-period deterministic cycles.

Example 4 revisited. Stationary sunspot equilibria can also be shown to exist in an economy of this sort, by a simple extension of the argument for Example 2. In this case, (π_a, π_b, z_a, z_b) constitute an equilibrium if and only if

$$\pi_a(z_a - a)u_1(a - z_a, z_a) + (1 - \pi_a)(z_a - a)u_1(a - z_a, z_b) = \pi_a z_a u_2(a - z_a, z_a) + (1 - \pi_a)z_b u_2(a - z_a, z_b)$$

$$\pi_b(z_b - a)u_1(a - z_b, z_b) + (1 - \pi_b)(z_b - a)u_1(a - z_b, z_a) = \pi_b z_b u_2(a - z_b, z_b) + (1 - \pi_b)z_a u_2(a - z_b, z_b)$$

One solution is again $z_a = z_b = z^*$, for arbitrary (π_a, π_b), where in this case the steady state z^* satisfies

$$(z^* - a)u_1(a - z^*, z^*) = z^* u_2(a - z^*, z^*)$$

The slope of the offer curve through the steady state in Figure 4 is

$$S = \frac{u_1 + z^* u_{21} + (a - z^*)u_{11}}{u_2 + z^* u_{22} + (a - z^*)u_{12}}$$

With these modifications, the argument goes through as before. Equations (4.5) and (4.7) hold as before, and so one again finds that two-state Marlov process equilibria exist arbitrary close to the steady state if and only if $-1 < S < 1$, i.e., exactly when the steady state is indeterminate.

This example shows that the possibility of stationary sunspot equilibria demonstrated by Azariadis does not depend upon peculiarities of the overlapping generations model of fiat money. Stationary sunspot equilibria may exist in an economy without fiat money. (The discussion of Example 6 below also illustrates this.) It also shows that the existence of one or more infinite lived agents need not prevent the existence of stationary sunspot equilibria. For Example 4 can be interpreted as describing an economy in which there exist infinite lived agents with an aggregate stationary endowment of a per period, which agents desire consumption only in period one.

Example 5 revisited. This case is exactly like that of Example 4, with the quantity a replaced by $-g$. Since no part of the argument above relied upon the restriction $a > 0$, the same argument goes through in this case. In the case of Example 5, an indeterminate monetary steady state exists even in the case of gross substitutes. It follows that the failure of the gross substitutes condition to hold is not necessary for the existence of stationary sunspot equilibria. An alternative method for proving the existence of an uncountably infinite number of stationary sunspot equilibria for this example is demonstrated in Farmer and Woodford [1984].

This example also indicates that stationary sunspot equilibria may exist even when no deterministic cycles exist. This shows that the characterization result of Azariadis and Guesnerie [1984]--that in the case of the exchange economy with passive government policy of Example 2, two-state Markov process sunspot equilibria exist if and only if a deterministic 2-period cycle exists--does not hold for more general economies. The search for equilibrium business cycle models which allow for stationary fluctuations in the absence of external shocks, initiated by Grandmont [1983], therefore ought not confine its attention to the investigation of the conditions under which deterministic equilibrium cycles are possible, as these are not the only cases in which stationary fluctuations driven purely by expectations are possible.

Example 1 revisited. It may seem puzzling that stationary sunspot equilibria exist in the case of Example 5, but not in Example 1. Example 1, after all, is the limiting case of Example 5 as g approaches zero. The problem arises from the requirement that $z_a, z_b > 0$. For $g > 0$, $z^{**} > 0$, and so for ε small enough, $z_a = z^{**} + m\varepsilon$ and $z_b = z^{**} + n\varepsilon$ will both be positive. As g approaches zero, z^{**} approaches zero, and this ceases to be the case. The above analysis would extend to the case $g = 0$ only if it were possible to choose $m > 0$, $n > 0$ so as to make the expressions in (4.5) and (4.7) both positive. But $m > 0$, $n > 0$ means that either m/n or n/m must be greater than one, so that both (4.5) and (4.7) cannot be positive; hence the construction used above does not continue to be possible when the sign restrictions on m and n are added.

It follows that stationary sunspot equilibria would be possible in the case of Example 1, if the condition $z_t > 0$ were not a necessary requirement for monetary exchange. One might imagine replacing money with government borrowing and lending at a stochastic real interest rate, where the interest rate is not revealed until the period in which repayment is due. In a rational expectations equilibrium, all agents would decide to save or borrow on the basis of correct knowledge of the distribution of real interest rates, and the government would balance its budget each period. Then if gross real interest rates are allowed to be either positive or negative (i.e., someone "lending" to the government in period t might be paid by the government in period $t + 1$, or might have to pay the government an additional sum), stationary rational expectations equilibria exist with consumption allocations fluctuating with the realizations of a sunspot variable even in the case of Example 1.[25/]

As an example, consider the utility function $u(y,z) = z + (1/2)(y - y^2)$, for consumption allocations satisfying $y < 1/2$. Any stochastic process for the $\{z_t\}$ such that $z_t > -1/2$ for all t, and $E_t(z_{t+1}) = z_t^2 + (1/2)z_t$, is then a rational expectations equilibrium. The monetary steady state for this economy is $z^* = 1/2$; the non-monetary steady state is $z_t = 0$ for all t. If, however, the government enters into contracts of the sort described in the previous paragraph, (π_a,π_b,z_a,z_b) will be a two-state Markov process equilibrium whenever

$$(1 - \pi_a)z_b = z_a^2 + (\tfrac{1}{2} - \pi_a)z_a$$

$$(1 - \pi_b)z_a = z_b^2 + (\tfrac{1}{2} - \pi_b)z_b$$

These equations may be combined to yield a quartic equation for either z_a or z_b. One solution is $z_a = z_b = 0$, another is $z_a = z_b = 1/2$; factoring these out leaves a quadratic equation for z_a or z_b. It has two real roots (so that additional two-state equilibria exist) if and only if $(1 - \pi_a)(1 - \pi_b) < 1/16$. The two additional equilibria are

$$z_a = (\pi_a - \frac{3}{4}) \pm \sqrt{\frac{1}{16} - (1 - \pi_a)(1 - \pi_b)}$$

$$z_b = (\pi_b - \frac{3}{4}) \mp \sqrt{\frac{1}{16} - (1 - \pi_a)(1 - \pi_b)}$$

There is thus a two-dimensional manifold of sunspot equilibria of this type, none of which satisfy $z_a, z_b \geq 0$. For $z_a, z_b \geq 0$ would require that $\sqrt{\frac{1}{16} - (1 - \pi_a)(1 - \pi_b)}$ be no greater than either $(\pi_a - 3/4)$ or $(\pi_b - 3/4)$, which in turn would imply $(1 - \pi_a), (1 - \pi_b) \leq 1/4$ and

$$\begin{aligned} \frac{1}{16} - (1 - \pi_a)(1 - \pi_b) &\leq (\pi_a - \frac{3}{4})(\pi_b - \frac{3}{4}) \\ &= [\frac{1}{4} - (1 - \pi_a)][\frac{1}{4} - (1 - \pi_b)] \\ &= \frac{1}{16} - \frac{1}{4}(1 - \pi_a) - \frac{1}{4}(1 - \pi_b) + (1 - \pi_a)(1 - \pi_b) \\ &\leq \frac{1}{16} - (1 - \pi_a)^2 - (1 - \pi_b)^2 + (1 - \pi_a)(1 - \pi_b) \\ &= \frac{1}{16} - (1 - \pi_a)(1 - \pi_b) - [(1 - \pi_a) - (1 - \pi_b)]^2 \end{aligned}$$

The latter inequalities imply $\pi_a = \pi_b$, which rules out sunspot equilibria. Note that it is possible to find sunspot equilibria that satisfy $z_a, z_b > -1/2$, so that the equilibrium allocation remains in the range in which there is no satiation in second period consumption. This will be satisfied, for example, for all $\pi_a, \pi_b < 1$ with both π_a and π_b sufficiently close to 1, and for all $\pi_a, \pi_b > 3/4$ with both π_a and π_b sufficiently close to $3/4$.

