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The $G_t/GI/s_t + GI$ Many-Server Fluid Queue: Longer Online Version with Appendix

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Abstract This paper introduces a deterministic fluid model that approximates the many-server $G_t/GI/s_t + GI$ queueing model, and determines the time-dependent performance functions. The fluid model has time-varying arrival rate and service capacity, abandonment from queue, and non-exponential service and patience distributions. Two key assumptions are that: (i) the system alternates between overloaded and underloaded intervals, and (ii) the functions specifying the fluid model are suitably smooth. An algorithm is developed to calculate all performance functions. It involves the iterative solution of a fixed-point equation for the time-varying rate that fluid enters service and the solution of an ordinary differential equation for the time-varying head-of-line waiting time, during each overloaded interval. Simulations are conducted to confirm that the algorithm and the approximation are effective.

Keywords queues with time-varying arrivals · nonstationary queues · transient behavior · many-server queues · deterministic fluid model · customer abandonment · non-Markovian queues

1 Introduction

Motivated by the need for tools to improve the performance of large-scale service systems, such as customer contact centers and healthcare systems, we

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introduce and analyze a deterministic fluid model that serves as an approximation for the many-server $G_t/GI/s_t + GI$ queueing model, which has customer abandonment (the $+GI$), time-varying arrival rate and staffing (the subscript t), unlimited waiting space, the first-come first-served service discipline and non-exponential service and patience distributions (the two GI's); see [3,37] and references therein for background on contact centers and healthcare systems, respectively. Abandonment is now recognized as an important feature, e.g., see [13,38]. Non-exponential service and patience distributions often do arise [8] and these features can strongly affect performance.

The analysis here applies to a system that alternates between overloaded (OL) and underloaded (UL) intervals. With time-varying arrival rates, such alternating behavior commonly occurs when it is difficult to dynamically adjust the staffing level in response to changes in demand. If the staffing cannot be changed rapidly enough, then system managers must choose fixed or nearly fixed staffing levels that respond to several levels of demand over a time interval. Then it may not be cost-effective to staff at a consistently high level in order to avoid overloading at any time. Then the fluid model introduced here may capture the essential performance.

Most queueing models are stochastic, because a primary cause of congestion is random fluctuation in arrivals and service. Deterministic fluid models can be useful when the systematic variation in the arrival rate and/or staffing dominates the stochastic variation in the arrivals and service, or at least is an important contributing factor. There is an established tradition of considering fluid models in queueing theory [15,29]. The present paper directly extends [36], which developed a deterministic fluid model to approximate the steady-state performance of a stationary $G/GI/s+GI$ queueing model. The accuracy of fluid models for capacity planning has been strongly supported by [5]. A novel feature here and in [36], compared to most fluid models, is that we consider a non-Markovian many-server fluid model, which involves two-parameter functions; e.g., the queue content at time t that has been in queue for a *duration* at most y, denoted by $Q(t, y)$, as a function of both t and y; see (2). The abandonment rate function and service completion rate function are driven by patience and service hazard-rate functions; see (7) and (9).

Our main goal here is to contribute to the techniques for analyzing service systems with the important and realistic feature of time-varying arrivals and staffing; see [14] for background. By focusing on the time-varying fluid model, we extend important work by Mandelbaum, Massey and Reiman [24], which established many-server heavy-traffic fluid and diffusion limits for the time-varying Markovian $M_t/M_t/s_t + M_t$ queueing model, and thus associated approximations; see also [25,26]. We make a significant step beyond [24–26] by considering non-exponential service and patience distributions as well as time-varying arrival rates and staffing.

Just as in [24–26], the approximations here are intended for systems with many servers and high arrival rate, so that mathematical support can be provided by many-server heavy-traffic limits. For the stationary Markovian $M/M/s + M$ model, such limits are established in [13]; for stationary non-

Markovian models, such limits are established in [18, 19]. A limit for a discrete-time model with time-varying arrival rates is given in §6 of [36]. However, here we do *not* establish stochastic-process limits. Instead, we are directly concerned with the fluid model itself. It is important to recognize that the fluid model can be considered directly as a legitimate model in its own right. By focusing on a continuous divisible quantity, which we call "fluid," our fluid model is a special case of a storage or dam model, as in [31].

Even though we do not establish stochastic-process limits here, the results here play an important role in a subsequent paper [22] in which we do establish such many-server heavy-traffic limits. The paper [22] establishes both a functional weak law of large numbers (FWLLN), showing convergence to the fluid model considered here, and a functional central limit theorem (FCLT), providing mathematical support for a refined Gaussian approximation, in the case of exponential service. The proof of the FWLLN, Theorem 4.1 in [22], uses the compactness approach, proving that the sequence is tight and that all convergent subsequences converge to the same limit. Theorem 3 here plays an important role in uniquely characterizing the limit of all convergent subsequences; see §6.6.2 of [22] for that part of the proof. The results in [22] also rely heavily on recent heavy-traffic limits for infinite-server queues in [30]. The connection to infinite-server queues plays a critical role here as well; see §§4, 5 and 7.1.

This paper makes important contributions even for the stationary $G/GI/s+GI$ fluid model introduced in [36]. Here we provide for the first time a full description of the transient behavior. The fundamental evolution equations, in (5) here, are the same as in (2.14) and (2.15) of [36], but the time-dependent performance when the system is overloaded actually depends on three features introduced for the first time here: First, for non-exponential service, the time-varying rate that fluid enters service is characterized as the unique solution to a fixed point equation; see (18) and Theorem 10. Second, the head-of-line waiting time is characterized here as the solution of an ordinary differential equation (ODE); see Theorem 3. Third, the potential waiting time, i.e., the virtual waiting time of an arrival at time t if that arrival would elect never to abandon, is characterized as the unique solution of an equation involving the head-of-line waiting time or by yet another ODE; see Theorems 5 and 6. To the best of our knowledge, none of this structure has been exposed previously.

There is an important modeling issue when we consider time-varying staffing. We need to carefully specify what happens when the service capacity is scheduled to decrease when all servers are busy. Do we require that customers in service stay in service with the same server until their service is complete? (Our analysis here applies to the case in which we allow the service in progress to be handed off to another available server.) Even with such server-assignment switching, there are issues: Do we alter the prescribed staffing function to avoid forcing a customer out of service? If we adhere to the given staffing function, as assumed here, then some customers are necessarily forced out of service in the stochastic system. (That can be prevented in the idealistic deterministic fluid model; see Assumption 4 and §9.) In the stochastic system, when customers

are forced out of service, which customers are forced out and what happens to them? Are these customers forced out of the system entirely? If so, is there service complete or do they retry? If customers are pushed back into the queue (as implicitly assumed in [24]), then where do they go in the queue, and what is their new abandonment behavior? Under regularity conditions, these realistic features will be asymptotically negligible in a many-server heavy-traffic limit, but these new considerations complicate the proof.

For the fluid model, we directly assume feasibility of the staffing function, but in §9 we show how to detect the first violation of feasibility of a staffing function and how to find the minimum feasible staffing function greater than or equal to the initial staffing function if that one is infeasible. In §10 we show how to construct a staffing function to stabilize delays at any fixed target value, contributing to prior work in [12,17].

The results have significant relevance for applications. First, service systems typically have arrival rates that vary significantly over time, and the results dramatically reveal the consequence, e.g., showing how the peak congestion lags behind the peak arrival rate, as discussed for the $M_t/GI/\infty$ stochastic model in [10,11]. Second, service systems often do have non-exponential service and patience distributions [8], and the results dramatically reveal the consequence. From [27,35,36,38], we know that the patience distribution beyond its mean has a significant impact. However, [35,36] show that the steady-state performance in the stationary $G/GI/s + GI$ model is relatively insensitive to the service-time cdf beyond its mean. In contrast, here we show that the service distribution beyond its mean can have a dramatic impact as well for the transient performance; see §2. Finally, the results in this paper have already been applied in [16] to create new effective real-time delay predictors for arriving customers in a service system with time-varying arrivals.

Here is how this paper is organized: We start in §2 by discussing an example, showing the results of the algorithm and how they compare to simulations of queueing systems. Next in §3 we carefully define the $G_t/GI/s_t + GI$ fluid model and specify key regularity conditions. In §4 we state important scale-proportionality results, which provide important simplification for UL intervals. In §5 we characterize performance during a UL interval.

In §6 we characterize the service content density during an OL interval. Subsections 6.1 and 6.2 are devoted to the special case of M service and non-M service, respectively. An explicit formula is available for M service; an iterative algorithm is developed for other cases. In §7 we characterize the queue performance functions. In §8 we summarize the resulting algorithm.

In §9 we show how to detect the first violation of feasibility of a staffing function and how to find the minimum feasible staffing function greater than or equal to any candidate one. In §10 we show how to construct a staffing function to stabilize delays at any fixed target value, In §11 we provide three postponed longer proofs, the proofs of Theorems 3, 5 and 6. Finally, in §12 we draw conclusions. Additional supporting material appears in a long appendix available from the authors' web sites.

2 An Example

We start with an example. We consider an $M_t/H_2/s + E_2$ fluid model with a sinusoidal arrival rate function: $\lambda(t) = 1+0.6sin(t)$, mean service time $1/\mu = 1$, mean patience $1/\theta = 1$, and fixed service capacity $s = 1$. (We consider other examples in the appendix.) Specifically, we let the service distribution be a two-phase hyperexponential (H_2) with probability density function (pdf)

$$g(x) = p \cdot \mu_1 e^{-\mu_1 x} + (1-p) \cdot \mu_2 e^{-\mu_2 x}, \quad x \geq 0,$$

with parameters $p = 0.5(1 - \sqrt{0.6})$, $\mu_1 = 2p\mu$ and $\mu_2 = 2(1-p)\mu$, which produces squared coefficient of variation (variance divided by the square of the mean) $c^2 = 4$. We let the patience distribution be Erlang-2 (E_2) with pdf

$$f(x) = 4\theta^2 x e^{-2\theta x}, \quad x \geq 0.$$

The E_2 distribution has $c^2 = 1/2$.

We relate the fluid model to associated queueing models by exploiting many-server heavy-traffic scaling, as discussed in [13,22,24,30,36]. Thus, the corresponding queueing model with n servers will have arrival rate function $\lambda_n(t) = n\lambda(t)$, $s_n = ns$ servers and the same service and patience distributions, Figure 1 shows plots of several key performance functions for $0 \leq t \leq T \equiv 17$, starting out empty, together with the specified arrival rate $\lambda(t)$: the head-of-line waiting time $w(t)$, the fluid content in queue $Q(t)$, the fluid content in service $B(t)$, the total fluid content in system $X(t) \equiv Q(t)+B(t)$, and the rate fluid enters service $b(t,0)$. All performance functions are continuous except for the rate-into-service function $b(t,0)$. In underloaded intervals, $b(t,0) = \lambda(t)$; in overloaded intervals, $b(t,0)$ is the unique solution of the fixed-point equation (18).

It is important that the fluid model provide useful approximations for stochastic queueing models. We apply simulation to show that the fluid approximation indeed is effective for that purpose. For very large queueing systems, the stochastic system behaves like the fluid model, having relatively small stochastic fluctuations. That is illustrated for an $M_t/H_2/s + E_2$ queueing system with 2000 servers in Figure 2. In the plot, the queueing content processes are scaled by dividing by $n = 2000$, so that s remains at 1. For the actual queueing system, the quantities $\lambda(t)$, $Q(t)$, $B(t)$, $X(t)$ and $b(t,0)$ should all be multiplied by $n = 2000$.

Figure 2 actually shows three plots. It also shows the fluid approximation for the corresponding $M_t/M/s + E_2$ model, having exponential service times with the same mean. For that alternative model, there is a more elementary algorithm, because it is not necessary to solve the fixed point equation for $b(t,0)$ in order to calculate $b(t,x)$. Figure 2 shows two things: First, it shows that the simulation sample path for the $M_t/H_2/s + E_2$ model agrees closely with the fluid performance. Second, Figure 2 shows that the service distribution can make a big difference in the time-dependent performance. The performance of the fluid model changes significantly when we change the service distribution

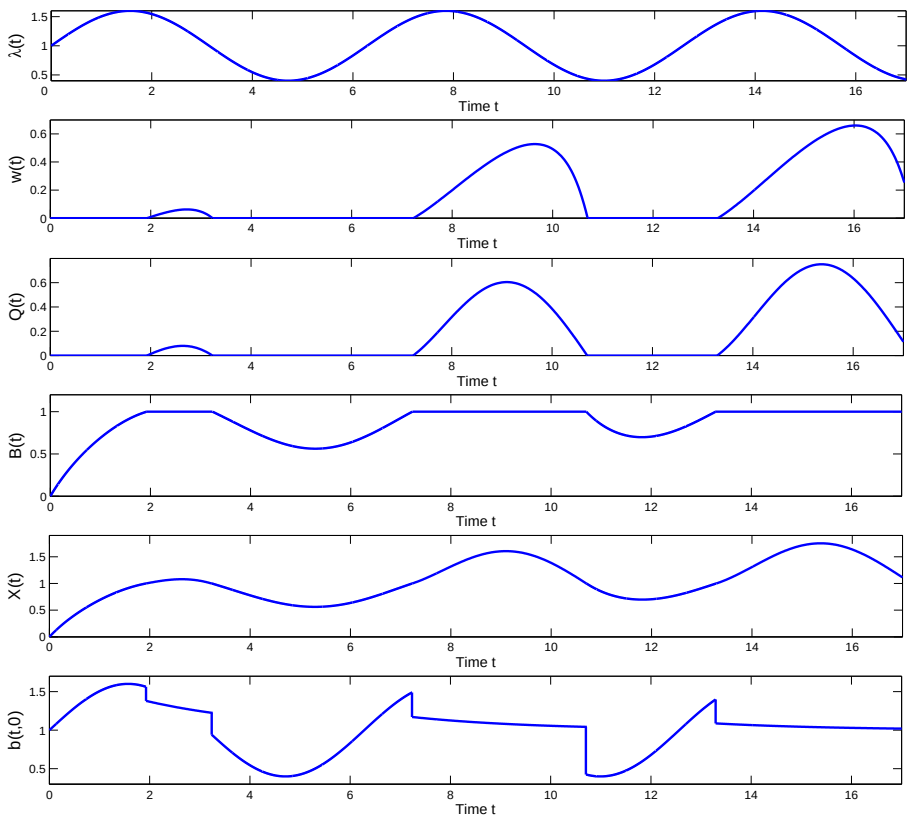


Fig. 1 The performance functions of the $G_t/H_2/s + E_2$ fluid model with sinusoidal arrival-rate function: (i) arrival rate $\lambda(t)$; (ii) head-of-line waiting time $w(t)$; (iii) fluid waiting in queue $Q(t)$; (iv) fluid in service $B(t)$; (v) total fluid in system $X(t)$; (vi) rate into service $b(t, 0)$.

from H_2 to M (with the same mean); e.g., look at $Q(t)$ at time $t = 3$. (We do not show a simulation path for the $M_t/M/s + E_2$ model, but it agrees closely with its fluid model for $n = 2000$. See the appendix.)

The impact of the service distribution may be surprising, because a major conclusion of [35,36] was that the steady-state performance is relatively insensitive to the service distribution beyond its mean. However, there is precedent for this phenomenon: In [9] we showed that the performance in the time-varying $M_t/GI/s/0$ loss model depends quite strongly on the service distribution beyond its mean, even though the steady-state distribution of the stationary $M/GI/s/0$ loss model has the well known insensitivity property, concluding that the standard steady-state performance measures do not depend at all on the service distribution beyond its mean.

Figure 2 suggests that the periodic models approach a periodic steady state as time evolves; that is proved for the fluid model with M service in [21]. (We conjecture that is also true with GI service under minimal regularity

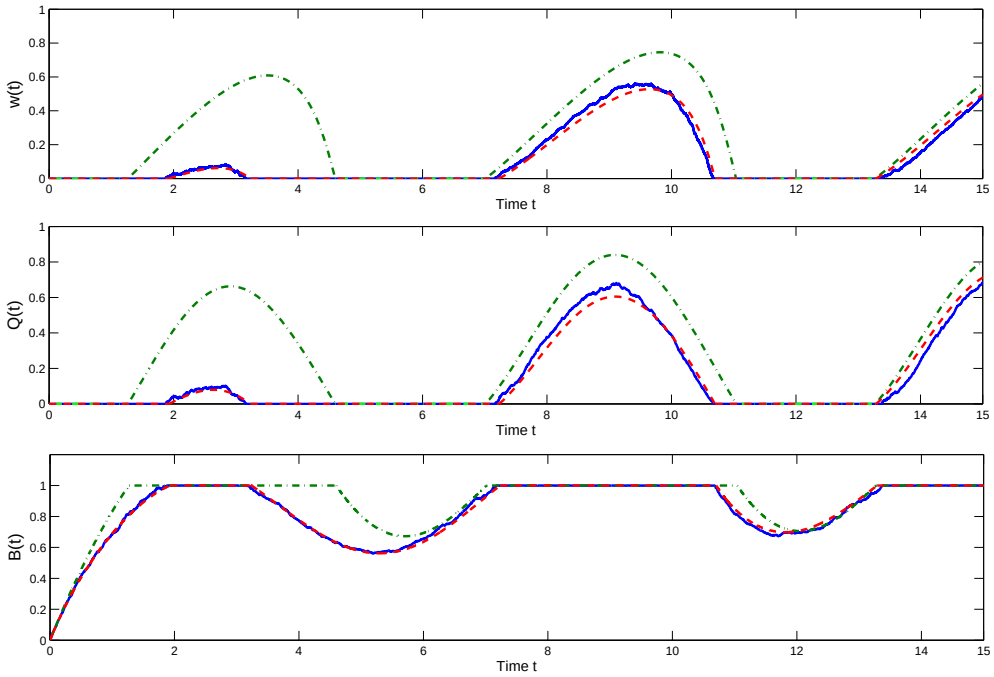


Fig. 2 Simulation comparison for the $M_t/H_2/s + E_2$ fluid model: (i) single sample paths in the scaled queueing model based on $n = 2000$ (blue solid lines), (ii) fluid functions (red dashed lines) and (iii) fluid functions assuming M service (green dashed lines).

conditions, but it has not yet been proved.) Figure 2 also shows that the impact of the service cdf G beyond its mean evidently is far greater at the beginning when the system is starting up, and then dissipates considerably as the system approaches its periodic steady state. That is consistent with intuition, because with H_2 service, there will be more very short service times and unusually long service times than would be the case of the exponential distribution. Hence, at the beginning starting empty, there are no old customers with long service times to compensate for many new customers with short service times in the H_2 case. As a consequence, the initial queue content is much less with H_2 than with M service. However, more supporting theory is needed.

