Letters to the Editor

A GUIDE TO THE APPLICATION OF LIMIT THEOREMS FOR SEQUENCES OF STOCHASTIC PROCESSES

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This is a bibliography for applications in operations research of the theory of weak convergence for sequences of probability measures on function spaces. The reasons for using this theory are mentioned, the major issues are outlined, and appropriate sources of information are provided.

RECENTLY, several stochastic models in operations research have been analyzed by considering limit theorems for sequences of stochastic processes in the context of the weak convergence theory for sequences of probability measures on function spaces. Examples in queuing theory, risk theory, and quality control appear in the papers by BOROVKOV (1967), IGLEHART (1965, 1969a,b), IGLEHART AND KENNEDY (1969), IGLEHART AND TAYLOR (1968), IGLEHART AND WHITT (1969a,c), and WHITT (1968, 1969b, 1970a). IGLEHART (1967) has also written a survey paper that discusses the basic method and its many possible applications.

The central idea is to use limit theorems for sequences of stochastic processes in applications. There are two ways in which this can be done. In the context of a sequence of random walks converging to a diffusion process, we can either use the limiting diffusion process as an approximation for a random walk, or we can use a random walk as an approximation for the limiting diffusion process. This example is typical in that a sequence of discrete-state, discrete-time processes are converging to a continuous-state, continuous-time process, but other possibilities can occur. Furthermore, both approaches may be useful. In a control problem, we may use stochastic control theory associated with a (continuous) diffusion process to approximate the control of a discrete process, or we may use dynamic programming associated with a discrete process to approximate the control of a continuous process. Recent surveys of the optimal stochastic control theory for continuous processes have been written by CHERNOFF (1968) and FLEMING (1969).

Given that we are considering limit theorems for sequences of stochastic processes, the central idea is to consider weak convergence for sequences of probability measures on function spaces; that is, we look at the stochastic processes as measures on function spaces and we consider modes of convergence appropriate to measures. Limit theorems for sequences of stochastic processes can be proved in other ways, but the weak convergence theory is particularly relevant because of the continuous mapping theorem [cf. reference 1, Theorems 5.1 and 5.5]. Weak convergence of a sequence of stochastic processes implies weak convergence of every sequence of stochastic processes or random variables that is obtained by a continuous mapping on the function space supporting the original stochastic processes. As a by-product, it is usually possible to obtain convergence of all the finite-dimensional distributions as well as limits for many related quantities. From the point of view of applications, weak convergence is useful, because it is thus often possible to obtain convergence for various cost or control features, as well as for the underlying probabilistic structure. An example is contained in the paper by IGLEHART AND TAYLOR (1968).

Until recently, it was necessary to refer to rather abstruse papers in order to gain access to the weak-convergence theory, but now an excellent account is available in the book by BILLINGSLEY (1968); *also see* PARTHASARATHY (1967), and GIKHMAN AND SKOROHOD (1969). However, even these sources are not entirely suitable for the more applied researcher, because the path to the applications is not as direct as it might be. The purpose of this letter is to provide a map leading directly through the main ideas to a position where applications of the weak convergence theory can be understood and carried out. We shall not provide an exhaustive survey of the weak-convergence theory, but, instead, we shall indicate the main points that should be visited on the way to the applications. A comprehensive bibliography of the theory has been provided by BILLINGSLEY (1968). We shall, however, try to give a comprehensive bibliography of the applications of the weak-convergence theory to stochastic models in operations research.

A MAP THROUGH THE WEAK-CONVERGENCE THEORY TO THE APPLICATIONS

(a) Sometimes we refer to random variables and stochastic processes and sometimes we refer to the measures they induce. In the case of stochastic processes, the measures are induced on a function space. The relation between random elements and the measures they induce is discussed in reference 1, pp. 22–24.

(b) As background to the study of general weak convergence, it is desirable to have in mind the possible modes of convergence for a sequence of random variables, in particular, convergence in law or in distribution. Useful sources are CHUNG (1968, Chapter 4, especially Section 4.4), GNEDENKO AND KOLMOGOROV (1968, Section 9), FELLER (1966, Chapter 8), LOÈVE (1963, pp. 178, 201), and LAMPERTI (1966, Section 12).