Example 6 revisited. Different methods must be employed for this example, as it is easily shown that a two-state Markov process equilibrium is impossible. When a stochastic rental rate for capital is allowed, the first order condition (1.2) becomes

$$E_t[R_{t+1}^{1-\gamma}] = n_t^{\gamma} w_t^{\gamma-1}$$

$$= k_{t+1}^{\gamma} w_t^{-1}$$

Equation (1.1) holds as before, and the factor-price frontier is the same, so that a rational expectations equilibrium is a stochastic process $\{k_t, R_t\}$ such that in each period

$$k_{t+1} = \frac{(1-\alpha)}{\alpha} R_t k_t \tag{4.8}$$

$$E_t[R_{t+1}^{1-\gamma}] = (1-\alpha)^{\gamma-1}\left(\frac{R_t}{\alpha}\right)^{\gamma+\frac{\alpha}{1-\alpha}} k_t^{\gamma} \tag{4.9}$$

One type of solution is one in which

$$R_{t+1}^{1-\gamma} = (1-\alpha)^{\gamma-1}\left(\frac{R_t}{\alpha}\right)^{\gamma+\frac{\alpha}{1-\alpha}} k_t^{\gamma} x_{t+1} \tag{4.10}$$

where x_{t+1} is an independently, identically distributed random variable, which always takes a positive value, and has mean one. Equations (4.8) and (4.10) then describe a Markov process on the continuous state space $\{k_t > 0,\ R_t > 0\}$. This will be a rational expectations equilibrium in which the sunspot variable $\{x_t\}$ affects allocations. It will be a stationary rational expectations equilibrium if there exists an invariant distribution on the state space $\{k_t > 0,\ R_t > 0\}$.

Again taking logarithms, the equations for the Markov process become

$$(4.11) \quad \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{R}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\gamma}{1-\gamma} & \frac{\gamma}{1-\gamma} + \frac{\alpha}{1-\alpha}\frac{1}{1-\gamma} \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{R}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{1-\gamma}\tilde{x}_{t+1} \end{bmatrix}$$

where

$$\tilde{k}_t \equiv \log k_t - \log k^* + \frac{1}{\gamma} E(\log x)$$

$$\tilde{R}_t \equiv \log R_t - \log R^*$$

$$\tilde{x}_t \equiv \log x_t - E(\log x)$$

and k^* and R^* are defined as in Section I. It is well known from the theory of stochastic linear difference equations that the process is stationary (an invariant distribution will exist, under weak restrictions an the variable $\tilde{x}$) if and only if the matrix in (4.11) has both eigenvalues inside the unit circle. But this is just the case $(\gamma > 2/(1 - \alpha))$ in which the non-stochastic steady state is indeterminate. Hence stationary sunspot equilibria exist in exactly the case that perfect foresight equilibrium is indeterminate, in the sense defined in Section I. (This provides a reason to be interested in indeterminacy in that sense. For there exists a large multiplicity of perfect foresight equilibria even when $\gamma < 2/(1 - \alpha)$, but there are no stationary sunspot equilibria in that case.)

Note that for a given sunspot variable $\{s_{t+1}\}$ with mean zero and finite variance, equation (4.11) with $\tilde{x}_{t+1} = \lambda s_{t+1}$ will be a stationary sunspot equilibrium for any λ. Hence for a given sunspot variable, there exists at least a one-parameter family of stationary sunspot equilibria, in the case that $\gamma > 2/(1 - \alpha)$. Different values of $|\lambda|$ will correspond to different degress of influence of the sunspot variable on equilibrium allocations. Varying $|\lambda|$ will affect not only the variance of k_t and R_t in the invariant distribution, but their means as well. For in the invariant distribution, $E(\tilde{k}_t) = E(\tilde{R}_t) = 0$, but Jensen's inequality implies that

$$\log E(k) = \log E(k) + \frac{1}{\gamma} \log E(x) > E(\log k) + \frac{1}{\gamma} E(\log x) = \log k^*$$

$$\log E(R) > E(\log R) = \log R^*$$

so that the means of both k and R are increased when sunspots affect the equilibrium.

We have seen that for each of the examples considered in Section I, indeterminary of a steady state is a sufficient condition for the existence of a large number of stationary sunspot equilibria. A reader might wonder, however, if the correct inference is not that "stability" of the forward perfect foresight dynamics (as discussed by Gale [1973], among others) is what guarantees the existence of stationary sunspot equilibria, rather than indeterminacy being sufficient when the two properties do not coincide. For in the examples of Section I, a steady state is indeterminate if and only if it is "stable", but in the case of dynamical systems with more than one "jump variable," steady states may be indeterminate without being "stable".

A consideration of a general first order linear system of difference equations indicates that indeterminacy suffices for the existence of stationary sunspot equilibria, rather than "stability" being necessary. Let us consider the system

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = M \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

where x_t is an m-vector of state variables, and y_t is an n-vector of "jump variables". Let us assume that the eigenvalues of M are distinct, and that none have modulus exactly equal to one. Let n^s be the number of eigenvalues of M that lie inside the unit circle, and let V be the stable shbspace of M, i.e., the n^s-dimensional subspace of R^{m+n} such that $v \in V$ implies $\lim_{t\to\infty} M^t v = 0$. Then the dimension of indeterminacy of perfect foresight equilibrium is $\max(n^s - m, o)$; the steady state $(x_t = y_t = 0$ for all $t)$ is indeterminate if $n^s > m$, buy only "stable" if $n^s = m + n$. It is evident that indeterminacy suffices for the existence of stationary sunspot equilibria. Such equilibria will be stochastic processes described by a stochastic difference equation of the form

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = M \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_{t+1} \end{bmatrix}$$

where ε_{t+1} is an n-vector of mean-zero finite-variance random variables, independent of ε_s for all $s \neq t + 1$, and drawn from the same joint distribution each period. The process described by this equation represents a stationary equilibrium if and only if

$$(4.12) \qquad \begin{bmatrix} 0 \\ \varepsilon_{t+1} \end{bmatrix} \in V$$

for all realizations of ε_{t+1}. If $n^s > m$, there will exist in general an $n^s - m$ dimensional space of n-vectors ε satisfying (4.12), and so $\{\varepsilon_{t+1}\}$ can be any mean-zero finite-variance random variable taking values in that linear space. On the other hand, if $n^s < m$, there will in general exist no vectors $\varepsilon \neq 0$ satisfying (4.12), so that, for a generic matrix M, indeterminacy is both necessary and sufficient for the existence of stationary sunspot equilibria.

We conjecture that a similar result holds for nonlinear systems as well. It is evident that, in general, existence of an indeterminate steady state is not necessary for the existence of stationary sunspot equilibria; the case depicted in Figure 6b is a counterexample.[26/] But in that case, all the two-state Markov process equilibria that exist are bounded away from the monetary steady state allocation (see Azariadis and Guesnerie [1984]). Our conjecture is therefore the following: under certain regularity conditions, stationary sunspot equilibria exist that remain within an arbitrarily small neighborhood of a deterministic steady state, if and only if that steady state is indeterminate. (Qualifications must be made about the type of sunspot equilibria that are possible in the case of a purely "hyperinflationary" indeterminacy, as in the discussion of Example 1, above.) We expect this to be true on the ground that the possible equilibria remaining within an arbitrarily small neighborhood of the steady state ought, in the case of a hyperbolic steady state, to be determined entirely by the linearized equilibrium conditions at the steady state. This is a subject of continuing research.