Of course, most service systems have far fewer servers than the number $n = 2000$ we considered. It is thus important that the fluid approximation can still be useful with fewer servers. With fewer servers, the stochastic fluctuations in the queueing stochastic processes play an important role. In that case, the fluid model can still be very useful by providing a good approximation for the *mean values* of the queueing stochastic processes. That is illustrated from the plot of the average of the scaled performance measures of 200 independent sample paths when there are only 30 servers in Figure 3. We also consider the case $n = 15$ in the appendix.

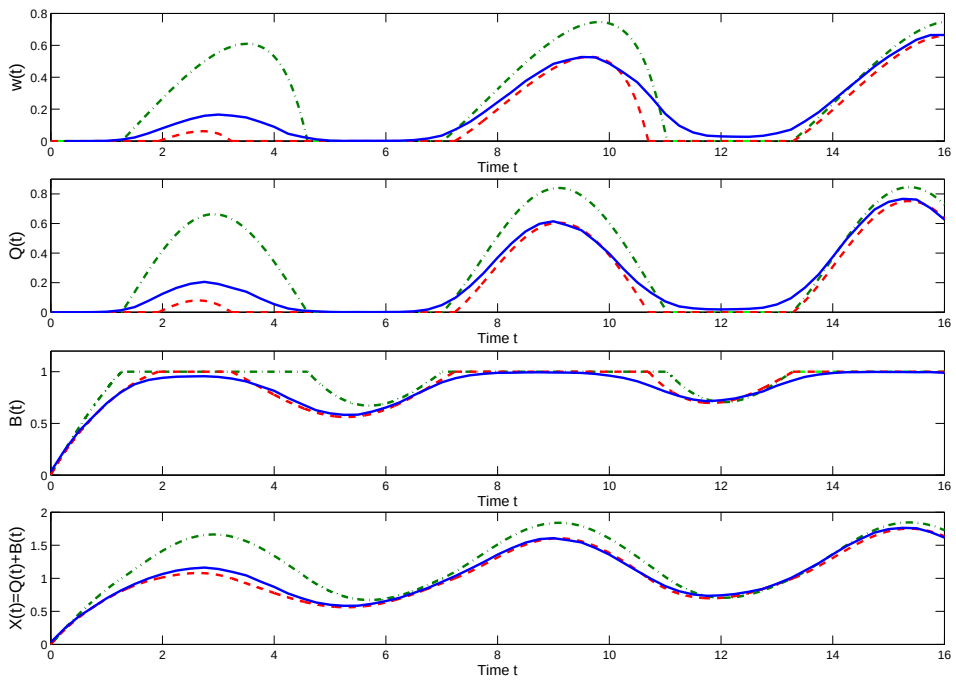


Fig. 3 Simulation comparison for the $M_t/H_2/s + E_2$ fluid model: (i) the averages of 200 sample paths of the scaled queueing model based on $n = 30$ (blue solid lines), (ii) fluid functions (red dashed lines) and (iii) fluid functions assuming M service (green dashed lines).

Work is in progress to investigate approximations for the full distributions at each time t, based on the new limits in [22]. A simple rough approximation for the distribution of $X(t)$ based on the approximation for the mean here is a normal distribution with variance equal to the determined mean; that is consistent with the exact Poisson distribution with the $M_t/GI/\infty$ model (and thus the stochastically equivalent $M_t/M/s_t + M$ model with $\theta = \mu$).

3 The Fluid Model

In this section we define the deterministic $G_t/GI/s_t + GI$ fluid model and specify important regularity conditions. There is a service facility with finite capacity (staffing function) $s \equiv \{s(t) : t \geq 0\}$ that is set exogenously and enforced. There also is waiting space with unlimited capacity. There is a deterministic arrival process, with input directly entering the service facility if there is space available; otherwise the input flows into the waiting room. Fluid may leave the service facility only by completing service. However, fluid may leave the queue either by entering service or abandoning (leaving directly from the queue without receiving service). These flows are deterministic as well. The

total input of fluid over the interval $[0,t]$ is $\Lambda(t) \equiv \int_0^t \lambda(u)\,du$, $t \geq 0$. We will be working with the time-dependent arrival-rate function $\lambda \equiv \{\lambda(t) : t \geq 0\}$.

There are service-time and abandon-time cdf's G and F, respectively, with pdf's g and f, satisfying

$$G(x) = \int_0^x g(u)du \quad \text{and} \quad F(x) = \int_0^x f(u)du, \quad x \geq 0. \tag{1}$$

Let $\bar{G}$ and $\bar{F}$ denote the associated complementary cdf's (ccdf's), defined by $\bar{G}(x) \equiv 1-G(x)$ and $\bar{F}(x) \equiv 1-F(x)$. We assume that the the random service and abandon times are unbounded above, so that $\bar{G}(x) > 0$ and $\bar{F}(x) > 0$ for all x. We assume that the mean service time is 1; that choice is without loss of generality, because we can measure time in units of mean service times. In the fluid model, the cdf's act as *proportions*. A proportion $G(x)$ of any quantity of fluid completes service and departs within time x of the time it starts service; a proportion $F(x)$ of any quantity of fluid abandons and departs without receiving service within time x of the time it arrives, providing that it has remained waiting in queue, and has not already been admitted to service.

The key performance descriptors are the two-parameter functions $B(t,y)$ and $Q(t,y)$: $B(t,y)$ is the quantity of fluid in service at time t that has been in service for time less than or equal to y; $Q(t,y)$ is the quantity of fluid waiting in queue at time t that has been in queue for time less than or equal to y. These functions will admit representations

$$Q(t,y) = \int_0^y q(t,x)\,dx \quad \text{and} \quad B(t,y) = \int_0^y b(t,x)\,dx, \quad y \geq 0, \tag{2}$$

where the fluid densities b and q are non-negative integrable functions. (See Proposition 2 in §5, Corollary 1 in §6.2, Proposition 6 in §7.1 and Corollary 5 in §7.2.)

Let $Q(t) \equiv Q(t,\infty)$ be the total fluid content in queue at time t, and let $B(t) \equiv B(t,\infty)$ be the total fluid content in service at time t. Let $X(t) \equiv B(t) + Q(t)$ be the total fluid content in the system at time t.

To fully specify the model, we also need to specify the initial conditions, describing the system state at time 0. The initial conditions are specified by the two functions $B(0,y)$ and $Q(0,y)$, which are defined as above, and also satisfy (2) with densities $b(0,x)$ and $q(0,x)$. Thus, the $G_t/GI/s_t + GI$ fluid *model data* consists of the six-tuple of functions $(\lambda, s, F, G, b(0,\cdot), q(0,\cdot))$.

We make several assumptions. The first is on the initial conditions.

Assumption 1 (*finite initial content*) $B(0) < \infty$ *and* $Q(0) < \infty$.

We develop a "smooth" model. For that purpose, let $\mathbb{C}_p$ be the set of *piecewise-continuous* real-valued functions, by which we mean that the function has only finitely many discontinuities in any finite interval, with left and right limits at each discontinuity point (within the interval); moreover, we assume that the function is right-continuous. Hence, $\mathbb{C}_p \subseteq \mathbb{D}$, where $\mathbb{D}$ is the space of right-continuous functions with left limits.

Assumption 2 (*smoothness*) $s, \Lambda, F, G, B(0,\cdot), Q(0,\cdot)$ *are differentiable functions with derivatives* $s', \lambda, f, g, b(0,\cdot), q(0,\cdot)$ *in* $\mathbb{C}_p$.

As a consequence of Assumption 2, $\Lambda(t) < \infty$ for all $t > 0$. (We use the fact that $\mathbb{C}_p \subset \mathbb{D}$ here to deduce that λ is bounded over finite intervals; see p. 122 of [7]; that implies that $\Lambda(t) < \infty$.) Together with Assumption 1, that implies the finite-content property in Assumption 1 holds for all t: $B(t) \le B(0)+\Lambda(t) < \infty$ and $Q(t) \le Q(0) + \Lambda(t) < \infty$ for all $t \ge 0$.

Whenever $Q(t) > 0$, we require there is no free capacity in service, i.e., $B(t) = s(t)$. Also, whenever $B(t) < s(t)$, then the queue is empty. These conditions are summarized in

Assumption 3 (*fluid dynamics constraints, FDC's*) *For all* $t \ge 0$,

$$(B(t) - s(t))Q(t) = 0 \quad \textit{and} \quad B(t) \le s(t). \tag{3}$$

In general, there is no guarantee that a staffing function s is feasible; i.e., having the property that the staffing function is set exogenously and adhered to, without forcing any fluid that has entered service to leave without completing service, because we allow s to decrease. (The fluid is assumed to be incompressible.) We directly assume that the staffing function we consider is feasible, but we also indicate how to detect the first violation and then construct the minimum feasible staffing function greater than or equal to the given staffing function; see §9.

Assumption 4 (*feasible staffing*) *The staffing function* s *is feasible, allowing all fluid that enters service to stay in service until service is completed; i.e., when* s *decreases, it never forces content out of service.*

We now consider the service discipline. We let the service discipline in the fluid model be first-come first-served (FCFS). We remark that there is much less motivation for considering other service disciplines, such as processor-sharing, with many servers than with few servers, because a few long service times can only make those few (of many) servers unavailable to other customers.

Assumption 5 (*FCFS service*) *Fluid enters service in order of arrival.*

As a consequence of Assumption 5, at time t there will be a boundary of the queue length density, which we call the boundary waiting time (BWT),

$$w(t) \equiv \inf\{y \ge 0 : q(t,x) = 0 \quad \text{for all} \quad x > y\}, \tag{4}$$

Clearly, first, $w(t) \ge 0$ and, second, $w(t) > 0$ if and only if $Q(t) > 0$. (Equation (4) is informal, because it is circular, with w depending on q, while q depends on w. We will carefully define and characterize the BWT w in §7.)

Based on the way the queueing system operates, we assume that q and b satisfy the following two fundamental evolution equations. Because of Assumption 5, fluid enters service from the queue from the right boundary of $q(t,x)$.

Assumption 6 (*fundamental evolution equations*) *For $t \geq 0$, $x \geq 0$ and $u \geq 0$,*

$$b(t+u, x+u) = b(t,x)\frac{\bar{G}(x+u)}{\bar{G}(x)},$$
$$q(t+u, x+u) = q(t,x)\frac{\bar{F}(x+u)}{\bar{F}(x)}, \quad 0 \leq x < w(t) - u. \tag{5}$$

The first equation in (5) says that the fluid in service that is not served remains in service (which requires that the staffing function be feasible, as in Assumption 4). The second equation in (5) says that the fluid waiting in queue that does not abandon and does not move into service, remains in queue.

Let $v(t)$ be the potential waiting time (PWT) at t, i.e., the virtual waiting time at t for an arriving quantum of fluid that has unlimited patience. The virtual waiting time at time t is the actual waiting time if there is positive input at time t; otherwise it is the waiting time of hypothetical input if it were to occur at time t. In order to simplify the analysis of the two waiting time functions w and v, we make extra assumptions: These extra assumptions will be introduced in §7.2 and §7.3.

We now turn to the flows. Let $A(t)$ be the total quantity of fluid to abandon in $[0,t]$; let $E(t)$ be the total quantity of fluid to enter service in $[0,t]$; and let $S(t)$ be the total quantity of fluid to complete service in $[0,t]$. Clearly we have the basic flow conservation equations

$$Q(t) = Q(0) + \Lambda(t) - A(t) - E(t) \quad \text{and} \quad B(t) = B(0) + E(t) - S(t), \quad t \geq 0. \tag{6}$$

These totals are determined by instantaneous rates. To define those rates, let $h_G(x) \equiv g(x)/\bar{G}(x)$ and $h_F(x) \equiv f(x)/\bar{F}(x)$ be the hazard-rate functions of the service and abandonment time distributions, respectively. Then

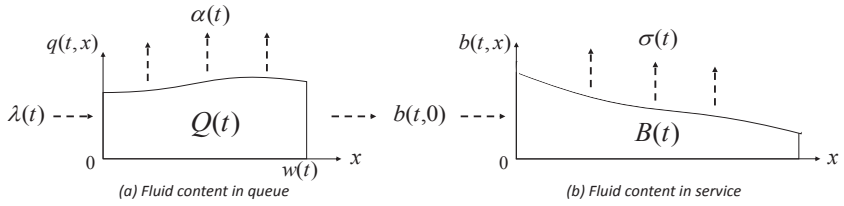


Fig. 4 (a) The fluid in queue, (b) The fluid in service.

$$A(t) \equiv \int_0^t \alpha(u)\, du, \quad \text{where} \quad \alpha(t) \equiv \int_0^\infty q(t,x) h_F(x) dx, \quad t \geq 0. \tag{7}$$

$$E(t) \equiv \int_0^t b(u,0)\, du, \quad t \geq 0. \tag{8}$$

$$S(t) \equiv \int_0^t \sigma(u)\,du, \quad \text{where} \quad \sigma(t) \equiv \int_0^\infty b(t,x)h_G(x)\,dx, \quad t \geq 0 \tag{9}$$

We have now completed the definition of the $G_t/GI/s_t + GI$ fluid model (with the exception of (w, q, v), for which more is given in §7; Figure 4 provides a pictorial summary. Our goal now is to fully characterize the six-tuple $(b, q, w, v, \sigma, \alpha)$ given the model parameters (λ, s, G, F) and the initial conditions $\{(b(0,x), q(0,x)) : x \geq 0\}$, where $q(0,x) > 0$ only if $Q(0) > 0$, which in turn, by Assumption 3, can hold only if $B(0) = s(0)$.

In doing so, we impose another regularity condition. We also assume that the system alternates between overloaded intervals and underloaded intervals, where these intervals include what is usually regarded as critically loaded. In particular, an *overloaded interval* starts at a time t_1 with (i) $Q(t_1) > 0$ or (ii) $Q(t_1) = 0$, $B(t_1) = s(t_1)$ and $\lambda(t_1) > s'(t_1) + \sigma(t_1)$, and ends at the *overload termination time*

$$T_1 \equiv \inf\{u \geq t_1 : Q(u) = 0 \quad \text{and} \quad \lambda(u) \leq s'(u) + \sigma(u)\}. \tag{10}$$

Case (ii) in which $Q(t_1) = 0$ and $B(t_1) = s(t_1)$ is often regarded as critically loaded, but because the arrival rate $\lambda(t_1)$ exceeds the rate that new service capacity becomes available, $s'(t_1) + \sigma(t_1)$, we must have the right limit $Q(t_1+) > 0$, so that there exists $\epsilon > 0$ such that $Q(u) > 0$ for all $u \in (t_1, t_1 + \epsilon)$. Hence, we necessarily have $T_1 > t_1$.

An *underloaded interval* starts at a time t_2 with (i) $B(t_2) < s(t_2)$ or (ii) $B(t_2) = s(t_2)$, $Q(t_2) = 0$, and $\lambda(t_2) \leq s'(t_2) + \sigma(t_2)$, and ends at *underload termination time*

$$T_2 \equiv \inf\{u \geq t_2 : B(u) = s(u) \quad \text{and} \quad \lambda(u) > s'(u) + \sigma(u)\}. \tag{11}$$

As before, case (ii) in which $Q(t_2) = 0$ and $B(t_2) = s(t_1)$ is often regarded as critically loaded, but because the arrival rate $\lambda(t_2)$ does not exceed the rate that new service capacity becomes available, $s'(t_2) + \sigma(t_2)$, we must have the right limit $Q(t_2+) = 0$. The underloaded interval may contain subintervals that are conventionally regarded as critically loaded; i.e., we may have $Q(t) = 0$, $B(t) = s(t)$ and $\lambda(t) = s'(t) + \sigma(t)$. For the fluid models, such critically loaded subintervals can be treated the same as underloaded subintervals. However, unlike an overloaded interval, we cannot conclude that we necessarily have $T_2 > t_2$ for an underloaded interval. Moreover, even if $T_2 > t_2$ for each underloaded interval, we could have infinitely many switches in a finite interval. We directly assume that those pathological situations do not occur.