(c) Just as there are several possible modes of convergence for a sequence of random variables, so there are also several possible modes of convergence for a sequence of stochastic processes; in fact, there are more for stochastic processes. The most common modes of convergence are defined and compared in WHITT (1968, Section 3.1). It is worth noting that taking limits of sequences of stochastic processes is not the same as taking limits for single stochastic processes as time is allowed to go to infinity.

(d) There are many possible applications of limit theorems for sequences of stochastic processes. For further discussion, *see* IGLEHART (1967), KINGMAN (1965), and KARLIN AND MCGREGOR (1964).

(e) In order to use the weak-convergence theory, a detailed knowledge of topological and metric spaces is not necessary; the prospective user should not be intimidated. However, an acquaintance with the basic notions of metric spaces and function spaces is necessary to get into BILLINGSLEY (1968). An excellent source is SIMMONS (1963, Chapters 2–4). But again the potential user should not be intimidated. As BILLINGSLEY (1968, p. 6) says, nothing of functional analysis is assumed beyond an initial willingness to view a function as a point in a space.

(f) It is of some value to compare the more familiar notion of weak convergence of distribution functions with weak convergence of probability measures on metric spaces. It turns out that these concepts coincide when the metric space is the real line or the Euclidean space R^k [cf. reference 1, pp. 2, 17–18]. Thus, the importance of weak convergence as a relatively new concept occurs when we are considering more general spaces, such as function spaces supporting stochastic processes.

(g) It appears that the weak-convergence theory developed from an invariance principle due to ERDÖS AND KAC (1946). The invariance principle was used to prove functional central limit theorems [cf. references 1, p. 72 and 37, p. 31]. DONSKER (1950) then showed that it was possible to prove functional central limit theorems for a large class of functions all at once. Then PROHOROV, SKOROHOD, BILLINGSLEY, and others put these results in the context of measures on general spaces. For a brief historical survey, *see* references 37, pp. 31–34 and 1, p. 6. As a consequence of the development outlined above, papers on weak convergence often have titles involving 'an invariance principle for' or 'functional central limit theorems for.' Since the limits in weak convergence theorems often are diffusion processes, 'diffusion approximations for' also frequently appears in the title of a paper involving weak convergence. The name 'weak convergence' itself occurs because the convergence we have in mind corresponds to the weak topology on the space of all probability measures [cf. references 1, p. 16 or 37, pp. 37–38].

(h) There are other, sometimes heuristic, procedures to help determine what the limit for a sequence of stochastic processes should be. In particular, it is often necessary to transform the time scale, translate, and normalize, but it is not generally clear how to do so. In the context of random walks converging to Brownian motion, one approach is given in FELLER (1957, pp. 323–327). Another approach involves calculating infinitesimal means and variances, cf. IGLEHART (1967, p. 238). Still another, more complicated, approach may be found in ITÔ AND MCKEAN (1965, pp. 10–12).

(i) Before considering weak convergence of sequences of probability measures and different equivalent definitions of such convergence, it is desirable to consider different but equivalent representations for probability measures. This is the topic of BILLINGSLEY (1968, Section 1). Equivalent definitions for weak convergence are discussed in reference 1, Section 2 and Theorem 5.2.

Classes of sets that determine a probability measure uniquely (if we know the probability of each set in the class) are called determining classes. Classes of sets that determine weak convergence of a sequence or probability measures are called convergence-determining classes. Similar definitions apply to classes of real-valued functions on the probability space if we consider the expectation (integral) of these functions with respect to the probability measures and whether or not these expectations determine representation and convergence of the measures. Again, *see* reference 1, Section 2. For examples that show determining classes need not be convergence-determining classes, *see* reference 1, pp. 15, 20.