V. Responses to the Problem

The previous sections have established that the problem of indeterminate perfect foresight equilibrium, and the related problem of indeterminate rational expectations equilibrium when "sunspot" variables are introduced, is quite robust, and cannot be avoided by simply objecting to one or another "pathological" feature of Example 1 of Section 1. As noted in the introduction, this means that the method of comparative statics cannot be used in all cases to investigate the effects of various shocks or policy interventions, without supplementary conditions being added to the theory of competitive equilibrium.

This section discusses possible responses of the economic theorist to this problem. The first subsection considers the possibility of a "technical" solution, i.e., one that merely reinterprets the equilibrium conditions or supplements them with additional stipulations, while still accepting rational expectations equilibrium as the correct solution concept for all competitive economies. The second subsection considers the alternative presented by "learning" theories and ad hoc expectation functions. The final subsection presents the point of view of greatest appeal to the author. This is that the determinacy or indeterminacy of perfect foresight equilibrium in a particular institutional context or under a particular policy regime should be an important desideratum in institutional design and policy selection.

1. Technical Solutions

To some, a purely formal restriction upon the set of equilibria, that succeeds in rendering equilibrium unique or at least locally unique, is much preferable to an admission that "arbitrary" or "non-economic" factors such as history or custom might determine which of a large set of possible equilibria actually occurs. Others, observing the large multiplicity of equilibria that often exists in strategic games without a competitive market structure, and the relative fruitlessness of the many efforts by game theorists to devise a satisfactory formal criterion that would render equilibrium unique, have long since ceased to find the problem unsettling. The present section considers some purely "technical" solutions that might appeal to persons with the first viewpoint.

It is sometimes suggested that the set of perfect foresight equilibria should be restricted to those equilibria that not only may be extended indefinitely into the future without contradiction of any equilibrium conditions or constraints, but that also may be extended indefinitely back into the past. The logic of the proposal is not clear; but, in any event, it does not resolve the problem. In Example 1, all the perfect foresight equilibria that start with $0 < z_1 < z^*$ may be extended backwards to $t = -\infty$; for all of them, $z_t \to z^*$ as $t \to -\infty$. In Example 2, all the perfect foresight equilibria with $z_1 \neq z^*$, such that $z_t \to z^*$ as $t \to \infty$, can be extended backward to

$t = -\infty$; all of them approach the deterministic 2-cycle (displayed in Figure 6a) as $t \to -\infty$. The same is true of Example 4. In Example 5, all the perfect foresight equilibria that start with $z^{**} < z_1 < z^*$ can be extended backwards to $t = -\infty$, and all have $z_t \to z^*$ as $t \to -\infty$. In Example 6, the dynamical equations (1.1) and (1.3) can be inverted to give k_t and R_t as functions of (k_{t+1}, R_{t+1}), and these functions map $(0,\infty) \times (0,\infty)$ into itself. This makes it clear that all the perfect foresight equilibria of that example can be extended indefinitely into the past.

This suggestion is doubtless related to another common argument, namely, that indeterminacy is no problem for equilibrium analysis because in any given period the economy is already on one perfect foresight equilibrium path or another, and so the "correct" equilibrium of any point in time is uniquely picked out by what has already happened.[27/] This position has various drawbacks. One is ambiguity about which perfect foresight equilibrium path "continues" the past. A standard exercise for which non-stationary perfect foresight equilibria are considered concerns the response of the economy to some unexpected shock, not expected to be repeated. In such a case, none of the possible perfect foresight equilibrium paths after the shock can properly be regarded as a continuation of the equilibrium that agents had anticipated before the shock. If, on the other hand, one supposes that the agents knew in advance that the shock might occur (with some low probability), and acted then on the basis of an expectation that a particular perfect foresight equilibrium would occur following the shock, then the correct continuation of the previous rational expectations equilibrium is whichever perfect foresight equilibrium agents expected to occur in the case of the shock. But that could be any of the possible perfect foresight equilibria. If the shock in question was expected to occur with a sufficiently low probability, the particular equilibrium expected to occur following the shock would have an insignificant effect on agents' actions prior to the shock, in which case the particular post-shock equilibrium expected could not be deduced from an observation of the economy's history prior to the shock.

Another objection is that in the case of an economy like that described in Example 1, where the monetary steady state is determinate, the initial price level (if it is assumed to be given by history) must be exactly right, or the economy will never reach the monetary steady state. That is, the monetary steady state must be regarded as "unstable", so that it is virtually inevitable that money will eventually lose all value. Many would find this interpretation of the model unpalatable. Hahn [1982] accepts it, however, and indeed argues that active policy should be implemented in order to render the monetary steady state indeterminate (i.e., "stable"), as a means to insure price level stability.[28/] If the conjecture of Section IV is correct, an economy subjected to Hahn-style "stabilization policy" would possess a large multiplicity of stationary sunspot equilibria, while the same economy without intervention would have none. Perhaps this is not too high a price, if one really believes that the economy is certain to approach autarchy without such a policy; but to us it seems that "stability" in this sense is not desirable if one believes in rational expectations equilibrium.[29]

There are further problems with the notion that history determines an initial price level for the post-shock perfect foresight equilibrium. What if a shift in endowments lowers the level of z^* in Figure 1, with the result that the initial price level determined by history (the previous steady-state price level) is lower than the new steady-state price level? Then there is no perfect foresight equilibrium consistent with the initial price level given by history. Presumably history does not pick out the initial price level in such a case; then how is it that history picks out the initial price level in other cases? Or consider the economy of Example 3. Here perfect foresight equilibrium is unique, so that in the event of a shock, the initial post-shock price of land must not be given by history, but rather by the forward-looking requirement that it be possible to continue the equilibrium indefinitely into the future. But why should the initial value of money be given by history in Example 1, if not the initial value of land in Example 3? This is particularly puzzling if one considers that Example 1 is a limiting case of Example 3, in which the land becomes completely unproductive but the title to land is still exchanged as a storable means of payment. And in Example 4 one finds again that the initial price of land must be given by history, if equilibrium is to be unique; but why in the case of preferences like those in Example 4 if not in the case of preferences like those in Example 3?

Geanakoplos and Polemarchakis [1983] calculate the effects of policy interventions, using the method of comparative statics, in the case of an overlapping generations model for which perfect foresight equilibrium is indeterminate (a more complicated version of Example 6). They resolve the indeterminacy by assuming that the initial period's nominal wage is predetermined (allegedly a "Keynesian" assumption); regardless of the policy intervention chosen, the perfect foresight equilibrium that is selected is the one with initial nominal wage w_o. However, they are sensitive to the objections raised here, and note that while there exists a rational expectations equilibrium in which this is true (and is expected by all agents to be true, in advance of the policy intervention), there also exist many other rational expectations equilibria in which other perfect foresight equilibria are picked out after the intervention. (For example, other prices or quantities might as well be predetermined, rather than the nominal wage). They are also careful to note that such a rational expectations equilibrium is only possible if there exist perfect foresight equilibria consistent with the same initial nominal wage w_o under all the different possible interventions; thus, for example, the existence of an uncountable set of perfect foresight equilibria is crucial in order for them to be able to stipulate the "Keynesian" predetermined nominal wage. With these qualifications, their analysis is correct. But it is not a resolution of the indeterminacy problem, in the absence of an explanation of why that particular rational expectations equilibrium is the "correct" one, rather than any other member of an extremely large set of equilibria. (There can even be an extremely large set of stationary rational expectations equilibria, in the case of stationary government interventions, as is shown by Farmer and Woodford [1984].) Hence policy evaluation is still not possible on the basis of a complete description of preferences, endowments, and technology alone; there is an additional role for the arbitrary expectations of agents, in picking out which rational expectations equilibrium actually occurs. (Of course, the predetermined nominal wage equilibrium considered by Geanakoplos and Polemarchakis may not represent a purely arbitrary expectation on the part of agents; perhaps reasons having to do with informational structure could be given that would explain why this equilibrium is feasible while others are not. But Geanakoplos and Polemarchakis offer no suggestions of this sort; nor is it clear that the introduction of "institutional detail" of that sort into the model would leave unchanged the perfect foresight equilibrium conditions that they assume must be satisfied.)