Assumption 7 (*finitely many switches between intervals in finite time*) *Each underloaded interval is of positive length, so that the positive half line* $[0, \infty)$ *can be partitioned into overloaded and underloaded intervals. Moreover, there are only finitely many switches between overloaded and underloaded intervals in each finite interval.*

For engineering applications, Assumption 7 is reasonable, but it is unappealing mathematically. We would like to have natural conditions on the model parameters under which the conclusion does hold. For the special case of M service and for the extension to time-varying Markovian service (M_t), we provide sufficient conditions for Assumption 7 to be satisfied in [20]. From a practical perspective, Assumption 7 provides no restriction, because we can discover violations when calculating the performance descriptions, and remove any violation that we discover by negligibly modifying either the arrival rate function λ or the staffing function s in a neighborhood of the problem time t to remove the problem. That is most easily done with the arrival-rate function λ, because we only require that it be piecewise-continuous. For t in a short interval $[a, b]$, we can replace $\lambda(t)$ by $\lambda(t) \pm \epsilon$. This will introduce new discontinuity points at the end points a and b (if they were not already discontinuity points), but that leaves $\lambda \in \mathbb{C}_p$.

All assumptions above are in force throughout this paper. We will introduce additional regularity assumptions as needed, starting in §6. We now determine the performance, first considering an underloaded interval.

4 Scale Proportionality

To treat an underloaded interval in the next section, we will exploit an important scale proportionality property of the $M_t/GI/\infty$ stochastic queueing model; see Remark 5 of [10]. For each $c > 0$, let $B_c(t, y)$ be the number of customers in service in the $M_t/GI/\infty$ stochastic model at time t that have been so for a duration at most y when the system starts empty at time 0 and the arrival-rate function is $\lambda_c(t) \equiv c\lambda(t)$, for some given arrival-rate function λ and service cdf. The following is proved like Theorem 1 of [10], using the two-parameter framework, as in [30].

Proposition 1 (scale proportionality in the $M_t/GI/\infty$ stochastic model) *For all $c > 0$, $B_c(t, y)$ has a Poisson distribution with mean*

$$m_c(t, y) \equiv E[B_c(t, y)] = cm_1(t, y) = c \int_0^{t \wedge y} \lambda(t - x)\bar{G}(x)\, dx. \qquad (12)$$

As a consequence of the SLLN for the Poisson distribution, we see that $c^{-1}B_c(t, y) \to m_1(t, y)$ as $c \to \infty$ for each t and y. In addition, we have the more general FWLLN in [30,32], which implies that $c^{-1}B_c(t, y) \to m_1(t, y)$, regarded as functions of t and y. Hence, the mean function $m_1(t, y)$ in the $M_t/GI/\infty$ stochastic queueing model directly coincides with the limit of the scaled process; i.e.,

$$m_1(t, y) \equiv E[B_1(t, y)] = B(t, y),$$

where $B(t, y)$ is the fluid content in service at time t that have been so for a duration at most y in the $M_t/GI/\infty$ fluid model. Thus, aside from scale, the

mean $m_c(t,y) \equiv E[B_c(t,y)]$ in the $M_t/GI/\infty$ stochastic model coincides with the corresponding fluid content in the deterministic fluid model.

Moreover, the conclusions above extend to the more general $G_t/GI/\infty$ models. First, the mean function in (12) above in the $G_t/GI/\infty$ stochastic model actually coincides with the mean function in the $M_t/GI/\infty$ stochastic model, provided that the arrival rate function is the same; this observation is made in Remark 2.3 of [28]. Second, the FWLLN in [30,32] actually holds for the $G_t/GI/\infty$ stochastic model, provided that the arrival process satisfies a FWLLN. To summarize, the mean function in the $M_t/GI/\infty$ stochastic model coincides with the fluid content in the corresponding $G_t/GI/\infty$ fluid model, assuming appropriate scale.

This scale proportionality in the infinite-server stochastic model actually extends to the more general $G_t/GI/s_t + GI$ fluid model. The following scale proportionality result is a consequence of the results in this paper.

Theorem 1 (scale proportionality in the $G_t/GI/s_t + GI$ fluid model) *If the vector* $(b_c(t,x), q_c(t,x), w_c(t), v_c(t), \alpha_c(t), \sigma_c(t))$ *is the performance at time* t *associated with model data* $(c\lambda, cs, F, G, cb(0,\cdot), cq(0,\cdot))$, *then*

$$(b_c, q_c, \alpha_c, \sigma_c) = c(b_1, q_1, \alpha_1, \sigma_1) \quad \textit{and} \quad (w_c, v_c) = (w_1, v_1).$$

5 An Underloaded Interval

We will consider the system over successive intervals, during each of which it is either underloaded or overloaded, as defined in §3. We start with the easier case, in which the system is underloaded. Without loss of generality, we assume that an underloaded interval starts at time 0 and terminates at a time T, defined in (11). We do not need to know in advance the termination time T. Instead, we can assume that the system is underloaded over the full interval $[0,\infty)$ and then calculate T.

If the $G_t/GI/s_t + GI$ fluid model is underloaded, then there is no queue, and so no abandonment. Then the model is is equivalent to the associated $G_t/GI/\infty$ fluid model.

Proposition 2 (service content in an underloaded interval) *For the fluid model with unlimited service capacity* $(s(t) \equiv \infty$ *for all* $t \geq 0)$, *the integral representation in* (2) *is valid for* B *and* b, *with*

$$\begin{aligned} B(t,y) &= \int_0^{t\wedge y} \bar{G}(x)\lambda(t-x)\,dx + \int_0^{(y-t)\vee 0} \frac{\bar{G}(x+t)}{\bar{G}(x)} b(0,x)\,dx, \\ b(t,x) &= \bar{G}(x)\lambda(t-x)1_{\{x\leq t\}} + \frac{\bar{G}(x)}{\bar{G}(x-t)} b(0,x-t)1_{\{x>t\}} \qquad (13) \\ B(t) &= \int_0^t \bar{G}(x)\lambda(t-x)\,dx + \int_0^\infty \frac{\bar{G}(x+t)}{\bar{G}(x)} b(0,x)\,dx \\ &\leq \Lambda(t) + B(0) < \infty, \quad 0 \leq t < T. \end{aligned}$$

If, instead, a finite-capacity system starts underloaded, then the same formulas apply over the interval $[0, T)$*, where the underload termination time is* $T \equiv \inf\{t \geq 0 : B(t) > s(t)\}$*, with* $T = \infty$ *if the infimum is never obtained. Hence,* $b(t, \cdot), b(\cdot, x) \in \mathbb{C}_p$ *for all* $t \geq 0$ *and* $x \geq 0$*, for* t *in the underloaded interval.*

Proof. The first term in the expression for $B(t, y)$ in (28) represents the content due to new input; it follows from §4. The second term represents the content to old content still in service; it follows from Assumption 6, along with Assumption 2. It is evident that, for each t, $B(t, y)$ is differentiable in y for all y except $y = t$. Thus, for each $t \geq 0$, $B(t, y)$ is absolutely continuous as a function of y and has the density $b(t, x)$ displayed above. In addition, by Assumption 2, $b(t, \cdot), b(\cdot, x) \in \mathbb{C}_p$ for all $t \geq 0$ and $x \geq 0$. ∎

During an underloaded interval, $b(t, x)$ depends upon the pair (λ, G) and the initial condition $b(0, x)$. There is no queue, so (q, F, w, v) play no role. The different roles of the two regimes are summarized in Figure 4. Hence, Proposition 2 fully describes the performance during underloaded intervals. The final piecewise-continuity conclusion ensures that the piecewise-continuity property assumed for $b(0, \cdot)$ will pass on to subsequent intervals when we consider successive intervals.

Remark 1 (discontinuity at $t = x$) From (28), we see that b inherits the smoothness of G, λ and $q(0, \cdot)$ except when $t = x$. That will be a persistent theme throughout our analysis. For general initial conditions, this discontinuity is fundamental, so we cannot expect greater smoothness. However, away from the set $\{(t, x) : t = x\}$, we can expect smoothness of the model parameters to be reflected in our performance descriptions.

Remark 2 (the generic scalar transport PDE) If, in addition to the assumptions of Proposition *2*, λ and $b(0, \cdot)$ are differentiable a.e. with respect to Lebesgue measure on $[0, \infty)$, then, for each t and x, $b(t, x)$ has first partial derivatives with respect to t and x a.e. with respect to Lebesgue measure on $[0, \infty)$. Moreover, b satisfies the following PDE a.e. with respect to Lebesgue measure on $[0, \infty) \times [0, \infty)$, a simple version of the generic scalar transport equation:

$$b_t(t, x) + b_x(t, x) \equiv \frac{\partial b}{\partial t}(t, x) + \frac{\partial b}{\partial x}(t, x) = -h_G(x) b(t, x).$$

with boundary conditions $\{b(t, 0) = \lambda(t) : t \geq 0\}$ and $\{b(0, x) : x \geq 0\}$; see Appendix §B.

We now give a monotonicity result comparing two underloaded fluid models. For this result, we exploit hazard rate order, writing $h_{G_1} \leq h_{G_2}$ if $h_{G_1}(x) \leq h_{G_2}(x)$ for all $x \geq 0$, for cdf's satisfying the assumptions in §3. It is easy to see that hazard rate order implies ordinary stochastic order via the representation

$$\bar{G}(x) = e^{-\int_0^x h_G(u)\, du}, \quad x \geq 0. \tag{14}$$

Proposition 3 (comparison result for b in an underloaded model) *Consider two underloaded fluid models. If $\lambda_1 \le \lambda_2$, $b_1(0,\cdot) \le b_2(0,\cdot)$ and $h_{G_1} \ge h_{G_2}$ as functions, then $b_1 \le b_2$, i.e., $b_1(t,x) \le b_2(t,x)$ for all $t \ge 0$ and $x \ge 0$, and $T_1 \le T_2$, where T_i is the underload termination time in model i.*

Proof. Apply (28) after applying (14) to write

$$\bar{G}(x)/\bar{G}(x-t) = \exp\{-\int_{x-t}^{x} h_G(u)\,du\}. \quad \blacksquare$$

The system could be in an underloaded period for an extended period of time. If so, it is often convenient to consider the system starting empty in the distant past. (That is done for the corresponding infinite-server queueing models in [10,28].) That allows us to directly construct stationary versions, including periodic versions, if that is warranted.

Proposition 4 (starting empty in the distant past) *Suppose the system started empty in the distant past (at $t = -\infty$) and has been underloaded up to time t. If $\int_0^\infty \bar{G}(x)\lambda(t-x)\,dx, < \infty$, then*

$$b(t,x) = \bar{G}(x)\lambda(t-x) \le \lambda(t-x), \quad B(t) = \int_0^\infty \bar{G}(x)\lambda(t-x)\,dx,$$
$$B(t,y) = B(t) - \int_0^\infty \bar{G}(x+y)\lambda(t-x-y)\,dx = \int_0^y \bar{G}(x)\lambda(t-x)\,dx$$

for $x \ge 0$ and $y \ge 0$. If the arrival-rate function λ is constant or periodic, then so are $b(t,\cdot)$, $B(t)$ and $B(t,\cdot)$.

As noted above, the expression for $B(t)$ coincides with the mean number of busy servers in the $M_t/GI/\infty$ model studied in [10,28]; see these sources for additional structural results. The expressions for the two-parameter function $B(t,y)$ and $b(t,x)$ coincide with the corresponding mean values in [30].

6 The Service Content Density in An Overloaded Interval

Without loss of generality, we assume that the overloaded interval begins at time 0 and ends at time T satisfying (10). Again, we do not need to know the end time T in advance, because we can calculate it while we are calculating the performance measures q and w. We proceed under the assumption that the arrival rate is sufficiently large that the system is overloaded throughout a specified interval $[0,T)$ (up to, but not including, time T), and afterwards detect violations before time T, if there are any, and then reduce the interval, if necessary.

6.1 The Special Case of M Service

The service content density is easy to compute if the service distribution is exponential, so we consider that case first. From (5), we can write down an expression for $b(t,x)$ during the overloaded interval:

$$\begin{aligned} b(t,x) &= b(t-x,0)\bar{G}(x)1_{\{x\le t\}} + b(0,x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}1_{\{x>t\}}, \\ &= b(t-x,0)e^{-x}(x)1_{\{x\le t\}} + b(0,x-t)e^{-t}1_{\{x>t\}}, \end{aligned} \tag{15}$$

where $b(0,x-t)$ is part of the initial conditions, but where $b(t-x,0)$ remains to be specified.

Since the service is exponential, the output rate, $\sigma(t)$, and thus the rate fluid enters service, $b(t,0)$, depend only on the staffing function s, in particular, on the values $s(t)$ and $s'(t)$. (Recall that the mean service time has been fixed at 1.)

Proposition 5 (the service content in an overloaded interval) *The departure (service completion) rate satisfies $\sigma(t) = B(t)$, $t \ge 0$, and, during each overloaded interval, the departure rate $\sigma(t)$ and rate fluid enters service $b(t,0)$ have the simple form*

$$\sigma(t) = B(t) = s(t) \quad \textit{and} \quad b(t,0) = s'(t) + s(t) \quad \textit{for all} \quad t, \tag{16}$$

depending only on the staffing function s. Then b is fully characterized by (15) *and* (16) *during an overloaded interval. Also $b(t,\cdot), b(\cdot,x) \in \mathbb{C}_p$ for all x, $t < T$.*

Proof. Apply (9).

6.2 General GI Service

We start with the general expression for the service content density given in (15), but it requires the rate into service $b(t,0)$, which is part of what we are trying to determine. Since the system is assumed to be overloaded over an initial interval $[0,T)$, the rate into service is determined by the rate service capacity becomes available. Thus, by (9), we have

$$b(t,0) = s'(t) + \sigma(t) = s'(t) + \int_0^\infty b(t,x)h_G(x)dx, \quad 0 \le t < T. \tag{17}$$

We now substitute equation (15) into equation (17) to obtain the following equation for the function $b(t,0)$:

$$b(t,0) = \hat{a}(t) + \int_0^t b(t-x,0)g(x)\,dx, \tag{18}$$

where

$$\hat{a}(t) \equiv s'(t) + \int_0^\infty \frac{b(0,y)g(t+y)}{\bar{G}(y)}\,dy. \tag{19}$$

From (19), we see that $\hat{a} \in \mathbb{C}_p \subseteq \mathbb{D}$ provided that the integral in (19) is finite. That will hold under regularity conditions, as we will explain below.

We now specify two ways to show that the fixed-point equation (18) has a unique solution and how it can be numerically calculated. First, we can recognize that equation (18) is a renewal equation, as in §V.2 of [4]. Thus, the existence of a unique solution to (18) follows from Theorem 2.4 on p. 146 of [4]. However, computation of the solution by this approach seems not elementary. One possible way is to apply Laplace transforms. From (18), we obtain the following equation for the associated Laplace transforms (replacing the variable t by s)

$$\mathcal{L}(b)(s,0) = \mathcal{L}(\hat{a})(s) + \mathcal{L}(b)(s,0)\mathcal{L}(g)(s), \tag{20}$$

which has explicit solution

$$\mathcal{L}(b)(s,0) = \frac{\mathcal{L}(\hat{a})(s)}{1 - \mathcal{L}(g)(s)}. \tag{21}$$

Given the transforms $\mathcal{L}(\hat{a})(s)$ and $\mathcal{L}(g)(s)$, the numerical values of the function $b(t,0)$ can be effectively computed by numerical transform inversion, e.g., using the Fourier-series method in [1,2]; see especially §13 of [1]. However, this requires computation of the transforms $\mathcal{L}(\hat{a})(s)$ and $\mathcal{L}(g)(s)$ for the required arguments s. Since only a few arguments s are required, this approach is feasible, but somewhat cumbersome.

We now present an alternative way to show that equation (18) has a unique solution and numerically calculate that solution. From (18), it is evident that $b(t,0)$ is a fixed point of the operator $\mathcal{T} : \mathbb{D} \to \mathbb{D}$, where

$$\mathcal{T}(u)(t) \equiv \hat{a}(t) + \int_0^t u(t-x)g(x)\,dx. \tag{22}$$

Under regularity conditions, we can show that there exists a unique solution to equation (18) by applying the Banach (contraction) fixed point theorem. We will use the complete (nonseparable) normed space $\mathbb{D}$ with the uniform norm over the interval $[0,T]$, i.e.,

$$\|u\|_T \equiv \sup_{0 \le t \le T} \{|u(t)|\}. \tag{23}$$

The proof of completeness follows the same argument used for the space $\mathbb{C}$; see pp. 150, 220 of [6].

We will require an additional bound on the tail of the initial service content density $b(0,\cdot)$. Recall that we have assumed that $\bar{G}(x) > 0$ for all x.

Assumption 8 (*tail of $b(0,\cdot)$*) *The tail of $b(0,\cdot)$ is bounded relative to the service-time pdf g via*

$$\tau(b,g,T) \equiv \sup_{0 \le s \le T} \int_0^\infty \frac{b(0,y)g(s+y)}{\bar{G}(y)}\,dy < \infty,$$

Assumption 8 warrants discussion, because it is unappealing. At first glance, it passes the requirement that the assumptions be on the model data, because the service density g, the associated cdf G and the initial fluid content in service $b(0,\cdot)$ are all part of the model data. However, in application we will be applying the algorithm recursively over several UL and OL intervals. We would thus not know in advance the function $b(0,\cdot)$ in all OL intervals after an initial one. It is thus important that we provide readily available sufficient conditions for Assumption 8 to hold; we do that after we state the theorem. For now, we point out that there is a simple practical condition implying Assumption 8 to hold: It suffices for the service hazard rate function h_G to be bounded. (See below.)

