Beyond Unbounded Beliefs: How Preferences and Information Interplay in Social Learning

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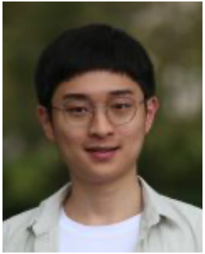


Motivation (1)

Sequential observational learning model (Banerjee '92; BHW '92)

- unknown state $\omega \in \Omega$
- each $n = 1, 2, \ldots$ takes action $a_n \in A$ (finite set)
  using private signal and (full) history of actions
- homogenous prefs $u(a_n, \omega)$

Many extensions, variations

Fundamental Q: does society eventually learn ω?

Received A: Unbounded vs. bounded beliefs (SS '00; AMF '21)

- $\forall \omega$, can posterior from a single signal $\approx \Pr(\omega) = 1$?
- $\forall \omega$, is posterior from single signal bounded away from $\Pr(\omega) = 0$?

Motivation (2)

Unbounded beliefs $\iff$ learning for **all** prefs

Bounded beliefs $\iff$ nonlearning for **all** (nontrival) prefs

Exhaustive (more or less) with two states $\rightsquigarrow$ most papers

But with multiple states, a significant gap

Suppose $\Omega = \{1, 2, 3\}$ and signals $\mathcal{N}(\omega, 1)$

- Can become certain about 1 or 3 but not 2
- Neither unbounded nor bounded!

So is there learning? Say for $u(a, \omega) = -(a - \omega)^2$

This Paper

For wide class of observational networks,

1. Excludability as a characterization of learning
   - simple cond over prefs & info
   - new perspective: learning requires agents' ability to displace wrong actions,
     not take the correct action (individually)
2. Permits study of learning for broad pref classes. Main application:
   - One-dim state: Single-crossing prefs & directionally unbounded beliefs
     covers quadratic loss, normal info e.g.
3. Methodology: General approach to learning + welfare

Literature

Most related

- Smith & Sørensen '00; Arieli & Mueller-Frank '21
- Acemoglu, Dahleh, Lobel, Ozdaglar '11; Lobel & Sadler '15

Other mechanisms for Bayesian learning

Non-Bayesian / Misspecified learning

Model

Environment

Countable set of states Ω ($|\Omega| \leq \infty$)

Signal space S (standard Borel)

- when MLRP is mentioned, both S and Ω are ordered

Signal/info structure $f(s|\omega)$ (R-N densities)

- no signal can exclude any state: $f(\cdot) > 0$

Action set A (standard Borel)

- can focus on finite

more general setup in paper: e.g., $\Omega = [0, 1]$ or non-full-support signals

The Game

Unobservable state ω drawn from prior pmf $\mu_0 \in \Delta\Omega$

Agents $1, 2, \ldots$ sequentially choose actions; each agent n observes both

- conditionally indep private signal $s_n \sim f(\cdot|\omega)$
- actions of all predecessors in her neighborhood $B(n) \subseteq \{1, \ldots, n-1\}$

$B(\cdot)$ defines social (observational) network structure (common knowledge)

- e.g., immediate predecessor or complete networks
- for talk, only **deterministic networks**; papers covers stochastic networks

Strategy $\sigma_n : S \times A^{B(n)} \to \Delta A$

All agents share bounded vNM utility $u : A \times \Omega \to \mathbb{R}$ (assm optimal action exists $\forall$ beliefs)

Bayes Nash equilibria (or refinements)

$\to$ no real strategic interaction

Learning

Full-information exp utility $u^*(\mu) := \sum_\omega \max_a u(a,\omega)\mu(\omega)$

Given prior μ_0 and eqm σ, agent n has ex-ante exp utility $\mathbb{E}_{\sigma,\mu_0} u_n$

Definition

There is adequate learning if for every prior μ_0 and every eqm σ, $\mathbb{E}_{\sigma,\mu_0} u_n \to u^*(\mu_0)$ as $n \to \infty$.

Adequate learning clearly impossible if

$$\exists K \in \mathbb{N} : |\{n : B(n) \subseteq \{1, \ldots, K\}\}| = \infty$$

Assumption (Expanding Observations)

$\forall K \in \mathbb{N}$, $|\{n : B(n) \subseteq \{1, \ldots, K\}\}| < \infty$.

Examples: complete and immediate predecessor networks (or any last M)

Under expanding obs, for what (u, f) is there adequate learning?