Another possible "technical" solution would rule out indeterminate equilibria by definition. That is, one might propose that equilibrium be defined to include only those perfect foresight equilibria such that no other perfect foresight equilibria exist which remain uniformly close to the given equilibrium. The justification might be a sort of "stability" argument--a candidate perfect foresight equilibrium would be considered "unstable" if a small perturbation in an expected price far in the future implies a very large change in the period one prices and allocations. (Perhaps a story about how agents come to have the common expectations required for a perfect foresight equilibrium could be told, such that the process would not converge to an equilibrium that is "unstable" in this sense.) This criterion would have the advantage of making the monetary steady state the unique equilibrium in Example 1.[30/]

Such a criterion might have some appeal if there always existed a unique determinate steady state, as in Example 1. But there need not exist any determinate steady state, as in Examples 2 and 4. In these cases, however, there exists a determinate 2-period cycle. (Or, if the offer curve bends back so sharply that the 2-period cycle is indeterminate, there exists a determinate 4-period cycle, and so on.) It might be thought sufficient that there exist some determinate equilibrium cycle, perhaps of a large period. The result of Grandmont [1983], that for a particular class of one-good models that he treats in details, no more than one determinate equilibrium cycle an exist, would seem encouraging.[31/] But there need not exist any determinate equilibrium cycle, as is shown by the following example.

Example 9. This is a special case of Example 2. Let the utility function be $u(y,z) = y + az - (b/2)z^2$, for allocations satisfying $z < a/b$. Then the perfect foresight equilibrium dynamics must satisfy $z_t = az_{t+1} - bz_{t+1}^2$. Consider the case $a = 4$. Then, using the change of variable $x_t = 2/\pi \text{ arc sin } \sqrt{bz_t/4}$, the perfect foresight equilibrium dynamics may be written

$$x_t = \begin{cases} 2x_{t+1} & \text{if} \quad 0 < x_{t+1} < \frac{1}{2} \\ 2 - 2x_{t+1} & \text{if} \quad \frac{1}{2} < x_{t+1} < 1 \end{cases}$$

It is easily verified that this economy possesses an infinite number of equilibrium cycles, including equilibrium cycles of every period. For example, the repeating sequence of values

$$x_t = \frac{2^{k-t(\text{mod } k)}}{2^k + 1}$$

is a k-period cycle, for any positive integer k. It is also easily verified that every cycle is unstable in the backwards perfect foresight dynamics (since the function which gives x_t as a function of x_{t+1} has slope ± 2 at all points except $x_{t+1} = 1/2$). Therefore every one of this infinite number of equilibrium cycles is indetermnate. Besides showing that perfect foresight equilibrium can be very indeterminate indeed, this example shows that the demand that equilibrium be determinate, to count as a proper equilibrium, may leave one with no equilibria at all.

On the other hand, the proposed criterion may fail to reduce the set of equilibria at all, even when that set is uncountably infinite. Consider the economy of Example 6, in the case that $\gamma < 2/(1 - \alpha)$. For such an economy, every perfect foresight equilibrium is determinate, in the sense that no other equilibrium remains uniformly close to it; any two perfect foresight equilibria diverge eventually. Yet for this economy, there exist an uncountably infinite number of perfect foresight equilibria. We conclude that there is no suitable formal criterion which resolves the problem of indeterminate equilibrium in a satisfactory manner.

2. Temporary Competitive Equilibrium

Another possible response would be to abandon perfect foresight equilibrium analysis altogether. The temporary competitive equilibrium analysis advocated by Grandmont [1983b], and others before him, would be an alternative. In the case of an economy like those of Examples 1 and 2, for instance, Grandmont proposes that the perfect foresight equilibrium condition

$$\frac{M}{p_t} + y\left(\frac{p_t}{p_{t+1}}\right) = 0$$

be replaced by the conditions

$$\frac{M}{p_t} + y\left(\frac{p_t}{p^e_{t+1}}\right) = 0 \tag{5.1}$$

$$p^e_{t+1} = \psi(p_t, p_{t-1}, \ldots, p_{t-T}) \tag{5.2}$$

That is, the market clears each period, but young agents act on the basis of expectations about the following period's price (p^e_{t+1}) that may or may not coincide with the price that clears the market in the following period (p_{t+1}). Agents' expectations in each period are a deterministic function of the price history; the forecasting function ψ is arbitrarily specified as a part of the model, it being considered one of "the important characteristics of the traders on the same level as preferences, endowments and the like" (p.21). Under relatively weak assumptions (the most important being a bound on the elasticity of p^e_{t+1} with respect to p_t), Grandmont proves the existence of a unique temporary competitive equilibrium price function, $p_t = W(p_{t-1},\ldots,p_{t-T})$, that solves (5.1) and (5.2). There is then no problem of indeterminacy of temporary competitive equilibrium; the past history of prices determines a unique sequence of future prices in a straightforward manner.

One justification for this approach is presumably the argument that there is no reason for rational agents to have correct expectations of future prices, outside of a stationary equilibrium. Thus Grandmont stipulates that his forecast function ψ be such that in a periodic equilibrium, expectations are always correct. However, the multiplicity of non-stationary perfect foresight equilibria can be understood to represent a multiplicity of <u>stationary</u> rational expectations equilibria. Let us suppose that a certain economy with an indeterminate steady state is subject to stochastic shocks that occur very infrequently (i.e., the mean frequency of shocks is low compared to the rate of convergence of perfect foresight equilibria to the indeterminate steady state). Then in any stationary rational expectations equilibrium of that economy, in which no random variables affect allocations except the infrequent random shock just mentioned, the dynamic path followed by prices and allocations in the time between any two shocks will be approximately a perfect foresight equilibrium path of the certainty economy. And the multiple non-stationary perfect foresight equilibria will indicate the many different paths the economy can take, following a shock, in a stationary rational expectations equilibrium. Hence a person who believes that expectations need only be correct in a stationary equilibrium many nonetheless believe that the multiplicity of perfect foresight equilibria in a case like Example 2 represents a genuine multiplicity of possible equilibria. This, in our view, reduces substantially the appeal of the temporary competitive equilibrium approach.

It should also be emphasized that if one adopts the temporary competitive equilibrium approach, one must give up all hope of finding conditions under which the welfare theorems of Walrasian equilibrium theory are valid. Even if the sequence of temporary competitive equilibria converges to a Pareto optimal steady state, the complete intertemporal allocation will not be Pareto optimal, so that there could be a role for Pareto-improving active policy outside the steady state. Furthermore, the sequence of temporary competitive equilibria need never converge to any stationary equilibrium in which expectations are correct. Grandmont proves (for the one-good model with equilibrium conditions (5.1) and (5.2)) that a perfect foresight equilibrium cycle must be locally stable in the temporary competitive equilibrium dynamics if:

(1) The forecast function ψ is consistent with all stationary equilibria of period k; i.e., given any periodic price history of period k, ψ forecasts that the periodic sequence will continue;

(2) The forecast function ψ is continuously differentiable, and its elasticity with respect to its first argument satisfies a certain bound;

(3) The forecast function ψ is non-decreasing in all its arguments, near any price history of period k; and

(4) The cycle of period k is a determinate perfect foresight equilibrium.