Theorem 2 (service content in the overloaded case) *Consider an overloaded interval $[0,T]$. If Assumption 8 holds, then the operator $\mathcal{T}$ in (22) is a monotone contraction operator on $\mathbb{D}$ with contraction modulus $G(T)$ for the norm $\|\cdot\|_T$ defined in (65), so that a finite function $b(t,0)$ is uniquely characterized via equation (18). Hence, for any $u \in \mathbb{D}$, the fixed point can be approximated by the n-fold iteration $\mathcal{T}^{(n)}$ of the operator $\mathcal{T}$ applied to u, with*

$$\|\mathcal{T}^{(n)}(u) - \hat{b}\|_T \leq \frac{G(T)^n}{1-G(T)}\|\mathcal{T}(u) - u\|_T \to 0 \quad as \quad n \to \infty \tag{24}$$

and, if $u \leq (\geq)\mathcal{T}(u)$, then $\mathcal{T}^{(n-1)}(u) \leq (\geq)\mathcal{T}^{(n)}(u) \leq (\geq)\hat{b}$ for all $n \geq 1$.

Proof. Clearly, Assumption 8 implies that $\|\hat{a}\|_T < \infty$, so that $\mathcal{T}$ maps $\mathbb{D}$ into $\mathbb{D}$. Moreover, the contraction property follows from

$$\begin{aligned}\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_T &= \sup_{0\leq t\leq T}\{\int_0^t (u_1(t-x) - u_2(t-x))g(x)\} \\ &\leq \quad \|u_1 - u_2\|_T \int_0^T g(x)\,dx = \|u_1 - u_2\|_T G(T). \quad \blacksquare\end{aligned}$$

Remark 3 Note we require $G(T) < 1$ in the proof of Theorem 10, which holds because we have assumed that $\bar{G}(x) > 0$ for all x. However, that requirement is actually not necessary, because we can always work in an interval $[0,\delta]$ as long as $G(\delta) < 1$ for some $\delta > 0$. We can show the uniqueness of $b(\cdot,0)$ for all $0 \leq t \leq T$ by recursively considering successive intervals of length δ.

We can deduce from Theorem 10 that an analog of Proposition 2 holds in OL intervals.

Corollary 1 (*the integral representation* (2) *in an OL interval*) *The integral representation in* (2) *is valid in OL intervals as well as in UL intervals.*

We now return to Assumption 8, which restricts the class of allowed service cdf's in a rather complicated way. We will show that it suffices for the service hazard rate h_G to be bounded. But even that is often not necessary in practice. It is important to note that Assumption 8 is always satisfied in a case of

principle interest: if there exists y_0 such that $b(0,y) = 0$ for all $y \geq y_0$. That case occurs whenever the system started empty at some (finite) time in the past. That case occurs if the overloaded interval of interest begins at time t, $0 \leq t < T$, after the system has begun empty with $b(0,y) \equiv 0$ for all y; then necessarily $b(t,y) = 0$ for all $y > t$, by virtue of Assumption 6. Then

$$\tau \leq B(0,T)g^{\uparrow}(2T)/\bar{G}(T) < \infty, \tag{25}$$

where $x^{\uparrow}(t) \equiv \sup\{x(s) : 0 \leq s \leq t\}$.

Nevertheless, other initial conditions are interesting. For example, for the stationary model, we might start with the stationary fluid content, which has the form we have $b(0,y) = \bar{G}(y)$, $y \geq 0$, because $\bar{G}$ is the stationary-excess or equilibrium-residual-lifetime density of the service-time distribution; see [36]. Thus we now present other sufficient conditions for Assumption 8.

Remark 4 (sufficient conditions for the bound when $B(t) - B(0,y) > 0$ for all y.) Clearly, we need to control the initial content density $b(0,y)$ and/or the service pdf $g(y)$ in order for Assumption 8 to hold. An easy sufficient condition directly related to the stationary fluid content density for the stationary model is for there to exist a constant K such that $b(0,y) \leq K\bar{G}(y)$ for all $y \geq 0$. Another easy sufficient condition for the bound in Assumption 8 is to have

$$\sup_{0 \leq t < T} \{\int_0^{\infty} b(0,y)h_G(y+t)\,dy\} < \infty. \tag{26}$$

In turn, three different sufficient conditions for (26) are:

(*i*) $\sup_{x \geq 0}\{h_G(x)\} < \infty$ (bounded hazard rate, using $B(0) < \infty$);

(*ii*) there exists $\beta > 0$ and K such that

$$\int_0^{\infty} b(0,y)e^{\beta y}\,dy < \infty \quad \text{and} \quad h_G(x) \leq Ke^{\beta}x \quad \text{for all} \quad x \geq 0.$$

(*iii*) $\limsup_{y \to \infty}\{b(0,y)/\bar{G}(y)\} < \infty$

(using $\sup_{0 \leq y \leq t} b(0,y) < \infty$ and $\sup_{0 \leq y \leq t} h_G(0,y) < \infty$ for all $t \geq 0$)

So far, we can only conclude that the function $b(t,0) \in \mathbb{D}$. We can obtain additional smoothness properties by imposing additional smoothness conditions on the model elements s and g. We use these properties for $b(\cdot,0)$ to establish properties of the ODE to calculate the BWT w in §4 of [20].

Corollary 2 (smoothness of service content in the overloaded case) *If s' and g are continuous, then $b(\cdot,0)$ is continuous as well. In that case, $b(\cdot,x)$ and $b(t,x)$ are elements of $\mathbb{C}_p$ for each $x \geq 0$ and $t \geq 0$.*

Proof. Under the extra smoothness conditions, we can apply the contraction fixed point theorem on the closed subspace $\mathbb{C}$ of continuous functions in $\mathbb{D}$, with the same uniform norm. Then the fixed point $b(t,0$ is necessarily in $\mathbb{C}$ as well, from which we deduce that $b(\cdot,x)$ and $b(t,x)$ are elements of $\mathbb{C}_p$ for each $x \geq 0$ and $t \geq 0$. ∎

We discuss alternative algorithms to calculate b in Appendix §C.

7 The Queue Performance Functions

We now turn to the queue during an overload interval. To do so, it is convenient to initially ignore the flow into service.

7.1 The Queue Content Ignoring Flow Into Service

Let $\tilde{q}(t,x)$ be $q(t,x)$ during the overload interval $[0,T)$ under the assumption that no fluid enters service from queue. We can once again invoke the connection to the $M_t/GI/\infty$ stochastic model, discussed in §4 to treat $\tilde{q}(t,x)$ just as we treated b in §5, because we can let the general patience cdf F play the role of the general service-time cdf G. Instead of (5), we can write

$$\tilde{q}(t+u,x+u) = \tilde{q}(t,x)\frac{\bar{F}(x+u)}{\bar{F}(x)}, \quad x \geq 0, \tag{27}$$

to obtain the following proposition. The proof is just like the proof of Proposition 2 for B.

Proposition 6 (*queue content without transfer into service in the overloaded case*) *In the overloaded case, the integral representation in* (2) *is valid for* $\tilde{Q}$ *and* $\tilde{q}$*, with*

$$\begin{aligned}
\tilde{Q}(t,y) &= \int_0^{t\wedge y} \bar{F}(x)\lambda(t-x)\,dx + \int_0^{(y-t)\vee 0} \frac{\bar{F}(x+t)}{\bar{F}(x)} q(0,x)\,dx, \\
\tilde{q}(t,x) &= \lambda(t-x)\bar{F}(x)1_{\{x\leq t\}} + q(0,x-t)\frac{\bar{F}(x)}{\bar{F}(x-t)}1_{\{t<x\}}, \\
\tilde{Q}(t) &= \int_0^{t} \bar{F}(x)\lambda(t-x)\,dx + \int_0^{\infty} \frac{\bar{F}(x+t)}{\bar{F}(x)} q(0,x)\,dx \\
&\leq \Lambda(t) + Q(0) < \infty, \quad 0 \leq t < T.
\end{aligned} \tag{28}$$

Remark 5 Just as we observed for b in an underloaded interval in Remark 2, in an overloaded interval $\tilde{q}$ *satisfies a version of the generic scalar transport PDE.*

Paralleling Proposition 3, we have the following comparison result, proved in the same way.

Proposition 7 (comparison result for $\tilde{q}$) *Consider two overloaded fluid models. If $\lambda_1 \leq \lambda_2$, $q_1(0,\cdot) \leq q_2(0,\cdot)$ and $h_{F_1} \geq h_{F_2}$ as functions, then $\tilde{q}_1 \leq \tilde{q}_2$, i.e., $\tilde{q}_1(t,x) \leq \tilde{q}_2(t,x)$ for all $t \geq 0$ and $x \geq 0$.*

We now derive q and w. The proper definition and characterization of the BWT w is somewhat complicated. We easily get an expression for q provided that we can find w.

Corollary 3 (*from $\tilde{q}$ to q*) *Given the BWT w,*

$$\begin{aligned} q(t,x) &= \tilde{q}(t-x,0)\bar{F}(x)1_{\{x\leq w(t)\wedge t\}} + \tilde{q}(0,x-t)\frac{\bar{F}(x)}{\bar{F}(x-t)}1_{\{t<x\leq w(t)\}} \\ &= q(t-x,0)\bar{F}(x)1_{\{x\leq w(t)\wedge t\}} + q(0,x-t)\frac{\bar{F}(x)}{\bar{F}(x-t)}1_{\{t<x\leq w(t)\}}. \end{aligned} \quad (29)$$

Moreover, $q(t,\cdot) \in \mathbb{C}_p$ for all $t \geq 0$.

Proof. Combine Proposition 6 and (29) to deduce that $q(t,\cdot) \in \mathbb{C}_p$ for all t, x.

7.2 The Boundary Waiting Time w

It now remains to define and characterize the BWT w. We can *define* the BWT w by exploiting flow conservation, in particular, by exploiting the fact that two expressions for the amount of fluid to enter service over any interval $[t, t+\delta]$ coincide; i.e.,

$$E(t+\delta) - E(t) \equiv \int_t^{t+\delta} b(u,0)\,du = I(t,w(t),\tilde{q},\delta) - A(t,t+\delta), \quad (30)$$

where

$$I \equiv I(t,w(t),\tilde{q},\delta) \equiv \int_{w(t)-\epsilon(t,\delta)}^{w(t)} \tilde{q}(t,x)dx \quad (31)$$

is the amount of fluid removed from the right boundary of $\tilde{q}$, starting at $x = w(t) - \epsilon(t,\delta)$ and ending at $x = w(t)$, during the time interval $[t, t+\delta]$ (where $\epsilon(t,\delta)$ is yet to be determined) and $A(t,t+\delta)$ is the amount of the fluid content in I that abandons in the interval $[t,t+\delta]$. We *define* the BWT w by letting $\delta \downarrow 0$ in (30). We will show in Theorem 3 below that, under regularity conditions, the relation in (30) determines an ODE for w that has a unique solution. Hence, we will show that the relation (30) serves to properly define w and characterize it.

We need two more regularity conditions. First, we assume that the initial value $w(0)$ for the interval we consider is finite. We will be representing w as the solution of an initial value problem involving an ODE, so this is needed.

Assumption 9 (*finite initial BWT*) $0 \leq w(0) < \infty$.

Second, we require that the functions $\lambda(t)$ and $q(0,x)$ be appropriately bounded away from 0.

Assumption 10 (*positive arrival rate and initial queue density*) *For all* $t \geq 0$,

$$\lambda_{\inf}(t) \equiv \inf_{0 \leq u \leq t} \{\lambda(u)\} > 0, \quad \textit{and}$$
$$q_{\inf}(0) \equiv \inf_{0 \leq u \leq w(0)} \{q(0,u)\} > 0 \quad \textit{if} \quad w(0) > 0.$$

By equation (**??**) for $\tilde{q}$, Assumption 10 for λ implies that $\tilde{q}(t,x) > \epsilon \bar{F}(x) > 0$ on $[0,T)$ for some positive ϵ. That is useful because $\tilde{q}(t,x)$ appears in the denominator in an expression for the derivative of w in (32) below. The BWT w can be discontinuous if these functions are 0 over subintervals; we give examples in Appendix E. We show that w can be discontinuous if $\lambda(t) = 0$ or $q(0,\cdot) = 0$ over a subinterval, while w can have an infinite derivative corresponding to zeros of these functions. However, we obtain the following positive result, proved in §11. Let $x(t+)$ and $x(t-)$ denote the right and left limits of a function x at t, respectively. We can obtain a more elementary statement and proof if we assume even more regularity conditions; see Appendix §D.

Theorem 3 (*the BWT ODE*) *Consider an overloaded interval* $[0,T)$. *If Assumptions* 9–10 *hold, then the BWT* w *is well defined being the unique solution of the initial value problem* (*IVP*) *on* $[0,T)$ *based on the ODE*

$$w'(t+) = \Psi(t, w(t)) \equiv 1 - \frac{b(t+,0)}{\tilde{q}(t, w(t)-)} \tag{32}$$

and any initial value $w(0)$. *In addition,* w *is Lipschitz continuous on* $[0,T]$ *with* $w(t+u) \leq w(t) + u$ *for all* $t \geq 0$ *and* $u \geq 0$ *with* $t + u \leq T$. *Moreover,* w *is right differentiable everywhere with right derivative* $w'(t+)$ *given in* (32) *and left differentiable everywhere* (*but not necessarily differentiable*) *with value*

$$w'(t-) = \tilde{\Psi}(t, w(t)) \equiv 1 - \frac{b(t-,0)}{\tilde{q}(t, w(t)+)}. \tag{33}$$

Overall, w *is continuously differentiable everywhere except for finitely many* t.

Remark 6 (different roles of $b(t,0)$ and F in shaping q) Our use of $\tilde{q}$ as an intermediate step in constructing q helps show the different roles played by $b(t,0)$ and F in producing q. First, the abandonment (F) controls the shape of $\tilde{q}(t,x)$ and thus $q(t,x)$ only for $x < w(t)$. Second, the transportation rate $b(t,0)$ controls only $w(t)$, the right boundary or the truncation of $\tilde{q}(t,x)$ on x; it does not affect $\tilde{q}(t,x)$ itself, and thus $q(t,x)$ for any $0 \leq x < w(t)$.

We give closed-form formulae for some special cases in the next corollary, proved in Appendix §D.

Corollary 4 *Suppose the system is overloaded for* $0 \leq t < T$ *and* $w(0) = 0$.

(*a*). *For the* $G_t/M/s_t$ *fluid model without customer abandonment* ($\bar{F}(x) = 1$ *for* $x \geq 0$),

$$w(t) = t - \Lambda^{-1}(\int_0^t b(y,0)dy), \quad 0 \leq t < \bar{t},$$

for $\Lambda^{-1}(x) \equiv \inf\{y > 0 : \Lambda(y) = x\}$, *and* $\bar{t} \equiv \inf\{t > 0 : \Lambda(t) = \int_0^t b(y,0)dy\}$.

(*b*). *For the* $G_t/M/s_t + M$ *fluid model, where the abandonment-time cdf is exponential* ($\bar{F}(x) = e^{-\theta x}$, $x \geq 0$),

$$w(t) = t - \tilde{\Lambda}^{-1}(\int_0^t b(y,0)e^{\theta y}dy), \quad 0 \leq t < \tilde{t}, \tag{34}$$

where $\tilde{\Lambda}(t) \equiv \int_0^t \lambda(y)e^{\theta y}dy$, $\tilde{\Lambda}^{-1}(x) \equiv \inf\{y > 0 : \tilde{\Lambda}(y) = x\}$, *and* $\tilde{t} \equiv \inf\{t > 0 : \tilde{\Lambda}(t) = \int_{t_1}^t b(y,0)e^{\theta y}dy\}$.

We conclude this section by combining Proposition 6, Corollary 3 and Theorem 3 to deduce that the integral representation in (2) is valid for Q as well as B and $\tilde{Q}$.

Corollary 5 (*integral representation for* Q) *In the overloaded case, the integral representation in* (2) *is valid for* Q *and* q, *with* w *in Theorem* 3 *and* q *in Corollary* 3.

7.3 The Potential Waiting Time

In the previous subsection, we characterized the dynamics of the BWT w. Now we want to connect w to the PWT v, the waiting time of an arriving quantum of fluid at time t that is infinitely patient.

As shown in [26], the PWT v can be defined as a first passage time, with abandonment after time t computed with the input turned off; also see [33]. Let $A_t(u)$ be the total fluid abandoning in the interval $[t, t+u]$ in our fluid model, modified by having the input shut off after time t. Paralleling (7),

$$A_t(u) \equiv \int_t^{t+u} \alpha_t(s)\,ds \quad \text{and} \quad \alpha_t(s) \equiv \int_{s-t}^{\infty} q(s,x)h_F\,dx, \quad s \geq t, \tag{35}$$

where $\alpha_t(s)$ is the abandonment rate of the fluid that arrives before time t, at time s.

With (35), we can define $v(t)$ as

$$v(t) \equiv \inf\{u \geq 0 : E(t+u) - E(t) + A_t(u) \geq Q(t)\}, \quad t \geq 0, \tag{36}$$

where $E(t)$ is the amount of fluid to enter service in the interval $[0,t]$, as in (8), i.e., $E(t) \equiv \int_0^t b(u,0)\,du$, $t \geq 0$. However, in general, so far, we have not assumed enough to guarantee that the PWT v is finite. It is possible for fluid to arrive and never be served; we need to rule that out.