Example

Unbounded Beliefs

Given belief μ, let $\mu_s(\omega)$ be posterior after signal s

Definition

Signal structure has unbounded beliefs if $\forall \mu \in \Delta\Omega$ with full support, $\forall \varepsilon > 0$:

$$\forall \omega,\ \Pr\{s : \mu_s(\omega) > 1 - \varepsilon\} > 0.$$

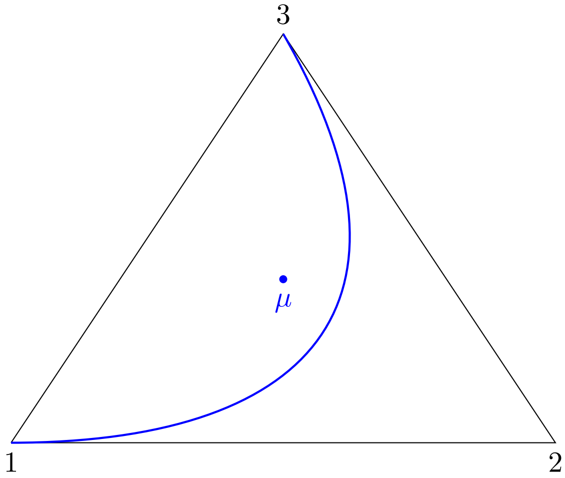
Unbounded beliefs $\implies$ adeq learning for all prefs

$\because$ every individual can take correct action

With only two states, adeq learning for **any** (nontrivial) pref $\implies$ unbounded beliefs

$\because$ if ω not distinguishable from ω', take prior $\mu_0(\omega') \approx 1$

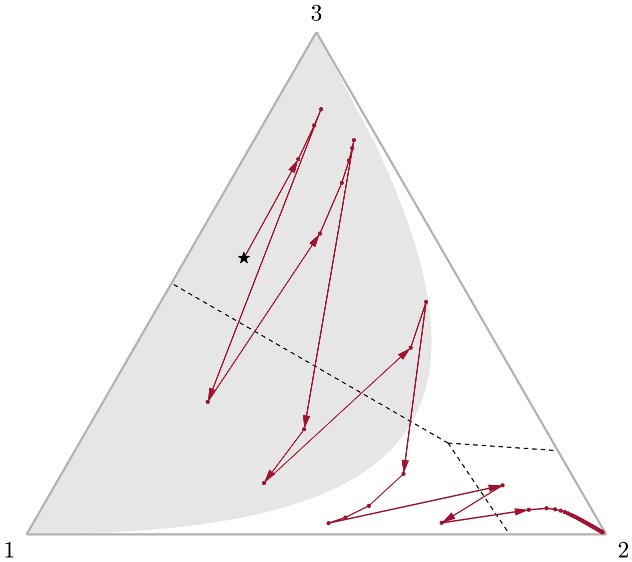
Learning without Unbounded Beliefs



Normal info: $s_n \sim \mathcal{N}(\omega, 1)$

fails unbounded beliefs

Learning without Unbounded Beliefs



Complete network

$\Omega = A = \{1, 2, 3\}$

$s_n \sim \mathcal{N}(\omega, 1)$

Consider realization $\omega = 2$

$\mu \in$ Gray region: no signal leads to correct action ($a = 2$)
$\rightarrow$ first few surely take wrong actions

But either wrong a can be displaced, eventually leading to correct action

Characterizations of Learning

Excludability

Definition

Ω' is distinguishable from Ω'' if $\forall \mu \in \Delta(\Omega' \cup \Omega'')$ with $\mu(\Omega') > 0$, $\forall \varepsilon > 0$:

$$\Pr\{s : \mu_s(\Omega') > 1 - \varepsilon\} > 0.$$

$\rightarrow$ can become $\approx$ certain about Ω' relative to all of Ω'', simultaneously

$\rightarrow$ e.g., $\Omega = \{1, 2, 3\}$, $s \sim \mathcal{N}(\omega, 1)$:

can become certain about 2 vs 1 and 2 vs 3 separately, but not simultaneously

so 2 is not distinguishable from $\{1, 3\}$

Excludability

Definition

Ω' is distinguishable from Ω'' if $\forall \mu \in \Delta(\Omega' \cup \Omega'')$ with $\mu(\Omega') > 0$, $\forall \varepsilon > 0$:

$$\Pr\{s : \mu_s(\Omega') > 1 - \varepsilon\} > 0.$$

$\rightarrow$ can become $\approx$ certain about Ω' relative to all of Ω'', simultaneously

If each $\omega' \in \Omega'$ is distinguishable from Ω'', then so is Ω'.

So Ω' distinguishable from Ω'' if [and only if, for finite Ω]:

$\forall \omega' \in \Omega'$:

$\exists\ (s_i)$ s.t. $\forall \omega'' \in \Omega''$, $\lim_{i \to \infty} f(s_i|\omega'')/f(s_i|\omega') = 0$.

(writing as if S countable)

Excludability and Learning

Theorem

Excludability $\implies$ adeq learning $\forall$ choice sets. If Ω finite, also the converse.