But some economies have no determinate equilibrium cycle, as is shown by Example 9. And even in the case of economies with a determinate cycle, there is no single forecast function ψ which satisfies properties (1), (2), and (3) for all possible cycle lengths. Hence there must be a fortunate coincidence between the forecast function ψ that agents happen to use, and the determinate periodic equilibrium that results from their preferences and endowments, if even local convergence of the sequence of temporary competitive equilibria is to be guaranteed.

3. Indeterminacy and Stabilization Policy

Another possible perspective would assume that perfect foresight equilibrium is the correct equilibrium concept, when perfect foresight equilibrium is determinate; but would hold that determinacy is a desirable state of affairs, and a reasonable object of economic policy. There are various possible reasons for wanting a determinate equilibrium. One might believe that it is possible for rational agents to compute the perfect foresight equilibrium, and so come to have the correct (and common) expectations necessary for the perfect foresight equilibrium to occur, when that equilibrium is at least locally unique, but believe that agents would be unable to coordinate their expectations in this way when perfect foresight equilibrium is indeterminate. Thus one might desire determinacy in order to insure that a perfect foresight equilibrium could in fact be achieved. This rationale is problematic for two reasons. First of all, the story about how one supposes that perfect foresight equilibria are achieved needs to be spelled out in greater detail, so that it can be verified that indeterminacy prevents any of the infinite set of indeterminate equilibria from being reached. Second, most economies which have one or more sets of indeterminate equilibria also have some determinate equilibrium. Thus, in Example 2, even though the steady state is indeterminate, the 2-period cycle cycle may well be determinate; if it is not, there exists a 4-period cycle that may well be determinate, and so on. From the point of view just suggested, the indeterminate monetary steady state in Example 2 would not mean that no perfect foresight equilibrium could be reached, only that it would not be one of the equilibria converging to the monetary steady state. Instead, perhaps, the determinate 2-period cycle would have to be considered the only attainable equilibrium of the economy; but this provides no justification for a stabilization policy, since that equilibrium is Pareto optimal.

A better reason for policy intervention to prevent indeterminacy of equilibrium is the desire to prevent sunspot equilibria. If, as conjectured in Section IV, indeterminacy always implies the existence of a large number of stationary sunspot equilibria, then--since sunspot equilibria are always inefficient--indeterminacy is undesirable. What one would really want, on these grounds, is a policy that could guarantee the impossibility of sunspot equilibria. But if the conjecture of Section IV is correct, eliminating the indeterminacy of a steady state will at least guarantee the impossibility of sunspot equilibria close to the steady state. This approach also provides a rationale for Grandmont's [1983b] interest in active policies to prevent equilibrium cycles. The equilibrium cycles themselves are Pareto optimal, in the model he considers; but if cycles exist, then stationary sunspot equilibria exist as well, as shown by Azariadis and Guesnerie [1984].

The stabilization policy proposed by Grandmont [1983b], for an economy like that of Example 2, works as follows. The monetary authority pays out new money to the old each period in proportion to the money balances they held at the beginning of the period. If the rate of nominal interest payments were predetermined, the set of perfect foresight equilibrium allocations would be unchanged by any such policy, as the proportional increase in money holdings could always be offset by a proportional increase in the price level. Grandmont proposes, however, that the rate of nominal interest payments be made a function of the price level in the period in which the payments are made, in addition to depending upon the history of the economy. Monetary policies of this sort are not neutral. The particular money supply rule he suggests is

(5.3) $$M_{t+1} = p_{t+1}\left(\frac{M_t}{p_t}\right)g\left(\frac{p_t}{p_t^*}\right)$$

where M_t is the nominal money stock in period t, p_t is the money price of the consumption good in period t, $\{p_t^*\}$ is a target sequence of prices chosen by the authority (a constant rate of inflation, which will correspond to the monetary steady state, with a constant rate of growth of the money supply through interest payments on money balances), and g is a smooth function satisfying $g(1) = 1$, $g'(v) < 0$ and $|g(v) - 1| < \varepsilon$ for all $v > 0$, where ε is a small positive number. The period one money stock is chosen so that $M_1/p_1^* = -y(1)$. That is, the target price level for period one will correspond to the level of real balances associated with the monetary steady state.

If a perfect foresight equilibrium exists under such a policy, the real rate of return on money balances held between periods t and $t + 1$ will be $g(p_t/p_t^*)$. But then the equilibrium condition in period t will be

$$\frac{M_t}{p_t} + y(g(\frac{p_t}{p_t^*})) = 0 \tag{5.4}$$

In period one, $p_1 = p_1^*$ satisfies this condition. Furthermore, it is the unique solution. For in Example 2, the offer curve is backward bending in the area of the monetary steady state, so that $y(g)$ is an increasing function of g for g near one; for ε chosen small enough, g can only vary over an interval on which $y(g)$ is an increasing function, so that $y(g(p_t/p_t^*))$ is a decreasing function of p_t. Hence the left hand side of (5.4), for $t = 1$, is a decreasing function of p_1, and the solution $p_1 = p_1^*$ is unique. But if in any period $p_t = p_t^*$, it follows that

$$\frac{M_{t+1}}{p_{t+1}} = \frac{M_t}{p_t} = -y(1)$$

so that $p_{t+1} = p^*_{t+1}$ will be the unique equilibrium price in period $t + 1$ as well. Hence the unique perfect foresight equilibrium is $p_t = p^*_t$ for all t.

Grandmont's object is to show that statilization policy can rule out equilibrium cycles. But it is clear that the proposed policy rules out indeterminacy as well. Furthermore, it prevents the existence of sunspot equilibria in this model, stationary or otherwise. For under the policy rule given by (5.3), the rate of return to holding money balances in period t is known with certainty, regardless of whether p_{t+1} is stochastic or not; but then (5.4) determines p_t uniquely, as discussed above. There is thus no point at which the realization of a sunspot variable can affect any prices or allocations.

Nor is this the only kind of active policy that can render the monetary steady state determinate in Example 2. Suppose that, in accordance with many familiar treatments of monetary policy, the monetary authority is not allowed to react so quickly to price movements as is required by the Grandmont policy. Suppose, in particular, that the money stock in period $t + 1$ must be determined by the history up to and including period t. Then if money supply variations occur through interest payments on money balances, monetary policy cannot alter the set of perfect foresight equilibria. If, on the other hand, the money supply is altered by means of lump-sum taxes and transfers, monetary policy has real effects.

Suppose that the transfers (if any) are made to the young each period, and similarly that taxes (if any) are levied upon the young. Then the budget constraint of a member of generation t is

$$p_t' y_t + p_{t+1}' z_{t+1} \leq M_t - M_{t-1}$$

Let the excess demand function of a young agent then be $y(p_t/p_{t+1}, (M_t - M_{t-1})/p_t)$. As before, let us assume that the excess demand function is smooth. The perfect foresight equilibrium condition for period t, in the case of active policy, is

$$\frac{M_{t-1}}{p_t} + y\left(\frac{p_t}{p_{t+1}}, \frac{M_t - M_{t-1}}{p_t}\right) = 0 \tag{5.5}$$

Let us consider a particularly simple class of monetary policies, $M_t = f(p_{t-1})$. Then (5.5) becomes

$$\frac{f(p_{t-2})}{p_t} + y\left(\frac{p_t}{p_{t+1}}, \frac{f(p_{t-1}) - f(p_{t-2})}{p_t}\right) = 0 \tag{5.6}$$

This is a third-order nonlinear difference equation for p_t. Let us suppose that there exists a price level $p^* > 0$ such that $f(p^*) = -p^* y(1,0)$. This price level corresponds to the monetary steady state.