First, we show that any initial fluid content in the system eventually must leave. Let $B_0(t)$ be the portion of the initial fluid content in service, $B(0)$, that is still in service at time t; let $Q_0(t)$ be the portion of the initial fluid content in queue, $Q(0)$, that is still in queue at time t.

Proposition 8 (dissipation of initial fluid content) *For $t \geq 0$,*

$$B_0(t) = \int_t^\infty b(0,y)\frac{\bar{G}(t+y)}{\bar{G}(y)}\,dy \to 0 \quad \textit{and}$$
$$Q_0(t) \leq \tilde{Q}(0) = \int_t^\infty \tilde{q}(0,y)\frac{\bar{F}(t+y)}{\bar{F}(y)}\,dy \to 0 \quad \textit{as} \quad t \to \infty.$$

Proof. The representation is immediate. It is elementary that $B_0(t) \leq B(0)$ and $\tilde{Q}_0(t) \leq \tilde{Q}(0) = Q(0)$. By Assumption 1, $B(0) < \infty$ and $Q(0) < \infty$. The convergence then follows from the Lebesgue dominated convergence theorem. ∎

However, the queue will not dissipate in finite time by abandonment alone, because $\bar{F}(x) > 0$ for all $x \geq 0$. Hence we need to have fluid enter service from the queue. Even if we invoke Assumption 9, and have $w(0) < \infty$, so that we have $w(t) \leq w(0) + t < \infty$ for all $t \geq 0$, we cannot guarantee that $v(0) < \infty$. Indeed, we would have $v(t) = \infty$ for all $t \geq 0$ if no fluid from queue were ever admitted into service. That in turn would be the case if we used the feasible staffing function $s(t) \equiv B_0(t)$, which is positive for all t when $B(0) > 0$, because $\bar{G}(x) > 0$ for all $x \geq 0$. In order to avoid such problems, we introduce two more regularity conditions:

Assumption 11 (*minimum staffing level*) *There exists a constant s_L such that $s(t) \geq s_L > 0$ for all $t \geq 0$.*

Assumption 12 (*minimum service hazard rate*) *There exists a constant $h_{G,L}$ such that $h_G(x) \geq h_{G,L} > 0$ for all $x \geq 0$.*

Theorem 4 (finite PWT) *Under Assumptions* 11 *and* 12*, the rate of service completion is bounded below: $\sigma(t) \geq s_L h_{G,L}$ for all $t \geq 0$. As a consequence,*

$$v(t) \leq \frac{Q(t) + s(t) - s_L}{s_L h_{G,L}} < \infty, \quad t \geq 0.$$

We give the proof in Appendix §D. Given that the PWT v is indeed bounded above as in Theorem 4, we can obtain it from our algorithm for w. The idea is simple: If, at time t, the elapsed waiting time of the quantum of fluid that is entering service is $w(t)$, then this quantum of fluid arrived in queue $w(t)$ units of time ago. That implies that the PWT at $t - w(t)$ is $w(t)$. We prove the following in §11.

Theorem 5 (the PWT v and the BWT w) *Consider an overloaded interval with Assumptions* 9-10 *holding and $w(0) = 0$. If $v(t) < \infty$ for all $t \geq 0$ (for which Assumption* 11 *is a sufficient condition, by Theorem 4), then v is the unique function in $\mathbb{D}$ satisfying the equation*

$$v(t - w(t)) = w(t) \quad \textit{or, equivalently,} \quad v(t) = w(t + v(t)) \quad \textit{for all} \quad t \geq 0, \tag{37}$$

as depicted in Figure 5*. Moreover, v is discontinuous at t if and only if there exists $\epsilon > 0$ such that $w(t + v(t) + \epsilon) = w(t + v(t)) + \epsilon$, which in turn holds if and only if $b(u, 0) = 0$ for $t + v(t) \leq u \leq t + v(t) + \epsilon$. If $b(\cdot, 0) > 0$ a.e. with respect to Lebesgue measure, then v is continuous.*

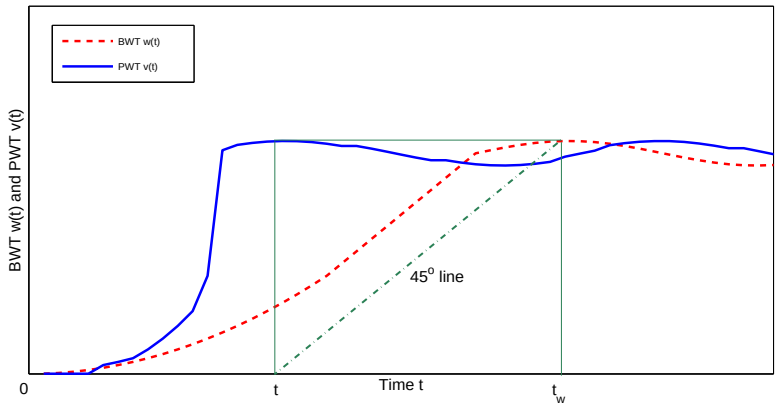


Fig. 5 Potential waiting time $v(t)$ and boundary waiting time $w(t)$.

The proof of Theorem 5 directly gives an algorithm to compute the PWT v given the BWT w. Similarly, the second equation in (37) can provide an algorithm to construct w given v. We now provide an alternative characterization of v via its own ODE, but this alternative characterization involves an extra condition. We give the proof in §11.

Theorem 6 (right derivative and ODE for v) *Under the conditions in Theorem* 5*, the right derivative of v always exists (except possibly infinite), with value*

$$\begin{aligned} v'(t+) &\equiv \lim_{\delta \downarrow 0} \frac{v(t+\delta)-v(t)}{\delta} = \Phi(t, v(t)) \equiv \frac{\tilde{q}(t+v(t), v(t)-)}{b((t+v(t))+, 0)} - 1 \\ &= \frac{\lambda(t+)\bar{F}(v(t))}{b((t+v(t))+, 0)} - 1 \geq -1. \end{aligned}$$

The right derivative at t is finite if and only if $b(t+v(t), 0) > 0$. If t is a continuity point of v, then the left derivative exists as well, with

$$v'(t-) = \tilde{\Phi}(t, v(t)) \equiv \frac{\tilde{q}(t+v(t), v(t)+)}{b((t+v(t))-, 0)} - 1 = \frac{\lambda(t-)\bar{F}(v(t))}{b((t+v(t))-, 0)} - 1 \geq -1.$$

If Φ is continuous at t, then v is differentiable at t, and v satisfies the first ODE. If, in addition, $b(t, 0) > 0$ for all t, then v is continuous. Then v is differentiable except at only finitely many t, and there exists a unique solution to the first ODE.

Remark 7 (*algorithm for v and w*) In an algorithm, it is convenient to avoid the complications for w and v that occur when $b(t, 0) = 0$. To do so, we can introduce an ϵ-approximation, letting $b_\epsilon(t, 0) \equiv b(t, 0) + \epsilon$, $0 \leq t \leq T$, only to be used in the calculation of w and v. Let w_ϵ be w and v_ϵ be v with $b(t, 0)$

replaced by $b_\epsilon(t,0)$. Since $w' \geq w'_\epsilon$ and $v' \geq v'_\epsilon$, we have $w_\epsilon \uparrow w$ and $v_\epsilon \uparrow v$ as $\epsilon \downarrow 0$.

We could also enforce a lower bound for $b(t,0)$ directly in our model by imposing a constraint on our staffing. We could require that $b(t,0) \geq b^* > 0$ for all t in order for the staffing function s to be feasible. Since $b(t,0) = s'(t)+\sigma(t)$, that translates into the staffing constraint

$$s'(t) \geq b^* - \sigma(t) = b^* - \int_0^\infty b(t,x)\,dx, \quad 0 \leq t < T. \tag{38}$$

In Appendix D we give closed-form formulae for the PWT v in some special cases, paralleling those for the BWT w given in Corollary 4.

8 Overview of the Total Algorithm

We now summarize the full algorithm for the $G_t/GI/s_t + GI$ fluid model. We alternately consider successive underloaded and overloaded intervals (under the assumption that any finite interval can be partitioned into finitely many of these, which can be verified in the computation). For each underloaded interval, we start with initial conditions as indicated in §3. We can compute the single key performance measure b directly by applying Proposition 2. We then end the underloaded interval the first time $B(t)$ exceeds $s(t)$. Since the queue is empty, the functions q, w and v do not appear.

8.1 An Overloaded Interval with M service

An overloaded interval is more complicated. There are two cases: (i) M service and (ii) non-M GI service. For M service, we do not need to solve the fixed point equation (18) for the rate fluid enters service from the queue, $b(t,0$. With M service (at rate 1), we know that $b(t,0) = s'(t) + s(t)$, by Proposition 5. The algorithm starts with initial conditions as in §3. The algorithm begins by calculating $\tilde{q}$ via Proposition 6 and b and $b(t,0)$ via Proposition 5. We then calculate w by solving the ODE (32) and then the function v via the equation (37), as explained in the proof of Theorem 5. We consider terminating the overloaded interval the first time that $w(t) = 0$. At that time we check to see if the interval actually remains overloaded, by looking at the net flow rate into the queue $r(t) \equiv \lambda(t) - s'(t) - \sigma(t)$ (see (10)). If $r(t) > 0$, then we continue the overloaded interval. Otherwise, we shift to the next underloaded interval. We present additional details about the algorithm for M service in Appendix G.

8.2 An Overloaded Interval with GI service

With non-M service, we need to solve the fixed point equation (18) for the rate fluid enters service from the queue, $b(t,0)$, in addition to the other steps

with M service. We now formally state the algorithm to compute all performance functions in an overloaded interval of the $G_t/GI/s_t + GI$ fluid model. Consider an interval $[0, T]$ and assume that the system is overloaded at $t = 0$, i.e., $Q(0) > 0$ and $B(0) = s(0)$. However, we typically do not know when the overloaded interval ends in advance. The objective is to determine the overload termination time T_1 defined in (10) with $t_1 = 0$ along with the other performance functions. Hence, we determine $q(t, \cdot)$ and $b(t, \cdot)$ for $0 \le t \le T \wedge T_1$. If $T_1 < T$, the system simply switches to an underloaded interval; otherwise, the system stays overloaded in $[0, T]$.

The input functions are the model parameters F, G, $\lambda(t)$ and $s(t)$ for $0 \le t \le T$ and initial condition $q(0, \cdot)$, $b(0, \cdot)$ and $w(0)$. We require that these conditions satisfy (i) $s(0) = B(0) = \int_0^\infty b(0, y)dy$ and (ii) $Q(0) = \int_0^{w(0)} q(0, y)dy > 0$. Applying the fixed-point operator discussed in §6, we have the following algorithm:

1: $u^{(0)}(t) \leftarrow 0, \quad a(t) \leftarrow s'(t) + \int_0^\infty b(0, y)\frac{g(t+y)}{G(y)}dy, \quad i \leftarrow 1$
2: $u^{(i)}(t) \leftarrow a(t) + \int_0^t u^{(i-1)}(y)g(t - y)dy \quad$ for $\quad 0 \le t \le T$
3: If $\|u^{(i)} - u^{(i-1)}\|_T > \epsilon$, then $i \leftarrow i + 1$ and go to Step 2;
otherwise $b(t, 0) \leftarrow u^{(i)}(t) \quad$ for $\quad 0 \le t \le T$
4: Solve the BWT ODE and determine T_1.
5: Compute $b(t, x)$ using (15) for $0 \le t \le T \wedge T_1$. End.

Note that ϵ is the (small positive) error threshold level that we specify in advance. Here we let the contraction iteration in Step 2 end when the uniform distance between the u functions in two consecutive iterations is small.

The algorithm above requires that the given staffing function s be feasible. However, we can also easily modify the algorithm so that infeasibility can be detected. That extension is discussed in Appendix G. With the algorithm above, we will see that s is infeasible (if it is) in Step 4 by observing that $b(t, 0) \le 0$ for some $0 \le t \le T$; see the next section.

9 Feasibility of the Staffing Function

So far, we have assumed that the staffing function s is feasible, yielding

$$b(t, 0) \ge s'(t) + \sigma(t) = s'(t) + \int_0^\infty b(t, x)h_G(x)\,dx \ge 0 \tag{39}$$

for all $t \ge 0$ such that $B(t) = s(t)$. This requirement is automatically satisfied in underloaded intervals when $B(t) = s(t)$, because in that case we require that $s'(t) + \sigma(t) \ge \lambda(t)$ where necessarily $\lambda(t) \ge 0$. Feasibility is only a concern during overloaded intervals, and then only when the staffing function is decreasing, i.e., when $s'(t) < 0$.

The first violation is easy to detect: Let t^* be the time of first violation. Let I_n be the n^{th} overloaded subinterval in $[0, \infty)$ determined under the assumption that the original staffing function s is feasible. Let I be the union of these

subintervals, i.e., the subset of $[0,\infty)$ during which the system is overloaded. Then

$$t^* \equiv \inf\{t \in I : b(t,0) < 0\}. \tag{40}$$

Even though we require (39), so far we have done nothing to prevent having $t^* < \infty$ (violation). Thus, we compute b and detect the first violation.

Correcting the staffing function is not difficult either (by which we mean replacing it with a higher feasible staffing function): We simply construct a new staffing function s^* consistent with turning off the input into the queue (setting $b(t,0) = 0$) starting at time t^* and lasting until the first time t after t^* at which $s^*(t) = s(t)$. (By the adjustment, we will have made $s^*(t^*+) > s(t^*+)$.) Since the system has operated differently during the time interval $[t^*, t]$, we must recalculate all the performance measures after time t, but we have now determined a feasible staffing function up to time $t > t^*$. By successive applications of this correction method (adjusting the staffing function s and recalculating b), we can construct the minimum feasible staffing function overall.

To make this precise, let $\mathcal{S}_{f,s}(t)$ be the set of all feasible staffing functions for the system over the time interval $[0,t]$, $t > t^*$, that coincide with s over $[0,t^*]$; i.e., with $\mathbb{C}_p^2(t)$ denoting the set of twice differentiable positive real-valued functions on $[0,t]$ with second derivatives in $\mathbb{C}_p$, let

$$\begin{aligned}\mathcal{S}_{f,s}(t) \equiv \{\tilde{s} \in \mathbb{C}_p^2(t) : b_{\tilde{s}}(u,0)1_{\{B_{\tilde{s}}(u)=\tilde{s}(u)\}} \geq 0, \quad 0 \leq u \leq t, \\ \text{and} \quad \tilde{s}(u) = s(u), \quad 0 \leq u \leq t^*\},\end{aligned} \tag{41}$$

for t^* in (40), where $b_{\tilde{s}}$ is the function b associated with the model with staffing function $\tilde{s}$.

Theorem 7 (minimum feasible staffing function) *Assume that $s \in \mathbb{C}_p^2$ and $b_{\tilde{s}}(\cdot,0)$ exists and is continuous for each $\tilde{s} \in \mathcal{S}_{f,s}(t))$. Then there exist $\delta > 0$ and $s^* \in \mathcal{S}_{f,s}(t^* + \delta)$ in (41) for t^* in (40) such that*

$$s^* = \inf\{\tilde{s} \in \mathcal{S}_{f,s}(t^* + \delta)\}; \tag{42}$$

i.e., $s^* \in \mathcal{S}_{f,s}(t^* + \delta)$ and $s^*(u) \leq \tilde{s}(u)$, $0 \leq u \leq t^* + \delta$, for all $\tilde{s} \in \mathcal{S}_{f,s}(t^* + \delta)$. In particular,

$$s^*(t^* + u) \equiv \int_u^\infty b_s(t^*, x-u)\frac{\bar{G}(x)}{\bar{G}(x-u)}dx, \quad 0 \leq u \leq \delta. \tag{43}$$

Moreover, δ can be chosen so that

$$\delta = \inf\{u \geq 0 : s^*(t^* + u) = s(t^* + u)\}, \tag{44}$$

with $\delta \equiv \infty$ if the infimum in (44) is not attained.

Proof First, since $b_s(\cdot, 0)$ is continuous for our original s, the violation in (40) must persist for a positive interval after t^*; that ensures that a strictly positive δ can be found.

We shall prove that $\tilde{s} \geq s^*$ over $[t^*, t^* + \delta]$ for s^* in (43) and any feasible function $\tilde{s}$, and we will show that s^* itself is feasible. For $0 \leq t \leq t^* + \delta$, suppose $\tilde{s}$ is feasible. Since the system is overloaded, system being in the overloaded regime implies that

$$\begin{aligned}
\tilde{s}(t^* + u) = B_{\tilde{s}}(t^* + u) &= \int_0^\infty b_{\tilde{s}}(t^* + u, x)dx \\
&= \int_0^u b_{\tilde{s}}(t^* + u - x, 0)\,\bar{G}(x)dx + \int_u^\infty b_s(t^*, x - u)\frac{\bar{G}(x)}{\bar{G}(x-u)}dx \\
&\geq \int_u^\infty b_s(t^*, x - u)\frac{\bar{G}(x)}{\bar{G}(x-u)}dx = s^*(t^* + u),
\end{aligned}$$

where equality on the second line holds because of the fundamental evolution equations in Assumption 6 and because $b_{\tilde{s}}(t^*, x) = b_s(t^*, x)$ for all x, and the inequality holds because $b_{\tilde{s}} \geq 0$. On the other hand, the equality holds when $b_{\tilde{s}}(t^* + u, 0) = 0$ for all u, which yields $B(t^* + u) = s^*(t + u)$. Therefore, the proof is complete. ∎

Corollary 6 (minimum feasible staffing with exponential service times) *For the special case of exponential service times, i.e., with* $\bar{G}(x) \equiv e^{-x}$, (43) *becomes simply* $s^*(t^* + u) = B(t^*)e^{-u}$, $0 \leq u \leq \delta$.