For converse, consider binary choice sets and extreme prior

Say that μ is stationary if $\exists a$ that is optimal no matter the signal

Say that μ has adequate knowledge if $\exists a$ that is optimal $\forall \omega \in \operatorname{Supp} \mu$

Straightforward: adeq learning $\implies$ all stationary beliefs have adequate knowledge

$\because$ at a stationary prior, there can be an immediate info cascade

Theorem

(Fix any choice set.) Adeq learning $\iff$ all stationary beliefs have adequate knowledge.

Excludability thm follows $\because$ excludability $\implies$ any inadeq knowledge belief μ is not stationary

$\rightarrow$ $a^*(\omega) \succ_\omega a^*(\mu)$, so $a^*(\mu)$ will be displaced ... perhaps never by $a^*(\omega)$

Excludability vs Unbounded Beliefs

Though a joint cond on prefs & info, excludability can usefully separate prefs and info classes

Excludability for all prefs $\iff$ info has unbounded beliefs

i.e., each ω is distinguishable from $\Omega \setminus \omega$

Corollary: adeq learning for all prefs $\iff$ unbounded beliefs

But unbounded beliefs very demanding when $|\Omega| > 2$

Remark

Assume $|\Omega| > 2$. MLRP $\implies$ NOT unbounded beliefs.

recall normal info

Main Application:

One-Dimensional State with Single-Crossing Prefs

Single-Crossing Differences

Now let $\Omega \subset \mathbb{R}$

$h : \mathbb{R} \to \mathbb{R}$ is single crossing if $\text{sign}[h]$ is monotonic

Definition

Utility $u : A \times \Omega \to \mathbb{R}$ has single-crossing differences (SCD) if

$$\forall a, a' : \; u(a, \omega) - u(a', \omega) \text{ is single crossing in } \omega.$$

- à la Milgrom & Shannon '94, but no order on A (Kartik, Lee, Rappoport '23)
- implied by supermodularity if A ordered

Directionally Unbounded Beliefs

Definition

There is directionally unbounded beliefs (DUB) if every ω is distinguishable from $\{\omega' : \omega' < \omega\}$ and also from $\{\omega' : \omega' > \omega\}$.

But need not distinguish ω simultaneously from both lower and higher states

Under MLRP, DUB $\iff$ "pairwise distinguishability" (e.g., normal info)

SCD-DUB Result

Proposition

SCD prefs & DUB info $\implies$ adeq learning.

Proof sketch (for finite Ω)

SCD $\implies$ $\forall a, a'$, $\min\{\omega : a \succ a'\} > \max\{\omega : a' \succ a\}$
or vice-versa

DUB $\implies$ disjoint upper and lower sets are distinguishable from each other

Apply Excludability Thm

SCD-DUB Result

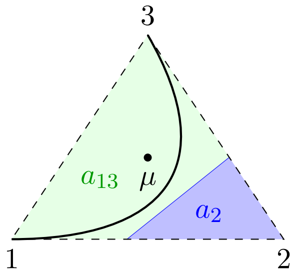
Proposition

SCD prefs & DUB info $\implies$ adeq learning. They are a minimal suff. pair (varying choice set).

Excludability for all SCD prefs $\implies$ DUB

→ Consider $a' \succ_\omega a''$ iff $\omega \geq \omega^*$. Excludability $\implies$ ω^* distinguishable from lower set

Absent SCD, excludability fails for some binary choice set under normal info ($\because$ MLRP)



μ is stationary and has inadeq knowledge

Application:

Multi-dimensional State with Intermediate Prefs

Multidimensional Application

- $\Omega, A \subset \mathbb{R}^d$
- Intermediate Prefs: $\forall a' \neq a''$, either $\Omega_{a',a''} = \emptyset$ or $\Omega_{a',a''} = \Omega$ or

$$\exists h \in \mathbb{R}^d \text{ and } c \in \mathbb{R} \text{ s.t. } \Omega_{a',a''} = \{\omega : h \cdot \omega > c\}.$$

Grandmont '78; Caplin & Nalebuff '88

e.g., **Weighted Euclidean**: $u(a, \omega) = -l((a - \omega)' W (a - \omega))$,
for some $d \times d$ sym. positive definite matrix W and str. $\uparrow$ loss function l

e.g., **CES**: $u(a, \omega) = (\omega_1 a_1^r + \cdots \omega_d a_d^r)^{1/r}$ with $r \neq 0$

- Location-shift info: $S = \mathbb{R}^d$, uniformly cts standard density $g : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$f(s|\omega) = g(s - \omega)$$

Say g is subexponential if $\exists p > 1$: $g(s) < \exp\left(-\|s\|^p\right)$ when $\|s\|$ large

e.g., g is multidim $\mathcal{N}(\omega, \Sigma)$

Multidimensional Application

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  $\exists h \in \mathbb{R}^d$ and $c \in \mathbb{R}$ s.t. $\Omega_{a',a''} = \{\omega : h \cdot \omega > c\}$.