The determinacy of the steady state is investigated by linearizing (5.6) around it. This yields

$$(1 - y_2)f'[p_{t-2} - p^*] + (y_2 f')[p_{t-1} - p^*] + (y_1 + y)[p_t - p^*]$$
$$- y_1[p_{t+1} - p^*] = 0$$

where y_1 and y_2 are the partial derivatives of y with respect to its first and second arguments, and y, y_1, y_2, f' are evaluated at the monetary steady state. We are therefore interested in the roots of the characteristic polynomial

$$P(\lambda) \equiv (-y_1)\lambda^3 + (y_1 + y)\lambda^2 + (y_2 f')\lambda + ((1 - y_2)f')$$

It follows from concavity of the utility function that $y_1 < yy_2$. The assumption that both periods' consumption are normal goods implies $0 < y_2 < 1$ and $y_1 + y < 0$. The assumption that the offer curve bends backward sharply enough for the monetary steady state to be indeterminate (under a passive monetary policy) implies $y_2 < 1/2$, $y_1 > 0$, and $2y_1 + y > 0$. These inequalities imply that $P(\lambda)$ has only real roots for all $f' > 0$. One root is zero for $f' = 0$, between zero and one for $0 < f' < f/p^*$, and greater than one for $f' > f/p^*$. A second root is zero for $f' = 0$, and between zero and $-(1 - y_2)/y_2$ (hence between zero and -1) for all $f' > 0$. The third root is equal to $(y_1 + y)/y_1$ for $f' = 0$, between $(y_1 + y)/y_1$ and -1 (hence between zero and -1) for $0 < f' < (2y_1 + y)/(2y_2 - 1)$, and less than -1 for $f' > (2y_1 + y)/(2y_2 - 1)$. It follows that the stable manifold of the monetary steady state is of dimension three for $0 < f' < (2y_1 + y)/(2y_2 - 1)$, of dimension two for $(2y_1 + y)/(2y_2 - 1) < f' < f/p^*$, and of dimension one for $f' > f/p^*$. (The borderline cases will not be treated here, as they require consideration of higher-order terms in the Taylor expansion of the equilibrium conditions (5.6).)

The perfect foresight dynamics must satisfy two initial conditions, in addition to satisfying (5.6) for $t = 3,4,\ldots$, namely

$$\frac{M_0}{p_1} + y\left(\frac{p_1}{p_2}, \frac{M_1 - M_0}{p_1}\right) = 0$$

$$\frac{M_1}{p_2} + y\left(\frac{p_2}{p_3}, \frac{f(p_1) - M_1}{p_1}\right) = 0$$

where M_0 is the nominal money holdings of generation zero at the beginning of period one, and M_1 is the nominal money holdings of generation one. (The monetary authority gives the young of generation one a lump-sum transfer in the amount $M_1 - M_0$; this transfer must be specified in addition to the function $f(p)$, since there is no previous period's price to use in period one. Alternatively, if one supposes that M_0 and M_1 are given by the same function $f(p)$, using historical prices p_{-1} and p_0, then there are again two initial conditions, the historical values of p_{-1} and p_0.) Hence a two-dimensional stable manifold is required for exact determinacy, and the monetary steady state is

- indeterminate under policies with $0 < f' < (2y_1 + y)/(2y_2 - 1)$;
- exactly determinate under policies with $(2y_1 + y)/(2y_2 - 1) < f' < f/p^*$;
- unstable under policies with $f' > f/p^*$.

Thus for an economy like that of Example 2, while a passive monetary policy (which means $f' = 0$) means that the monetary steady state will be indeterminate, there exist active policies under which the monetary steady state will be determinate, even if the transfers or taxes in each period are required to be predetermined. Note that in the notation used above, the slope of the offer curve through the monetary steady state in Figure 2 is $S = (y_1 + y)/y$. Thus, as S approaches -1, $2y_1 + y$ approaches zero, so that the lower bound on f' consistent with determinacy approaches zero. The degree to which M_t must respond to the previous period's price level is greater the more flat the offer curve at the monetary steady state; if the slope is more negative than -1, $f' = 0$ is consistent with determinacy.

Note also that the policy needed in order to insure determinacy of the monetary steady state for a given economy need not be finely adjusted to the properties of the economy. There is an entire range of possible policies that will suffice, for any given economy; hence no very precise knowledge of the preferences and endowments of agents is needed by the monetary authority in order to insure determinacy.

It is not easy to check whether stationary sunspot equilibria exist under an active policy of the sort just considered. Certainly they exist in the case of a passive monetary policy, for in that case there exist two-state Markov process equilibria, as shown in Section IV. Two-state Markov process equilibria do not exist in the case of the active policies, because of the lags introduced into the equilibrium conditions by the dependence of each period's money supply on the previous period's price level. But this does not mean that stationary sunspot equilibria of other sorts do not exist; more powerful methods will be required to address this question. If the conjecture of Section IV is correct, then the policies that suffice to render the monetary steady state determinate in the class of perfect foresight equilibria will also rule out stationary sunspot equilibria near the monetary steady state. There might still exist other stationary sunspot equilibria. This is a topic for further research.

Conclusion

It has been shown that the problem of indeterminate perfect foresight equilibrium in overlapping generations models is robust. The problem is not simply a reflection of pathological aspects of a model of fiat money in which such money serves only as a store of wealth; it can occur in models without money, and in the case of monetary equilibria in which money has a value bounded away from zero forever. It can occur in production economies as well as exchange economies, in economies with non-depreciating assets as well as in economies where all commodities are perishable, and even in economies in which some agents behave like infinite lived consumers.

On the other hand, and just as important, it has been shown that indeterminacy is not ubiquitous. There exist robust examples of infinite horizon models with overlapping generations of consumers in which there do not exist multiple perfect foresight equilibria (e.g., Example 3 of Section I). This is important because it allows for the possibility of using determinacy of equilibrium as a criterion for the choice of policy regime. For among the parameters which determine whether equilibrium is indeterminate or not may well be parameters which are policy instruments. The two examples in Section VI show that this may be the case.

This survey has also suggested, through a consideration of several examples, that indeterminacy of perfect foresight equilibrium is a sufficient condition for the existence of stationary sunspot equilibria. This is an important topic for further research. The connection, if it can be established in general, will be important for several reasons. On the one hand, it would provide further reasons for regarding indeterminacy of perfect foresight equilibrium as a serious problem, and for seeking to insure determinacy through active policy if necessary. It shows that indeterminacy is a problem even if one believes that agents can only be expected to have correct expectations in the case of a stationary equilibrium; that the possible influence of self-fulfilling expectations upon equilibrium allocations in the case of indeterminacy is not a merely transient one; and that indeterminacy implies the existence of a large number of inefficient rational expectations equilibria. On the other hand, it would allow many things to be learned about stationary sunspot equilibria--conditions for their existence, types of policies that make them possible or that rule them out -- through an analysis of the much more tractable problem of perfect foresight equilibrium, of the sort taken up in Section II.

Footnotes

*My understanding of the issues treated here has developed over the course of numerous conversatins with Costas Azariadis, Jess Benhabib, Guillermo Calvo, Roger Farmer, John Geanakoplos, Jean-Michel Grandmont, Tim Kehoe, David Levine, Walter Muller, Heraklis Polemarchakis, and Karl Shell. Errors and eccentricities of the present exposition are mine alone. I would also like to acknowledge the generous support of the John D. and Catherine T. MacArthur Foundation.

1/ Gale makes a similar distinction between the two steady states, but he calls the non-monetary steady state stable and the monetary steady state unstable, and appears to believe that for this reason the monetary steady state should be disregarded, as unlikely to actually be reached. This interpretation of perfect foresight equilibrium dynamics is discussed in Section V.

2/ Certain general propositions mentioned below, however, might fail to be true under a more comprehensive definition of indeterminacy. For example, it may be established that a Pareto optimal steady state cannot be indeterminate in the sense proposed here unless income effects are sufficiently important. Yet there may exist an uncountable number of perfect foresight equilibria in a model where income effects are modest, that do not converge asymptotically to the steady state. See the discussion of Example 6.