We have constructed a minimal feasible staffing function by requiring that the new staffing function agree with the original one up until the time of the first violation. We have shown that assumption leads to a unique minimum feasible staffing function. However, it may be desirable to consider other approaches to feasibility, where we have the freedom to revise the staffing function before t^* as well as afterwards. It is natural to frame the issue as an optimization problem; e.g., as in productions smoothing, we might want to impose costs for for fluctuations of the staffing function as well high values. We leave such investigations for future work.

10 Staffing the $G_t/GI/s_t + GI$ Fluid Model to Stabilize Delays

So far, we have discussed the performance analysis of the $G_t/GI/s_t + GI$ fluid model with the staffing function s regarded as a given function. In this section, we assume that we are free to choose the staffing function s, and do so with the objective of stabilizing the potential waiting time v at some (constant) target $v^* > 0$. This delay stabilization problem is a variant of one considered previously for many-server queueing models with time-varying arrival rates in [12]. In [12], the goal was to stabilize the probability an arrival experiences any delay. in contrast, here we stabilize the delay of all fluid at precisely $v^* > 0$. Now everybody must wait, but only v^*.

As a consequence of Theorem 5, we see that, in order to stabilize v at v^*, it suffices to stabilize w at v^*. By Theorem 3, we see that we will be able to do so if and only if we can find a staffing functions s for which the resulting performance satisfies the equation

$$0 = w'(t) = 1 - \frac{b(t,0)}{q(t,v^*)}, \quad t \geq 0 \tag{45}$$

which implies that we must have $b(t,0) = q(t,v^*)$ when $w(t) = v^*$.

Suppose that the system is initially empty, i.e., $b(0,x) = q(0,x) = 0$ for all $x > 0$. Thus, we do not start staffing the service facility until time v^*, so that no input enters service during $[0, v^*]$; i.e., we let $b(t,0) = 0$ for $0 \leq t \leq v^*$, in order to let w increase from 0 to v^*. At time v^*, the input at time 0 is sent to the queue, after waiting precisely time v^*.

With the initial conditions $q(t,0) = \lambda(t)$ and $q(0,x) = 0$, the queue instantly becomes overloaded at time 0, and we can apply Proposition 6 and Corollary 3 (or (5)) to obtain

$$q(t,x) = \bar{F}(x)\lambda(t-x)1_{\{0 \leq x \leq t\}}, \quad 0 \leq t \leq v^*. \tag{46}$$

Combining (45) and (46), we obtain the transportation rate after $t = v^*$:

$$b(t,0) = q(t,v^*) = \bar{F}(v^*)\lambda(t-v^*)1_{\{t > v^*\}}.$$

With the explicit expression of $b(t,0)$ and $b(0,x) \equiv 0$, $x \geq 0$, (5) implies that

$$b(t,x) = \bar{G}(x)\bar{F}(v^*)\lambda(t-x-v^*)1_{\{0 \leq x \leq t-v^*\}}, \quad t \geq 0 \quad \text{and} \quad x \geq 0. \tag{47}$$

Therefore, we can easily compute $B(t)$, $\sigma(t)$, $q(t,x)$, $Q(t)$ and $\alpha(t)$ for $t > v^*$. We have just proved the following theorem.

Theorem 8 *Consider the $G_t/GI/s_t + GI$ fluid model with a general arrival-rate function λ. Suppose the system is initially empty. For any specified constant $v^* > 0$, we can make the system overloaded such that the PWT is fixed at v^*, i.e., $v(t) = v^*$ for all $t \geq 0$, by (i) not allowing any input to enter service until time $t = v^*$, (ii) letting the service-capacity function be*

$$s(v^*,t) \equiv s^*(t) = \bar{F}(v^*) \int_0^{t-v^*} \bar{G}(x)\lambda(t-v^*-x)dx \cdot 1_{\{t > v^*\}} \tag{48}$$

and (iii) operating the queue in the usual FCFS manner after time v^* with $b(t,0) > 0$. If we do so, then $w(t) = v^*$ for $t \geq v^*$ and $w(t) = t$ for $t \leq v^*$,

$$B(t) = s^*(t), \quad b(t,0) = \bar{F}(v^*)\lambda(t-v^*) \cdot 1_{\{t > v^*\}},$$

$$Q(t) = \int_0^t \bar{F}(x)\lambda(t-x)dx \cdot 1_{\{0 \leq t \leq v^*\}} + \int_0^{v^*} \bar{F}(x)\lambda(t-x)dx \cdot 1_{\{t > v^*\}},$$

$$\sigma(t) = \bar{F}(v^*) \int_0^{t-v^*} \lambda(t-v^*-x)g(x)dx \cdot 1_{\{t > v^*\}},$$

$$\alpha(t) = \int_0^t \lambda(t-x)f(x)dx \cdot 1_{\{0 \leq t \leq v^*\}} + \int_0^{v^*} \lambda(t-x)f(x)dx \cdot 1_{\{t > v^*\}}, \quad t \geq 0.$$

If λ is a periodic function, then so are $b(\cdot, x)$, $B(\cdot) = s^*(\cdot)$, σ, $q(\cdot, x)$, $Q(\cdot)$ and α after time v^*, with the same period.

Remark 8 (connection to the QED regime when $v^* = 0$) All the analysis in this section can be extended to the delay target $v^* = 0$. In this case, the staffing function in Theorem 8 is just sufficient to guarantee that all fluid enters service immediately upon arrival (thus with 0 delay in the queue) and that the system is CL for all t (the service capacity is fully occupied, i.e., $B(t) = s(t)$). This scenario corresponds to the heavy-traffic QED system regime.

Remark 9 (*general initial conditions or no delay*) Theorem 8 is based on starting empty. However, it is possible to stabilize delays with arbitrary initial conditions. We present the details in Appendix H. We can also achieve the minimum staffing level so that there is no delay at all by simply staffing at the fluid content $B(t)$ in the underloaded regime. These two variants may involve having an atom of initial fluid content enter service at time 0, so that we leave the smooth framework.

11 Proofs of the Main Results

Proof of Theorem 3. We establish the different results in turn:

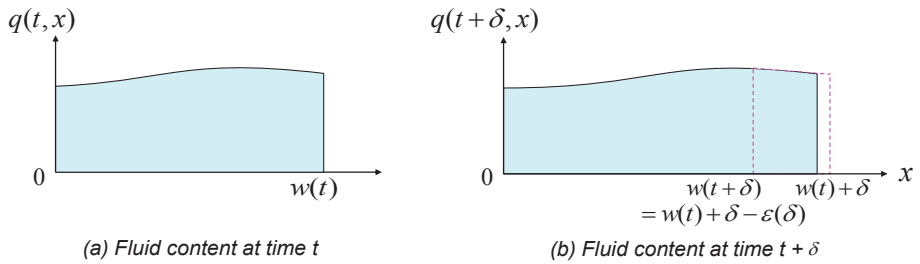
(a) (rate of growth) Consider an interval $[t, t+\delta]$ that is overloaded. If no fluid enters service during this interval, i.e., if $b(s, 0)$=0 for $t \le s \le t+\delta$, then the waiting time of a quantum of fluid at the front of the queue will increase with rate 1, i.e., $w(t+\delta) = w(t)+\delta$, provided that quantum does not abandon. Hence, we have the claimed bound on the rate of growth: $w(t+u) \le w(t)+u$ for all $t \ge 0$ and $u \ge 0$ with $t+u \le T$. A more formal argument follows from (5) in Assumption 6.

(b) (characterization) However, we will have $w(t+\delta) < w(t)+\delta$ if $b(t, 0) > 0$ because the FCFS service discipline implies that the queue is being eaten away from the head. In other words, fluid is being transported from the queue to the service facility from the right boundary of $q(t, x)$. Therefore,

$$w(t+\delta) = w(t) + \delta - \epsilon(t, \delta), \tag{49}$$

where $\epsilon(t, \delta)$ is the amount of boundary waiting time $w(t)$ that is pushed back (eaten up) by $b(t, 0)$ from t to $t+\delta$, see Figure 6. (Note that $\delta > 0$ and $\epsilon(t, \delta) \ge 0$.) To determine $\epsilon(t, \delta)$, we apply (30), with (31). We will bound $\epsilon(t, \delta)$ in (51) below.

(c) (controlling the abandonment term) We will show that the abandonment term $A(t, t+\delta)$ in (30) is asymptotically negligible, so that it can be ignored when computing the derivative, but we use it to establish Lipschitz continuity. Even though $A(t, t+\delta)$ is somewhat complicated, we can easily bound it above. Moreover, we can do so uniformly in t over the entire interval $[0, T]$. First let $w^\uparrow \equiv \sup\{w(t) : 0 \le t \le T\}$. We necessarily have $w^\uparrow \le w(0)+T < \infty$ by virtue of the bound on the growth rate growth determined above. Next let



(a) Fluid content at time t (b) Fluid content at time $t + \delta$

Fig. 6 The boundary of the waiting time $w(t)$ under FCFS.

$h_F^{\uparrow} \equiv \sup\{h_F(x) : 0 \le x \le w^{\uparrow}\}$ which necessarily is finite, since $f \in \mathbb{C}_p$ and $\bar{F}(w^{\uparrow}) > 0$; and let $\tilde{q}^{\uparrow} \equiv \sup\{\tilde{q}(t,x) : 0 \le x \le w^{\uparrow}\}$, which again necessarily is finite because $\tilde{q}(t,\cdot) \in \mathbb{C}_p$. We thus have the bound

$$A(t, t+\delta) \le h_F^{\uparrow} \tilde{q}^{\uparrow} \epsilon(t,\delta) = C_1 \delta \tag{50}$$

for $0 \le t \le t+\delta \le T$, where $C_1 \equiv h_F^{\uparrow} \tilde{q}^{\uparrow} w^{\uparrow}$, because $\epsilon(t,\delta) \le w^{\uparrow}\delta$.

(d) (Lipschitz continuity) By (49), we can show that w is Lipschitz continuous by showing that $\epsilon(t,\delta) \le C\delta$ for some constant C. Recall that $b(\cdot,0)$ is an element of $\mathbb{D}$ by Theorem 10. Hence, $\|b(\cdot,0)\|_T < \infty$, so that there exists a constant C_2 such that $E(t+\delta) - E(t) \le C_2\delta$ for $0 \le t \le t+\delta \le T$. Together with (50), that implies that the integral $I(t, w(t), \tilde{q}, \delta)$ is bounded above by $C\delta$ for $0 \le t \le t+\delta \le T$, where $C \equiv C_1 + C_2$. Since the integrand of I is bounded below by $c > 0$ by virtue of Assumption 10,

$$c\epsilon(t,\delta) \le I(t, w(t), \tilde{q}, \delta) \le (E(t+\delta) - E(t)) + A(t, t+\delta) \le C\delta \tag{51}$$

for $0 \le t \le t+\delta \le T$, so that indeed

$$|w(t+\delta) - w(t)| \le \delta + \epsilon(t,\delta) \le (1 + (C/c))\delta \quad \text{for} \quad 0 \le t \le t+\delta \le T.$$

as claimed.

(e) (the derivative) Since w is Lipschitz continuous, w necessarily is differentiable a.e., but we will establish a stronger result. Given that $\epsilon(t,\delta) = c\delta + o(\delta)$ as $\delta \downarrow 0$, from the first inequality in (50) we see that $A(t, t+\delta) = O(\delta^2) + o(\delta^2)$, so that the abandonment term can be ignored when we consider the derivative. Together with (30) and (31), that implies that a right derivative of w exists at t with value in (32). The convergence as $\delta \downarrow 0$ in the definition of that right derivative will be uniform over a neighborhood of t if $\tilde{q}(t,x)$ is continuous function of x at $x = w(t)$, but not otherwise.

To show (33) is similar. We consider an interval $[t-\delta, t]$ that is overloaded. Similarly, we have

$$w(t) = w(t-\delta) + \delta - \epsilon(t-\delta, \delta), \tag{52}$$

and

$$E(t) - E(t-\delta) \equiv \int_{t-\delta}^{t} b(u,0)\,du = J + K - A(t, t+\delta)),$$

where

$$J \equiv J(t, w(t), \tilde{q}) \equiv \int_{w(t)}^{w(t)+\epsilon(t-\delta,\delta)} \tilde{q}(t,x)dx, \tag{53}$$

and

$$\begin{aligned} K \equiv K(t, w(t), \tilde{q}) &\equiv I(t-\delta, w(t-\delta), \tilde{q}, \delta) - J(t, w(t), \tilde{q}) \\ &= \int_{w(t-\delta)-\epsilon(t-\delta,\delta)}^{w(t-\delta)} \tilde{q}(t-\delta, x)dx - \int_{w(t)}^{w(t)+\epsilon(t-\delta,\delta)} \tilde{q}(t,x)dx. \end{aligned}$$

A closer look at K implies

$$\begin{aligned} K &= \int_{w(t)-\delta}^{w(t)+\epsilon(t-\delta,\delta)-\delta} \tilde{q}(t-\delta,x)dx - \int_{w(t)}^{w(t)+\epsilon(t-\delta,\delta)} \tilde{q}(t-\delta, x-\delta)\frac{\bar{F}(x)}{\bar{F}(x-\delta)}dx \\ &= \int_{w(t)-\delta}^{w(t)+\epsilon(t-\delta,\delta)-\delta} \tilde{q}(t-\delta,x)dx - \int_{w(t)-\delta}^{w(t)+\epsilon(t-\delta,\delta)-\delta} \tilde{q}(t-\delta, y)\frac{\bar{F}(y+\delta)}{\bar{F}(y)}dy \\ &= \int_{w(t)-\delta}^{w(t)+\epsilon(t-\delta,\delta)-\delta} \tilde{q}(t-\delta,y)\left(1 - \frac{\bar{F}(y+\delta)}{\bar{F}(y)}\right)dy, \end{aligned}$$

where the first equality follows from (52) and fundamental evolution equations, the second equality holds by change of variable. It is easy to see that $K = o(\delta)$ as $\delta \downarrow 0$. Therefore, together with (53), that implies that a left derivative of w exists at t with value in (33).

The stronger differentiability conclusion depends on the discontinuities of $\tilde{q}(t,x)$. From Proposition 6, all discontinuity points lie on finitely many 45 degree lines in the upper right quadrant $[0,\infty) \times [0,\infty)$; i.e., in the set $\{(t,x) : x = t + c \quad \text{and} \quad c \in \mathcal{S}\}$ where $\mathcal{S}$ contains $c = 0$ and the finite set of discontinuities of λ for $c < 0$ and the finite subset of discontinuities of $q(0,\cdot)$ for $c > 0$. Since $w(t+u) \le w(t) + u$ for $0 \le t \le t+u \le T$, the trajectory of $\tilde{q}(t, w(t))$ crosses over each of these lines at most once. Moreover, it stays on each line for at most a finite interval. If the trajectory immediately crosses over the line, then the crossing time t constitutes the sole discontinuity point for w' associated with that line. If the trajectory stays on the line for an interval, then the two endpoints constitute discontinuity points for w' associated with that line.

(f) (existence of a solution) The solution can be constructed by considering the successive intervals between discontinuity points and piecing together the solutions. The function Ψ in (32) is continuous in each continuity interval. Hence, existence follows from Peano's theorem; see §2.6 of [34]. We apply Assumption 9 to ensure that $w(0) < \infty$.

(g) (uniqueness of a solution) Under extra regularity conditions, the function Ψ in (32) will be locally Lipschitz on each continuity interval of w', so

that each piece constructed in the existence argument above will be unique, by virtue of the classical Picard-Lindelöf theorem; e.g., Theorem 2.2 of [34]. Specifically, it suffices to assume that λ and $q(0,\cdot)$ (already assumed to be in $\mathbb{C}_p$) are differentiable on the subintervals where they are continuous with derivatives in $\mathbb{C}_p$ over these subintervals.