(j) There are a few very useful theorems associated with weak convergence. We have already mentioned one, the continuous mapping theorem. The most important theorems seem to be Theorems 4.1, 4.4, and 5.1 in reference 1. Also important are Theorems 3.1, 3.2, 5.2, and 5.5 in reference 1, Theorem 3.10 in reference 37, or Lemma 3.1 in reference 21, Theorem 1 in reference 22, and an approach using almost everywhere convergence, cf. SKOROHOD (1956), BREIMAN (1968, Section 13.9), and PYKE (1968).

(k) Limit theorems for sequences of stochastic processes often involve weak convergence of sequences of probability measures on function spaces. The two most common function spaces in the weak-convergence literature are $C[0, 1]$ and $D[0, 1]$. An investigation of these spaces makes up Chapters 2 and 3 of reference 1. The same theory applies to the spaces $C[a, b]$ and $D[a, b]$, where a and b are arbitrary constants. Function spaces with semi-infinite time intervals have been investigated by STONE (1963) and WHITT (1969a, 1970b). Any function space would do, but the function space must be a complete separable metric space in order to apply the results available in reference 1. Weak convergence on more general spaces has also been studied, cf. DUDLEY (1966) and VARADARAJAN (1961). The general technique for obtaining weak convergence on function spaces is described in reference 1, p. 35.

(l) There is a variety of functional central limit theorems that are perhaps the major results of the weak-convergence theory and that can be used to generate other results. The first functional central limit theorem is DONSKER'S theorem for partial sums of independent and identically distributed random variables [reference 1, Theorems 10.1 and 16.1]. Similar theorems exist for dependent sequences [reference 1, Chapter 4], partial sums from triangular arrays [reference 30, p. 220], random sums [reference 1, Section 17, reference 19, Section 2, and reference 40, Section 2], renewal processes and other counting processes [reference 1, Theorem 17.3 and reference 22, Theorem 1], and birth and death processes [reference 34]. For a rather compact discussion of such theorems, *see* reference 37, Section 3.2.5.

(m) The limit in the functional central limit theorems mentioned above is usually Brownian motion. There are many detailed and involved accounts of Brownian motion and general diffusion processes; a nice introduction that is sufficient for most purposes is KARLIN (1966, Chapter 10).

(n) With even only a rudimentary understanding of the points above, applications of the weak-convergence theory to stochastic models in operations research should be fairly easy to understand and not too difficult to construct. The most extensive work so far has been done with queues [cf. references 3, 15, 18, 19, 21, 23, 37, 39, and 40]. Most of the weak-convergence theorems for queues have been obtained by using the general functional central limit theorems in (l) together with the weak convergence tools in (j).

The weak-convergence theorems for queues have so far been mainly of two kinds. The first yields limit theorems for a single queue in heavy traffic, that is, when the queue is unstable and a steady-state is never achieved. The second yields limit theorems for sequences of queuing systems, which may be stable, to an unstable limit. The first approach gives a description of unstable queues and the second approach gives possible approximations for stable queues. Extensive surveys of earlier work on these two problems appear in KINGMAN (1965) and WHITT (1968, Chapter 2). We might add in closing that these approaches are not limited to heavy traffic, cf. IGLEHART (1965), IGLEHART AND KENNEDY (1969), and forthcoming work by IGLEHART AND KENNEDY.

There have been some, but fewer, applications of the weak-convergence theory to stochastic models other than queues; more can be expected in the future. Work is currently in progress to determine rates of convergence [cf. reference 37, Section 4.3], and computational evaluations of approximations [cf. GAVER (1968)], but much more needs to be done. Efforts are also being made to use the weak-convergence theory more for control problems in the manner of reference 20.

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A NOTE ON A DISTRIBUTION PROBLEM

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A certain class of distribution problems has been treated incorrectly and ambiguously, respectively, in two major and widely read texts on linear programming. This note traces the nature of this mistreatment to a paradoxical solution property of this class of problems, namely, even if all the shipping costs are nonnegative, an optimal solution may ship more than the minimum demand to a destination.

A VARIANT of the distribution model of linear programming that often arises in practical situations has received an erroneous algorithmic treatment in HADLEY,[3] and an ambiguous passage in SIMMONARD[4] is susceptible to an interpretation that supports the same incorrect approach. The approach to which we