3/ A common justification for the inclusion of infinite-horizon maximizers is the Barro [1974] suggestion that if agents care about the welfare of their descendants, and expect them in turn to care about the welfare of their own descendants, and so on, the intertemporal consumption and savings choices of the members of such a "dynasty" may mimic those of a single infinite-lived consumer.

4/ The "land" considered here is thus different from the asset that exists in the first example of Calvo [1978]. Calvo's land is a non-depreciating fixed factor of production, but labor input is also required for production. Scheinkman [1980] considers an asset which is the same as Calvo's land. The "land" considered here is one of the types of "rent" considered by Tirole [1984].

5/ For a proof in the case of a general stationary overlapping generations exchange economy, see Muller and Woodford [1983]. In the language of Tirole [1984], the existence of "rents" of this sort rules out the existence of speculative "bubbles", of which valued fiat money would be an example. Tirole provides further discussion of some situations in which valued fiat money can coexist with "rents". He shows that there may exist equilibria with valued fiat money if the population is growing while the amount of land in existence is fixed, if there is technological progress, or if the quantity of "rents" in existence grows at the same rate as the population but the new "rents" in each period are part of the endowment of agents born in that period.

6/ For a proof that the existence of any amount of land rules out inefficient equilibria in the overlapping generations model with constant population, see Muller and Woodford [1983]. Tirole [1984] states a similar result in the context of his model, and Karl Shell advises us that this result has attained the status of a "folk theorem." Scheinkman [1980] proves a similar result, for a one-good model, in the case of a non-depreciating fixed factor of production. A theorem of this sort applicable to non-stationary economies as well is given by Wilson [1981], if one interprets "land" as the endowment of an infinite lived agent, as discussed below. Wilson shows that if there exists an $\varepsilon > 0$ such that the yield of the land is more than fraction ε of the total available good each period, equilibrium will be Pareto optimal.

7/ In fact, the basic idea of the example, that inflation-financed government expenditure can lead to indeterminacy of perfect foresight equilibrium, is due to Black [1974], who demonstrates this in the case of an ad hoc money demand function, rather than in an overlapping generations model.

8/ Sargent calls steady state z^{**} "stable", and z^{*} "unstable", thus following Gale's usage, and seeming to imply that z^{**} is the steady state to which the economy must tend. In this, curiously enough, he ignores the arguments of Sargent and Wallace [1973].

9/ Note, however, that the indeterminate monetary steady state in this example is not Pareto optimal. In fact, as is discussed in Section II, it is necessary that income effects be important if a Pareto optimal steady state is to be indeterminate.

10/ Geanakoplos and Polemarchakis discuss the perfect foresight equilibrium that exist for this model with valued fiat money.

11/ Gale [1973] distinguishes these two classes, calling the former "golden rule" steady states, and the latter "balanced" steady states. Grandmont [1983a] follows this terminology. Kehoe and Levine [1982, 1983a, 1983b] call the former "nominal" steady states, and the latter "real" steady states. We follow here the terminology of Muller and Woodford [1983, 1984], which coincides with the traditional distinction between "monetary" and "non-monetary" perfect foresight equilibria. It would perhaps be more accurate to characterize the "non-monetary" steady states as equilibria in which there is no outside money; agents may lend to and borrow from one another in a steady state of this sort.

12/ The matrix G in Kehoe and Levine [1982, 1983b] is closely related to this one, but not identical. Their matrix is the linearization of the mapping which gives (p_2, p_3) as a function of (p_1, p_2). Hence stable eigenvalues, for them, are eigenvalues of modulus less than β, rather than eigenvalues inside the unit circle. The notation used here allows use of the more familiar stability criterion.

13/ The criterion for Pareto optimality used here is a special case of that derived in Balasko and Shell [1980].

14/ This argument assumes that $1/\beta$ is not a repeated eigenvalue. Kehoe and Levine [1983b] show that generically the eigenvalues of G are distinct.

15/ In fact, $q_t'a - p_t'z(p_{t-1}, p_t) = 0$ along any perfect foresight equilibrium path, regardless of whether it converges asymptotically to a steady state. This is why a monetary equilibrium is impossible when there exists non-depreciating land and population is constant. See Muller and Woodford [1983].

16/ Various qualifications must be made to this claim. For example, if there exist production technologies which allow consumption goods within a single period to be transformed into one another, and these technologies are used in the steady state under consideration, then each such technology effectively reduces the number of goods per period by one, and as a result the upper bound on the possible dimension of indeterminacy is correspondingly reduced. As another example, if there exist independent constant returns to scale production technologies, all used in the steady state under consideration, equal in number to the number of goods per period, then the conditions for each of these technologies to earn zero profits each period suffice to determine the entire sequence of prices, given any initial prices p_1. It can then be the eigenvalues of G must be such that exactly half have modulus less than $\beta^{-1/2}$, as in the literature on "Hamiltonian dynamical systems" in optimal growth theory. This makes indeterminacy impossible, in the case of Pareto optimal steady states $(\beta < 1)$. However, the existence of any number less than n of independent technologies used in the steady state, implies no restrictions on the possible dimension of indeterminacy. See Muller and Woodford [1984].

17/ Muller and Woodford [1983] prove that <u>some</u> steady state must always exist for a stationary economy with both finite and infinite lived agents, but it need not be one in which any of the infinite lived agents consume. It should be obvious that a steady state in which infinite lived agents consume will not be possible, if they all discount the future to a sufficiently great extent.

18/ If there exist $H > 1$ infinite lived agents with distinct preferences and endowments, then there will remain $H - 1$ budget constraints that must be imposed as equilibrium conditions, in addition to the conditions corresponding to (2.12) and (2.13). However, because there are an additional $H - 1$ state variables in that case as well (a distinct λ_t for each infinite lived agent), the results derived below as to the possible dimension of indeterminacy are unchanged. See Muller and Woodford [1984].

19/ Muller and Woodford [1984] extend this analysis to the case of an economy with an arbitrary number of production technologies. In the case with production, one need not have $n^s = n$, but one can show that exactly half the eigenvalues will have modulus less than $\beta^{-1/2}$, so that $n^s < n$. Thus indeterminacy remains impossible, although it is possible for the steady state to be unstable, as in the example of Benhabib and Nishimura [1984], discussed in Section III.

20/ Grandmont speaks not of determinacy, but of the stability of the backward perfect foresight dynamics. Thus his "ϕ-stability" corresponds to determinacy of equilibrium, in the terminology of this paper. He is not concerned with determinacy because he does not consider the perfect foresight dynamics to represent the actual dynamics of the economy, other than in the case of a stationary equilibrium. He considers the "ϕ-stability" of equilibrium cycles only as a device used to learn about the stability of temporary competitive equilibrium dynamics.

21/ However, there may be very many such equilibria. Grandmont [1983b] shows that there exist robust examples with an infinite set of periodic equilibria.

22/ Thus Calvo [1978] states that the problem of indeterminacy indicates that the "general equilibrium approach is not sufficient for uniquely determining the path of income distribution," but then notes that "since in our examples all equilibrium paths converge to the same steady state, they do not necessarily contradict the view that the Walras-Fisher approach is sufficient for uniquely determining income distribution in the long run." The examples of the present section, especially the reconsideration of Example 6, show that, to the contrary, the relative rewards of capital and labor may be indeterminate forever.

23/ He states the condition as one of "local stability" rather than of indeterminacy.

24/ Nor are finite-state Markov processes the only possible sunspot equilibria. Sunspot equilibria corresponding to Markov processes on a continuous state space can be shown to exist for this model using the method demonstrated in Farmer and Woodford [1984].

25/ I am indebted to Karl Shell for this insight.