However, we can actually prove uniqueness without resorting to extra assumptions. To do so, we exploit the special structure of the ODE in (32). By (29) in Corollary 3, $q(t, w(t)-)$ in the denominator or (32) takes one of two forms, depending on whether $w(t) \leq t$ or not. Our proof applies to both cases in the same way, so we only consider one case: we suppose that $w(t) \leq t$. Then $q(t, w(t)-) = \lambda((t - w(t))-)\bar{F}(w(t))$. Then ODE (32) implies that

$$\frac{b(t+,0)}{\bar{F}(w(t))} = \lambda((t-w(t))-)(1-w'(t)) = \frac{d}{dt}\left(\int_{t_1}^{t-w(t)} \lambda(y)dy\right),$$

$$\text{so that} \quad \int_{t_1}^{t} \frac{b(y,0)}{\bar{F}(w(y))}dy = \int_{t_1}^{t-w(t)} \lambda(y)dy, \quad t_1 \leq t \leq t_2. \tag{54}$$

Now suppose there is another function $\tilde{w}$ that also satisfies ODE (32) with $\tilde{w}(t_1) = 0$. Then, by the same reasoning, we get

$$\int_{t_1}^{t} \frac{b(y,0)}{\bar{F}(\tilde{w}(y))}dy = \int_{t_1}^{t-\tilde{w}(t)} \lambda(y)dy, \quad t_1 \leq t \leq t_2. \tag{55}$$

Equations (54) and (55) imply that

$$\int_{t_1}^{t} b(y,0)\left(\frac{1}{\bar{F}(w(y))} - \frac{1}{\bar{F}(\tilde{w}(y))}\right)dy = \int_{t-\tilde{w}(t)}^{t-w(t)} \lambda(y)dy, \quad t_1 \leq t \leq t_2. \tag{56}$$

Now suppose function w and $\tilde{w}$ are different. Since $w(t_1) = \tilde{w}(t_1) = 0$, let $\tilde{t} \equiv \inf\{t > t_1 : w(t) \neq \tilde{w}(t)\}$, which implies that $w'(\tilde{t}) \neq \tilde{w}'(\tilde{t})$. Without loss of generality suppose that $w'(\tilde{t}) < \tilde{w}'(\tilde{t})$, hence there exists a $\delta > 0$ such that $w(t) < \tilde{w}(t)$ for all $\tilde{t} < t \leq \tilde{t} + \delta$. Then we have $1/\bar{F}(w(t)) < 1/\bar{F}(\tilde{w}(t))$ for all $\tilde{t} < t \leq \tilde{t} + \delta$ and $\tilde{t} + \delta - \tilde{w}(\tilde{t} + \delta) < \tilde{t} + \delta - w(\tilde{t} + \delta)$. Therefore, (56) implies that

$$0 > \int_{\tilde{t}}^{\tilde{t}+\delta} b(y,0)\left(\frac{1}{\bar{F}(w(y))} - \frac{1}{\bar{F}(\tilde{w}(y))}\right)dy = \int_{\tilde{t}+\delta-\tilde{w}(\tilde{t}+\delta)}^{\tilde{t}+\delta-w(\tilde{t}+\delta)} \lambda(y)dy > 0,$$

which is a contradiction. Hence the solution to ODE (32) must be unique. ∎

Proof of Theorem 5. To show that the two equations in (37) are equivalent, make the change of variables $s \equiv t - w(t)$. Then the first equation gives $v(s) = w(t) = w(s + w(t)) = w(s + v(s))$, which is the second equation. The other direction is similar.

For a given w, we shall do three things: (i) construct v given the first equation in (37), (ii) show that this construction gives a function v that is right

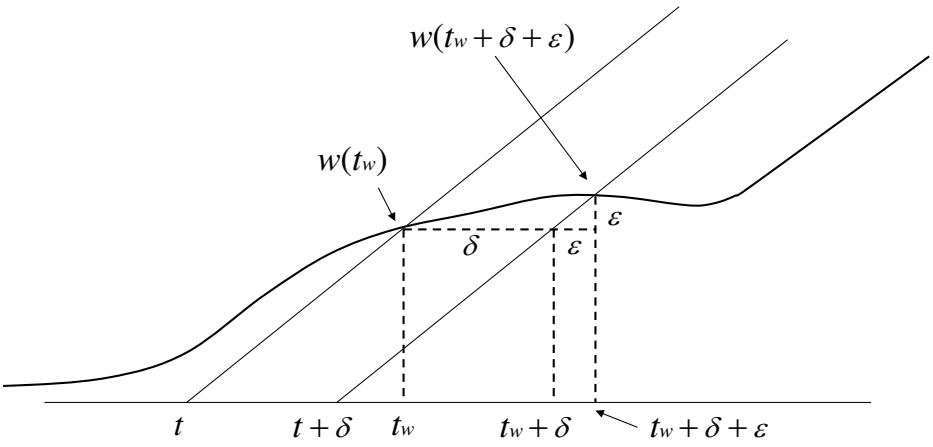


Fig. 7 Potential waiting time $v(t)$ is right continuous and has limits from the left.

continuous and has limits from the left, and (iii) show that the construction in (i) is the unique one that satisfies (ii).

For an arbitrary t, we draw a 45-degree ray starting from point $(t, 0)$: $L(s) = s - t$, $s \geq t$. Let $v(t)$ be the largest t_w such that $L(t_w) = w(t_w)$, as shown in Figure 5. We first show that there necessarily exists at least one time $t_w \geq t$ such that $L(t_w) = w(t_w)$. If $w(t) = 0$, then $t_w = t$ is a solution. Otherwise, we have $w(t) > 0 = L(t)$, and w starts above the line L at time t. By Theorem 3, w is a continuous function. In general, we could have $w(t) > L(t)$ for all t, but then we would have $v(t) = \infty$. Since $v(t) < \infty$, there necessarily is a time t_w such that $L(t_w) = w(t_w)$.

By Theorem 3, $w'(t) \leq 1$. Therefore, once $L(t_w) = w(t_w)$ for the first time, it either stays there or leaves, never to return. In other words, there are two cases: First, as always occurs if $w'(t_w) < 1$, there may be a unique $t_w \geq t$ such that $L(t_w) = w(t_w)$. Second, there may exist an interval $I \equiv [t_1, t_2]$ such that $L(t) = w(t)$ for $t \in I$, i.e., $L(t_1) = w(t_1)$ and $w'(t) = 1$ for $t \in I$; see Figure 5. In the first case, we let $v(t) \equiv t_w$; in the second case, we let $v(t) \equiv w(t_w)$ where $t_w \equiv \inf\{s > t_1 : L(s) \neq w(s)\}$. That completes our construction.

Next we show right continuity. For any $\epsilon > 0$, our construction shows that it is possible to choose $\delta > 0$ sufficiently small that $v(t + \delta) = w(t_w + \delta + \epsilon)$ such that $w(t_w + \delta + \epsilon) - w(t_w) = \epsilon$, where $\epsilon \equiv \epsilon(t, \delta)$, as shown in Figure 7. Our construction implies that

$$\epsilon = w(t_w + \delta + \epsilon) - w(t_w) = w'(\hat{t})(\delta + \epsilon)$$

for some $t_w \leq \hat{t} \leq t_w + \delta + \epsilon$ and $w'(\hat{t}) < 1$, which implies that

$$\epsilon \equiv \epsilon(t, \delta) = \frac{w'(\hat{t})\,\delta}{1 - w'(\hat{t})} \to 0, \quad as\ \delta \to 0.$$

Therefore, as $\delta \to 0$,

$$v(t+\delta) - v(t) = w(t_w + \delta + \epsilon) - w(t_w) \to 0,$$

by the continuity of w. Therefore, v is right continuous. Similarly, we can show that v has limits from the left.

It is evident that, by this construction, we have ensured that v is right continuous with left limits and unique. Moreover, v is discontinuous at t if and only if we are in the second case with an interval of solutions. ∎

Proof of Theorem 6. For $\delta > 0$, the second equation in (37) yields

$$\begin{aligned}\frac{v(t+\delta)-v(t)}{\delta} &= \left(\frac{w(t+\delta+v(t+\delta))-w(t+v(t))}{v(t+\delta)-v(t)+\delta}\right)\left(\frac{v(t+\delta)-v(t)+\delta}{\delta}\right)\\ &= \left(\frac{w(t+v(t)+\epsilon(t,\delta))-w(t+v(t))}{\epsilon(t,\delta)}\right)\left(\frac{v(t+\delta)-v(t)}{\delta}+1\right),\end{aligned}$$

where $\epsilon(t,\delta) \equiv v(t+\delta) - v(t) + \delta$. Simple algebra implies that

$$\frac{v(t+\delta)-v(t)}{\delta} = \frac{1}{1-\frac{w(t+v(t)+\epsilon(t,\delta))-w(t+v(t))}{\epsilon(t,\delta)}} - 1.$$

Letting $\delta \downarrow 0$, we obtain

$$\begin{aligned}v'(t+) &= \lim_{\delta\downarrow 0}\left(\frac{v(t+\delta)-v(t)}{\delta}\right) = \frac{1}{1-\lim_{\delta\downarrow 0}\left(\frac{w(t+v(t)+\epsilon(t,\delta))-w(t+v(t))}{\epsilon(t,\delta)}\right)} - 1\\ &= \frac{1}{1-w'((t+v(t))+)} - 1 = \frac{\tilde{q}(t+v(t), w(t+v(t))-)}{b((t+v(t))+,0)} - 1\\ &= \frac{\tilde{q}(t+v(t), v(t)-)}{b((t+v(t),0)} - 1 = \frac{\lambda(t+)\bar{F}(v(t))}{b(t+v(t)+,0)} - 1,\end{aligned}$$

where the second equality holds since right continuity of v implies that $\epsilon(t,\delta) \to 0$ as $\delta \to 0$, the third equality follows from ODE (32), the fourth equality follows from the second equation in (37), the last equality holds because the system being overloaded at time $t + v(t)$ implies that $\tilde{q}(t+v(t), v(t)) = q(t,0)\bar{F}(v(t)) = \lambda(t)\bar{F}(v(t))$. The similar argument applies to the left derivative with $(v(t) - v(t-\delta))/\delta$ when t is a continuity point of v.

By Theorem 5, v is continuous under the extra condition that $b(t,0) > 0$ for all t. That clearly makes the right derivative finite for all t. Hence, v is differentiable wherever Φ is continuous. We can now exploit Theorem 3 and its proof. Since $b(t,0) > 0$ for all t, there will be a one-to-one correspondence between the finitely many points where Ψ in (32) is discontinuous and the points where Φ is discontinuous. Now we have the relations (for the right derivatives everywhere)

$$v'(t) = \frac{w'(t+v(t))}{1-w'(t+v(t))} \quad \text{and} \quad w'(t) = \frac{v'(t-w(t))}{v'(t-w(t))+1}, \quad t \geq 0, \qquad (57)$$

with the denominators positive in both cases. Directly, we can establish existence and uniqueness of a solution to the ODE by the same reasoning as used for ODE (32) for w. ∎

12 Conclusions.

In this paper we have characterized all the standard performance functions for the $G_t/GI/s_t + GI$ fluid model. Our results were obtained under two important regularity conditions: (i) Assumption 2, requiring that we have a smooth model, and (ii) Assumption 7, requiring that there be only finitely many switches between overloaded (OL) and underloaded (UL) intervals in finite time. There also is a restriction on the service distribution in Assumption 8 in order to guarantee that the fixed point equation (18) for the rate of flow from queue into service, $b(t, 0)$, has a unique solution that can be computed iteratively. It suffices for either (i) the service hazard function h_G to be bounded or (ii) the system to have started empty at some time in the (finite) past; see §6. Additional regularity conditions were imposed in §7 to obtain results for the waiting times.

For M service, the relatively simple algorithm primarily requires solving the ODE for the BWT w in Theorem 3 and the equation for the PWT v in Theorem 5 during each OL interval. For non-exponential service, in addition we must solve the fixed point equation (18) for the flow rate into service $b(t, 0)$, which is needed to determine the full service content density $b(t, x)$. The algorithm is summarized in §8. We characterized the model, as just reviewed, under the assumption that the staffing function s is feasible, but in Theorem 7 we also characterized the minimum feasible staffing function greater than or equal to any given staffing function, provided that it is not changed prior to the first infeasibility time. In §10 we showed that we can construct a staffing function to stabilize the potential waiting time v at any desired target $v^* > 0$.

The fluid model is well defined directly, but it is intended to serve as an approximation for large-scale many-server queueing systems. We performed extensive simulation experiments to confirm that the fluid model can provide a useful approximation for such stochastic queueing systems. One of these experiments is described in §2; others are described in the appendix, available on the authors' web pages. The simulation results show that, first, the fluid approximation is essentially exact for very large queueing systems and, second, it can be effective as an approximation for mean values even when the scale is not too large; e.g., the number of servers might be only 20. The approximation tends to be more accurate when the system is either overloaded or underloaded, rather than critically loaded, as illustrated by Figure 3.

The results here even contribute to our understanding of the stationary $G/GI/s + GI$ fluid model introduced in [36]. For the special case of the $G/M/s + GI$ fluid model, building on the present paper, in [21] we prove that the time-dependent performance of the fluid model converges to that steady-state performance as time evolves for any finite initial condition. More-

over, we provide bounds on the rate of convergence. In [21], we also establish convergence to a periodic steady state for periodic models and we establish asymptotic loss of memory (ALOM) for more general time-varying models.

There are many directions for future research: (i) It remains to consider alternative approaches to obtaining feasible staffing functions. The method in §9, detects any infeasibility of a candidate staffing function and removes the problem by increasing the staffing after the violation point. Alternative methods could modify the entire staffing function, aiming to achieve minimum cost subject to constraints. (ii) It remains to establish existence, uniqueness and algorithm results for the more general model in which many of the regularity conditions imposed here are relaxed. (iii) It remains to extend the model to represented more complicated service systems with multiple service pools and multiple customer classes. Building on the present paper, a first step has been made for single-class networks of queues with time-varying Markovian routing among the queues in [20]. (iv) Finally, it remains to develop alternative approximations and many-server heavy-traffic limits for $G_t/GI/s_t + GI$ systems that tend to be nearly critically loaded at all times, instead of switching back and forth between OL and UL intervals.

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APPENDIX

to

The $G_t/GI/s_t + GI$ Many-Server Fluid Queue

by

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A Overview.

This appendix contains material supplementing the main paper. We start with results for the fluid model and conclude with simulation experiments.

First, §B explains why the service content density $b(t, x)$ satisfies the transport PDE in an underloaded (UL) interval, as noted in Remark 2. In §C we supplement §6 by presenting alternative algorithms for the service content density b during an overloaded (OL) interval. This leads to a another PDE for $b(t, x)$ under extra smoothness assumptions.

In §D we present additional results for the BWT w and the PWT v during an OL interval, thus supplementing §7. We begin by providing a more elementary proof of Theorem 3 for the ODE for the BWT w under additional smoothness regularity conditions. Then we prove Corollary 4, which provides explicit formulas for the BWT in special cases. We also state an analog of Corollary 4 for the PWT v. We also prove Theorem 4, which established conditions for the PWT v to be finite. In §E we discuss the structure of the BWT function w. Theorem 3 requires the positivity $\lambda_{inf} > 0$ in Assumption 10. We now consider cases in which $\lambda(t) = 0$ for some $t \geq 0$. We show that the BWT w can have more complicated structure when the the zero set has zero Lebesque measure or positive Lebesque measure.

In §F we say more about the flows, i.e., the service-completion-rate function, σ, and the abandonment-rate function, α, defined in (7) and (9). In §G we supplement §8, which summarizes the algorithm, by providing more discussion of the algorithm. In particular, we provide a formal statement of the algorithm for M service. We also specify the algorithm to adjust for an initially infeasible staffing function s and illustrate its performance. In §H we present additional material related to §10 on choosing staffing functions to stabilize delays. In particular, we show how to stabilize delays with general initial conditions. (In §10 we assumed that the system starts empty.)

Finally, in §I we supplement §2 in the main paper by presenting additional comparisons of the fluid model to simulations of large-scale queueing systems. These additional simulations confirm the observations in §2: First, for very large queueing systems, with thousands of servers, the individual sample paths of the scaled queueing processes have negligible stochastic fluctuations and agree closely with the computed fluid model performance functions. Second, for smaller queueing systems, e.g., with about 20 servers, the fluid model performance functions still provide remarkably accurate approximations for the mean values of the queueing processes.

B The Transport PDE for b in an Underloaded Interval

In Remark 2 we observed that the service content density b satisfies a version of the generic scalar transport equation in the underloaded case. We provide more details here. The same reasoning applies to the queue content density $\tilde{q}(t, x)$, during an overloaded interval, ignoring flow into service; see §7.1.

Proposition 9 (transport pde) *In the underloaded region, if $b(0, \cdot)$ is differentiable in x, then the service content function b is differentiable for $t \neq x$ and satisfies the the following pde, a simple version of the generic scalar transport equation:*

$$b_t(t, x) + b_x(t, x) \equiv \frac{\partial b}{\partial t}(t, x) + \frac{\partial b}{\partial x}(t, x) = -h_G(x)b(t, x). \tag{58}$$

Proof. Since λ and $b(0,\cdot)$ are both differentiable, then it is easy to see that $b(t,x)$ is differentiable for $t \neq x$. If we let $p(u) \equiv b(t+u, x+u)$, we have that

$$\begin{aligned} b_t(t,x) + b_x(t,x) = p'(0) &= \lim_{u\to 0}\left(\frac{p(u)-p(0)}{u}\right) \\ &= \lim_{u\to 0}\left(\frac{b(t+u,x+u)-b(t,x)}{u}\right) \\ &= \lim_{u\to 0}\left(\frac{\bar{G}(x+u)-\bar{G}(x)}{u}\right)\left(\frac{b(t,x)}{\bar{G}(x)}\right) \\ &= -\frac{g(x)}{\bar{G}(x)}b(t,x) = -h_G(x)b(t,x), \end{aligned}$$

where we apply the chain rule of calculus and the fundamental evolution equation for b in (5).

Solving pde (58) with initial conditions $\lambda(t)$ and $b(0,x)$, yields Proposition 2. To verify that, recall that the general solution to pde (58) is $b(t,x) = e^{-\int_0^x h_G(u)du}\phi(t-x) = \bar{G}(x)l(t-x)$, where function ϕ is any differentiable function. Here we have $\phi(t) = \lambda(t)1_{\{t\geq 0\}}$. By the initial condition, $b(0,x) = \phi(-x)\bar{G}(x)$ when $x \geq 0$. Therefore we see that the claim is valid.