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e.g., **CES**: $u(a,\omega) = (\omega_1 a_1^r + \cdots \omega_d a_d^r)^{1/r}$ with $r \neq 0$

Proposition

In this setting, there is excludability (hence adeq learning) if g is subexponential.

Intuition: $\{\omega : h \cdot \omega > c\}$ and $\{\omega : h \cdot \omega < c\}$ can be distinguished $\because$ g has thin tail.

Methodology

Backbone Result

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

Proof idea ($\Longleftarrow$): (elaborate)

1. If agent's social belief distr is not close to stationary, can achieve a min utility improvement
   - $\to$ $\Phi^S \subset \Phi^{BP} \subset \Delta\Delta\Omega$; and Φ^{BP} is compact (weak topology; $\Delta\Delta\Omega$ may not be compact)
   - $\to$ complement of ε-nbhd of Φ^S is a closed (hence compact) subset (Prohorov metric)
   - $\to$ exp utility / improvement is cts in belief, also cts on distrs

2. Expanding observations $\implies$ improvement principle: these min improvements propogate (e.g., consider immediate-predecessor network); so they can occur only finitely often
   - $\to$ eventually as if every agent has arb. close to stationary social belief
   - $\to$ eventual exp utility is at least that of the *worst* stationary belief distr: Theorem 3
   - $\to$ when all stationary beliefs have adeq knowledge, there is adeq learning

Backbone Result

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

Recall this characterization is for any given action set A

- Excludability is sufficient for learning; necess requires varying choice sets

Subsumes existing learning results (and "info diffusion"; Lobel & Sadler '15)

- Including "responsive prefs" with infinite action spaces (Lee '93; Ali '18)
  - → E.g., if $\Omega = \{0,1\}$, $A = [0,1]$, and $u(a,\omega) = -(a-\omega)^2$, then given any informative signal structure, only stationary beliefs are $\{0,1\}$
- Suppose only 2 states and finite actions, as much of the literature
  - → Adeq knowledge means knowing the state (modulo trivialities)
  - → So unbounded beliefs

Discussion

Most-Related Papers

Complete network: Smith & Sørensen '00 (two states)
Arieli & Mueller-Frank '21 (general)

- unbounded beliefs characterizes learning for all prefs
- AMF '21: "vanishing value of private information", analogous to our Backbone Lemma
  - → Martingale approach, which fails for general networks

General networks, but only **two states and two actions**

- Acemoglu, Dahleh, Lobel, Ozdaglar '11: introduce improvement principle approach
- Lobel & Sadler '15 introduce "info diffusion" (and correlated networks)
  - → Both rely critically on two states & actions to derive minimum improvement
  - → Our methodology using compactness/continuity works generally

Study of broad pref classes is new to social learning (but classical approach!)

- AMF '21 have example with a special utility

Conclusion

Std condition for learning, unbounded beliefs, very demanding with > 2 states

For a given pair of prefs and info, excludability characterizes learning
in general environment with social networks satisfying expanding observations

Permits a study of learning for canonical classes of prefs

- SCD prefs + DUB info
- Intermediate prefs + subexponential location-shift info

Beyond learning, general welfare bound

Interesting future directions:

- Other pairs of suff conds
- Heterogenous prefs
- Speed of convergence
- DUB in other contexts

Thank you!

More on Backbone Result

$\boxed{u_*(\mu_0) := \inf_{\varphi \in \Phi^S} u(\varphi)}$, where $\Phi^S \subset \Phi^{BP} \subset \Delta\Delta\Omega$ is set of Bayes-Plausible stationary distrs

Theorem

In any equilibrium σ, $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u_*(\mu_0)$.

When all stationary beliefs have adeq knowl, u_* is full-information utility, so adeq learning.

Proof idea: (ideas)

Φ^{BP} is compact, even when $\Delta\Delta\Omega$ is not ($\Delta\Delta\Omega$ metrized by Prohorov)

Fix small $\varepsilon > 0$ and let Φ^S_ε be an ε-nbhd of Φ^S

Expected improvement $I(\varphi)$ is cts, so attains minimum $\delta(\varepsilon) > 0$ over $(\Phi^S_\varepsilon)^c$ (closed hence compact)

Whereas for $\varphi \in \Phi^S_\varepsilon$, $u(\varphi) > u_* - \gamma(\varepsilon)$, with $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ (using unif cont of u)

By an improvement principle, $\liminf_n \mathbb{E}u_n \geq u_* - \gamma(\varepsilon)$ (this step adapts ADLO '11)

- E.g., consider immediate-predecessor network
- Each $\mathbb{E}u_n \geq \min\{u_* - \gamma(\varepsilon), \mathbb{E}u_{n-1} + \delta(\varepsilon)\}$
- Iterate

Result follows $\because$ $\varepsilon > 0$ is arbitrary