26/ However, in that case, one of the 2-period deterministic cycles is indeterminate. Hence no counterexamples are known to the proposition that stationary sunspot equilibria exist only when there exists some indeterminate periodic equilibrium of period one or greater.

27/ This is presumably the reason for the usage of Gale [1973] and others, who describe indeterminate steady states of the sort- discussed in Section I as "stable." This point of view is explicitly assumed by Hahn [1982], and a related view is expressed by Geanakoplos and Polemarchakis [1983]. Both of these papers assume that the perfect foresight equilibrium that occurs is picked out by some predetermined initial price level, and that this price level is unchanged by a policy intervention that alters the set of equilbria. Geanakoplos and Polemarchakis, however, qualify their interpretation of this solution in ways discussed below.

28/ Hahn [1982], pp. 14-15.

29/ If one uses, instead, a temporary competitive equilibrium framework, as discussed in the following subsection, then stability is desirable--but in the temporary competitive equilibrium dynamics, not in the forward perfect foresight equilibrium dynamics. Indeed, Grandmont [1983b] shows that it is instability of the forward perfect foresight dynamics (i.e., determinacy) that guarantees stability of temporary competitive equilibrium dynamics for a wide class of forecasting functions. See the discussion in subsection 2 below.

30/ It would also correspond to a common practice in rational expectations macroeconomics (never clearly justified), of selecting the rational expectatins equilibrium path that converges to a steady state with the "saddle point" property as "the" equilibrium, even when there are no explicit equilibrium conditions (such as transversality conditions) that are violated by other paths. See the discussion by Calvo [1978], who seems to state that the rational expectations equilibrium methodology can only be applied to a model with a determinate equilibrium, so that the assumption that the world is like that becomes a testable implication of the rational expectations hypothesis.

31/ There is no reason, however, to expect such a result to be true for multi-good models. In fact, the class of economies Grandmont considers is somewhat restrictive, even among one-good overlapping generations models. The example of Figure 6b, for instance, shows an economy with two determinate cycles (the steady state and one of the 2-period cycles).

References

Azariadis, C. [1981], "Self-fulfilling Prophecies," Journal of Economic Theory 25, pp. 380-396.

Azariadis, C. and R. Guesnerie [1982], "Prophéties créatrices et persistance des théories," Revue Economique 33, pp. 787-806.

Azariadis, C. and R. Guesnerie [1984], "Sunspots and Cycles," CARESS Working Paper No. 83-22R, revised March 1984.

Bailey, M.J. [1956], "The Welfare Cost of Inflationary Finance," Journal of Political Economy 64, pp. 93-110.

Balasko, Y. [1983], "Extrinsic Uncertainty Revisited," Journal of Economic Theory 31, pp. 203-210.

Balasko, Y., D. Cass, and K. Shell [1980], "Existence of Competitive Equilibrium in a General Overlapping Generations Model," Journal of Economic Theory 23, pp. 307-322.

Balasko, Y. and K. Shell [1980], "The Overlapping Generations Model, I: The Case of Pure Exchange Without Money," Journal of Economic Theory 23, pp. 281-306.

Barro, R. [1974], "Are Government Bonds Net Wealth?" Journal of Political Economy 82, pp. 1095-1118.

Benhabib, J. and K. Nishimura [1979], "The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimum Economic Growth," Journal of Economic Theory 21, pp. 421-444.

Benhabib, J. and K. Nishimura [1984], "Competitive Equilibrium Cycles," C.V. Starr Center Research Report No. 83-30, revised January 1984.

Black, F. [1974], "Uniqueness of the Price Level in Monetary Growth Models With Rational Expectations," Journal of Economic Theory 7, pp. 53-65.

Brock, W.A. and J.A. Scheinkman [1980], "Some Remarks on Monetary Policy in an Overlapping Generations Model," in J.H. Kareken and N. Wallace, eds., Models of Monetary Economies, Federal Reserve Bank of Minneapolis, pp. 211-232.

Calvo, G. [1978], "On the Indeterminacy of Interest Rates and Wages with Perfect Foresight," Journal of Economic Theory 19, pp. 321-337.

Calvo, G. and D.A. Peel [1983], "Growth and Inflationary Finance: Variations on a Mundellian Theme," Journal of Political Economy 91, pp. 880-887.

Cass, D., M. Okuno, and I. Zilcha [1979], "The Role of Money in Supporting the Pareto Optimality of Competitive Equilibrium in Consumption-Loan Type Models," Journal of Economic Theory 20, pp. 41-80.

Diamond, P. [1982], "Aggregate Demand Management in Search Equilibrium," Journal of Political Economy 90, pp. 881-894.

Diamond, P. and D. Fudenberg [1983], "An Example of Rational Expectations Business Cycles in Search Equilibrium," unpublished manuscript, February 1983.

Farmer, R.E.A. and M. Woodford [1984], "Self-fulfilling Prophecies and the Business Cycle," unpublished manuscript, April 1984.

Gale, D. [1973], "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory 6, pp. 12-36.

Geanakoplos, J.D. and H.M. Polemarchakis [1983], "Walrasian Indeterminacy and Dynamic Macroeconomic Equilibrium: The Case of Certainty," unpublished manuscript, August 1983.

Grandmont, J.-M. [1983a], Money and Value: A Reconsideration of Classical and Neoclassical Monetary Theories, Cambridge University Press.

Grandmont, J.-M. [1983b], "On Endogenous Competitive Business Cycles," CEPREMAP Working Paper No. 8316, revised September 1983. Forthcomig in Econometrica.

Guckenhelmer, J. and P. Holmes [1983], Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag.

Hahn, F. [1982], Money and Inflation, Basil Blackwell, Oxford.

Irwin, M.C. [1980], Smooth Dynamical Systems, Academic Press.

Kehoe, T.J. and D.K. Levine [1982], "Comparative Statics and Perfect Foresight in Infinite Horizon Economies," MIT Working Paper No. 312, December 1982. Forthcoming in Econometrica.

Kehoe, T.J. and D.K. Levine [1983a], "Indeterminacy of Relative Prices in Overlapping Generations Models," MIT Working Paper No. 313, April 1983.

Kehoe, T.J. and D.K. Levine [1983b], "Regularity in Overlapping Generations Exchange Economies," MIT Working Paper No. 314, revised April 1983. Forthcoming in Journal of Mathematical Economics.

Muller, W.J. and M. Woodford [1983], "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: I. Existence of Equilibrium," MIT Working Paper No. 325, August 1983.

Muller, W.J. and M. Woodford [1984], "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: II, Determinacy of Equilibrium," MIT Working Paper No. 326, revised March 1984.

Peck, J. [1984], "On the Existence of Sunspot Equilibria in an Overlapping Generations Model," unpublished manuscript, April 1984.

Samuelson, P.A. [1958], "An exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," Journal of Political Economy 66, pp. 467-482.

Sargent, T.J. [1984], "Consumption, Loans, and Currency: II," unpublished lecture notes, January 1984.

Sargent, T.J. and N. Wallace [1973], "The Stability of Models of Money and Growth with Perfect Foresight," Econometrica 41, pp. 1043-1048.

Scheinkman J.A. [1980], "Notes on Asset Trading in an Overlapping Generations Model," unpublished manuscript, revised July 1980.

Shell, K. [1977], "Monnaie et Allocation Intertemporelle," CNRS séminaire de E. Malinvaud, November 21, 1977.

Spear, S.E. [1983], "Sufficient Conditions for the Existence of Sunspot Equilibria," unpublished manuscript, revised December 1983.

Tirole, J. [1984], "Asset Bubbles and Overlapping Generations," CERAS Working Paper No. 25, revised February 1984.

Wilson, C.A. [1981], "Equilibrium in Dynamic Models with an Infinity of Agents," Journal of Economic Theory 24, pp. 95-111.