C Alternative Algorithms for b in an Overloaded Interval

We now discuss alternative algorithms to calculate the service content density b in an overloaded interval.

If Assumption 8 holds, then a finite function b is uniquely characterized via equation (15), where

$$b(t,x) = \hat{b}(t,x)/h_G(x), \quad 0 \leq x \leq t < T, \tag{59}$$

with $\hat{b}$ being the unique solution of the equation

$$\hat{b}(t,x) \equiv \hat{a}(t,x) + g(x)\int_0^{t-x} \hat{b}(t-x,y)\,dy, \quad 0 \leq x \leq t < T, \tag{60}$$

where

$$\hat{a}(t,x) \equiv g(x)s'(t-x) + g(x)\int_0^\infty \frac{b(0,y)g(y+t-x)}{\bar{G}(y)}\,dy \in \mathcal{F}_T. \tag{61}$$

We can establish the existence of a unique solution to equation (60) by applying the Banach fixed point theorem on an appropriate space of functions of *two* variables.

Although this new fixed-point equation is more complicated, it can lead to a PDE characterization of b. This PDE representation follows directly by differentiating in the equation (60). (Convenient cancelation occurs.)

Theorem 9 (*PDE for* $\hat{b}$) *Under the assumptions of Theorems* 10 *and* 11*, wherever* $\hat{b}$ *has first partial derivatives with respect to* t *and* x*, it satisfies the PDE*

$$\hat{b}_t(t,x) + \hat{b}_x(t,x) = \hat{y}(t,x) + \hat{z}(x)\hat{b}(t,x), \quad 0 \leq x \leq t \leq T, \tag{62}$$

where

$$\hat{y}(t,x) \equiv \hat{a}_t(t,x) + \hat{a}_x(t,x) - \frac{g'(x)}{g(x)}\hat{a}(t,x) \quad \textit{and} \quad \hat{z}(x) \equiv \frac{g'(x)}{g(x)} \tag{63}$$

for $\hat{a}(t,x)$ *in* (68). (*The functions* $\hat{y}$ *and* $\hat{z}$ *in* (63) *are well defined by the assumptions in Theorem* 11.) *Associated with the PDE is the boundary condition*

$$\hat{b}(t,t) = \hat{a}(t,t) = g(t)s'(0) + g(t)\int_0^\infty b(0,y)h_G(y)\,dy, \quad 0 \leq t \leq T, \tag{64}$$

which is finite by (26).

We now continue with the two-parameter functions $b \equiv b(t,x)$. To apply the Banach fixed point theorem in this setting, we use the space $\mathcal{F}_{T,1}$ of measurable real-valued functions of the pair of real variables (t,x) over the "triangular" domain $0 \le x \le t \le T$, for which the norm

$$\|u\|_{T,1} \equiv \sup_{0\le t\le T} \int_0^t |u(t,x)|\,dx. \tag{65}$$

is finite. The norm $\|\cdot\|_{T,1}$ is an L_1 norm in one coordinate and an L_∞ norm in the other; it makes $\mathcal{F}_{T,1}$ a Banach space.

Theorem 10 (service content in the overloaded case) *Consider an overloaded interval $[0,T)$. If Assumption 8 holds, then a finite function b is uniquely characterized via equation (15), where*

$$b(t,x) = \hat{b}(t,x)/h_G(x), \quad 0 \le x \le t < T, \tag{66}$$

with $\hat{b}$ being the unique fixed point of the operator $\mathcal{T} : \mathcal{F}_{T,1} \to \mathcal{F}_{T,1}$ defined by

$$\mathcal{T}(u)(t,x) \equiv \hat{a}(t,x) + g(x)\int_0^{t-x} u(t-x,y)\,dy, \quad 0 \le x \le t < T, \tag{67}$$

where

$$\hat{a}(t,x) \equiv g(x)s'(t-x) + g(x)\int_0^\infty \frac{b(0,y)g(y+t-x)}{\bar{G}(y)}\,dy \in \mathcal{F}_T. \tag{68}$$

Moreover, the operator $\mathcal{T}$ is a monotone contraction operator on $\mathcal{F}_{T,1}$ with contraction modulus $G(T)$ for the norm $\|\cdot\|_{T,1}$ defined in (65), so that, for any $u \in \mathcal{F}_{T,1}$, the fixed point can be approximated by the n-fold iteration $\mathcal{T}^{(n)}$ of the operator $\mathcal{T}$ applied to u, with

$$\|\mathcal{T}^{(n)}(u) - \hat{b}\|_{T,1} \le \frac{G(T)^n}{1-G(T)}\|\mathcal{T}(u) - u\|_{T,1} \to 0 \quad as \quad n \to \infty \tag{69}$$

and, if $u \le (\ge)\mathcal{T}(u)$, then $\mathcal{T}^{(n-1)}(u) \le (\ge)\mathcal{T}^{(n)}(u) \le (\ge)\hat{b}$ for all $n \ge 1$. Finally, $\hat{b}(t,t) = \hat{a}(t,t) = g(t)b(0,0)$.

Proof. First, we show that $\hat{b}$ in (66) is a fixed point of the operator $\mathcal{T}$, i.e., that $\mathcal{T}(\hat{b}) = \hat{b}$. To see that, multiply (15) through by $h_G(x)$, noting that (i) $h_G(x)\bar{G}(x) = g(x)$ and (ii) we are interested in the case $x \le t$. We get $\hat{b}(t,x) = b(t,x)h_G(x) = b(t-x,0)g(x)$. Next we successively apply (17), (5) and a change of variables to get

$$\begin{aligned}
\hat{b}(t,x) &= b(t-x,0)g(x) = s'(t-x)g(x) + g(x)\int_0^\infty b(t-x,y)h_G(y)\,dy \\
&= s'(t-x)g(x) + g(x)\int_{t-x}^\infty b(t-x,y)h_G(y)\,dy + g(x)\int_0^{t-x} b(t-x,y)h_G(y)\,dy \\
&= s'(t-x)g(x) + g(x)\int_{t-x}^\infty b(0,y-(t-x))\frac{\bar{G}(y)}{\bar{G}(y-(t-x))}h_G(y)\,dy \\
&\qquad + g(x)\int_0^{t-x} \hat{b}(t-x,y)\,dy \\
&= s'(t-x)g(x) + g(x)\int_0^\infty b(0,y)\frac{g(y+t-x)}{\bar{G}(y)}\,dy + g(x)\int_0^{t-x}\hat{b}(t-x,y)\,dy \\
&= \hat{a}(t,x) + g(x)\int_0^{t-x}\hat{b}(t-x,y)\,dy = \mathcal{T}(\hat{b})(t,x),
\end{aligned} \tag{70}$$

where $\hat{a}(t,x) = \hat{c}(t,x) + \hat{d}(t,x)$ with

$$\hat{c}(t,x) \equiv g(x)s'(t-x) \quad \text{and} \quad \hat{d}(t,x) \equiv g(x)\int_0^\infty b(0,y)\frac{g(y+t-x)}{\bar{G}(y)}\,dy.$$

We next show that $\|\hat{a}\|_{T,1} < \infty$. First, $\|\hat{c}\|_{T,1} \leq G(T)\|s'\|_T < \infty$ because $s' \in \mathbb{C}_p \subset \mathbb{D}$. Because of the factor $g(x)$, $\|\hat{d}\|_{T,1}$ is bounded by the integral term. Taking the supremum over x and t with $0 \leq x \leq t \leq T$ of the integral in the expression for $\hat{d}$ yields the term τ in Assumption 8, which we have assumed is bounded. Hence $\|\hat{d}\|_{T,1} < \infty$, and so $\|\hat{a}\|_{T,1} < \infty$.

Next note that $\mathcal{T}$ is indeed a contraction operator on $(\mathcal{F}_{T,1}, \|\cdot\|_{T,1})$, because

$$\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_{T,1} \leq \sup_{0\leq t\leq T} \int_0^t g(x) \left(\int_0^{t-x} |u_1 - u_2|(t-x, y)\, dy \right) dx \leq G(T)\|u_1 - u_2\|_{T,1},$$

and we have assumed that $G(T) < 1$ for all T. The geometric rate of convergence in (69) is the standard conclusion from the Banach fixed point theorem, and the subsequent ordering follows from the monotonicity of $\mathcal{T}$. Finally, $\hat{b}(t,t) = \hat{a}(t,t)$ because the subset of u in $\mathcal{F}_{T,1}$ for which $u(t,t) = \hat{a}(t)$ is closed, and $\mathcal{T}$ maps that subset into itself, because $\mathcal{T}(u)(t,t) = \hat{a}(t,t)$, $0 \leq t \leq T$, for all u in $\mathcal{F}_{T,1}$. By (17), $\hat{a}(t,t) = g(t)b(0,0)$. ∎

We now provide conditions for $\hat{b}(\cdot, x)$ and $b(\cdot, x)$ to be in $\mathbb{C}_p$ for all $x \geq 0$. (We use these properties for $b(\cdot, 0)$ to establish properties of the ODE to calculate the BWT w in §4 of [**?**].) We first introduce extra smoothness conditions.

Assumption 13 (*extra smoothness for g and s*) *g and s' are differentiable with derivatives g' and s'' in $\mathbb{C}_p$.*

We next impose additional regularity conditions on the service-time pdf g. For that purpose, let $\|g\|_\infty$ be the uniform norm, i.e., $\|g\|_\infty \equiv \sup_{x\geq 0} \{|g(x)|\}$.

Assumption 14 (*extra regularity for g*) *The service-time pdf g satisfies: $g(x) > 0$ for all x, $\|g\|_\infty < \infty$ and there exists K such that $g(x) \leq g(0)e^{Kx}$ for all $x \geq 0$.*

We will use the last inequality in Assumption 14 in its equivalent form: $|g'(x)| \leq Kg(x)$ for all x. (To see the equivalence, Divide by $g(x)$, integrate and take the exponential.)

Theorem 11 (smoothness of service content in the overloaded case) *If Assumptions* 8–14 *all hold, then $\hat{b}(\cdot, x)$ and $b(\cdot, x)$ are differentiable functions for each $x \geq 0$, almost everywhere equal to their partial derivatives with respect to t, for b in* (66) *and $\hat{b}$ in* (67)*. Hence, $\hat{b}(\cdot, x), b(\cdot, x) \in \mathbb{C}_p$ for all $x \geq 0$.*

Proof. We again apply the Banach fixed point theorem, but now on a subspace of $\mathcal{F}_{T,1}$ with a new norm. Consider the subspace of measurable real-valued functions u of the pair of real variables (t, x) over the same triangular domain $0 \leq x \leq t \leq T$ that are differentiable with respect to the variable t, and equal almost everywhere to the integral of its partial derivative u_t, with finite norm $\|u\|_{T,2}$, where

$$\|u\|_{T,2} \equiv \sup_{0\leq t\leq T} \{ \int_0^t (|u(t,x)| + |u_t(t,x)|)\, dx, \} \tag{71}$$

which is like the Sobolev norm on the Sobolev space $\mathcal{W}^{1,\infty}(0,t)$. The functions in $\mathcal{F}_{T,2}$ are Lipschitz continuous in the first variable t for each x in $0 \leq x \leq t \leq T$. Reasoning as in the proof of Theorem 10, we will show that $\|\hat{a}\|_{T,2} < \infty$, and then we will show that $\mathcal{T}$ maps $\mathcal{F}_{T,2}$ into itself.

Then,

$$\|\hat{a}\|_{T,2} \leq \|\hat{a}\|_{T,1} + G(T) \left(\|s''\|_T + K \sup_{0\leq s\leq T} \int_0^\infty (b(0,y)g(s+y)/\bar{G}(y))\, dy \right) < \infty$$

by the proof of Theorem 10 and the conditions in Assumptions 8, 13 and 14. (Since $\mathbb{C}_p \subset \mathbb{D}$, $\|s''\|_T < \infty$.) Next, $\|\mathcal{T}(u)\|_{T,2} \leq \|\hat{a}\|_{T,2} + G(T)(\|u\|_{T,1} + \sup_{0\leq t\leq T} \{|u(t,t)\} + \|u_t\|_{T,1}) < \infty$. Then we see that $\mathcal{T}$ is again a contraction operator on $(\mathcal{F}_{T,2}, \|\cdot\|_{T,2})$ with modulus $G(T)$. We can ignore the term involving $|u_1(t,t) - u_2(t,t)|$, because, as noted at the end of Theorem 10, we can restrict attention to the closed subspace $\mathcal{F}_{T,2}$ containing only u for which $u(t,t) = g(t)b(0,0)$; as a consequence, $u_1(t,t) = u_2(t,t)$ for all t. Hence, the fixed point $\hat{b}$ is an element of $\mathcal{F}_{T,2}$, and so has the claimed smoothness properties. ∎

D More on the Performance in Overloaded Intervals

We now present additional material on the queue performance functions during an OL interval.

D.1 More on the BWT w

Alternate Proof of Theorem 3: the ODE for the BWT w. If we assume additional smoothness, then we can obtain a simple direct proof of Theorem 3. In particular, we can obtain the expression for the ODE describing the evolution of the BWT $w(t)$ by differentiating in the basic flow conservation equation in (6). Consider an overloaded interval that starts out with the queue empty, so that $Q(0) = 0$. Then, when we differentiate with respect to t in (6), we get

$$\frac{d}{dt}Q(t) \equiv Q'(t) = \lambda(t) - \alpha(t) - b(t,0), \tag{72}$$

where, from (7) and Corollary 3, by a change of variable,

$$\alpha(t) \equiv \int_0^\infty q(t,x)h_F(x)\,dx = \int_0^{w(t)} \lambda(t-x)f(x)\,dx = \int_{t-w(t)}^t \lambda(x)f(t-x)\,dx. \tag{73}$$

and,

$$Q(t) = \int_0^{w(t)} \lambda(t-x)\bar{F}(x)\,dx = \int_{t-w(t)}^t \lambda(x)\bar{F}(t-x)\,dx. \tag{74}$$

Then, assuming that w is differentiable (as well as $\bar{F}$), we can differentiate under the integral in (74) to get

$$Q'(t) = \lambda(t) - \tilde{q}(t,w(t))(1-w'(t)) + \int_{t-w(t)}^t \lambda(x)f(t-x)\,dx. \tag{75}$$

We remark that the standard conditions to justify differentiation under the integral, i.e., differentiation of

$$I(t) \equiv \int_{a(t)}^{b(t)} h(t,x)\,dx \tag{76}$$

is to have (i) the partial derivative of $h(t,x)$ with respect to t be well defined, (ii) $h(t,x)$ and $\partial h(t,x)/\partial t$ both be continuous in the two variables t and x in some region including $\{(t,x) : a(t) \le x \le b(t), t_1 \le t \le t_2\}$, and (iii) a and b to have continuous derivatives in the region $\{t : t_1 \le t \le t_2\}$. Under these conditions,

$$I'(t) = h(t,b(t))b'(t) - h(t,a(t))a'(t) + \int_{a(t)}^{b(t)} \frac{\partial h(t,x)}{\partial t}\,dx. \tag{77}$$

Equation (75) is an application of (77) to (74).

Inserting (75) into (72) and making appropriate cancelations ($\lambda(t)$ and $\alpha(t)$ appear on both sides), we get

$$b(t,0) = \tilde{q}(t,w(t))(1-w'(t)), \tag{78}$$

which yields

$$w'(t) = 1 - \frac{b(t,0)}{\tilde{q}(t,w(t))}. \tag{79}$$

The more complicated analysis in our main proof is needed because we do not have all the smoothness conditions. ∎

Proof of Corollary 4: explicit expressions for the BWT w. Since the proofs to (a) and (b) are similar, we will only prove (b). ODE (32) implies that

$$b(t,0)e^{\theta t} = \lambda(t-w(t))e^{\theta(t-w(t))}(1-w'(t)) = \frac{d}{dt}\left(\int_0^{t-w(t)} \lambda(y)e^{\theta y}dy\right),$$

which implies

$$\tilde{\Lambda}(t-w(t)) = \int_0^t b(y,0)e^{\theta y}dy,$$

and inverting function $\tilde{\Lambda}(\cdot)$ yields (34). Moreover,

$$\tilde{t} \equiv \inf\{t>0 : w(0)=0\} = \inf\{t>0 : \tilde{\Lambda}(t) = \int_{t_1}^t b(y,0)e^{\theta y}dy\}. \quad \blacksquare$$

D.2 More on the PWT v

We now give closed-form formulae for the PWT v in some special cases, paralleling those for the BWT w in Corollary 4. We omit the proof, which is similar to the proof of Corollary 4, which is given in the next subsection.

Corollary 7 *Suppose $v(0)=0$, the system is overloaded for $0<t<\delta$, $b(t,0)>0$.*
(a). If there is no abandonment, i.e., if the model is $G_t/M/s_t$, then

$$v(t) = \Gamma^{-1}(\int_0^t \lambda(y)dy) - t,$$

for $0 \le t < \bar{t}$, where $\Gamma(t) \equiv \int_0^t b(y,0)dy$, $\Gamma^{-1}(x) \equiv \inf\{y>0 : \Gamma(y)=x\}$, and $\bar{t} \equiv \inf\{t > 0 : \Gamma(t) = \int_0^t \lambda(y)dy\}$.
(b). If the abandonment-time distribution is exponential ($\bar{F}(x) = e^{-\theta x}$ for $x \ge 0$), i.e., if the model is $G_t/M/s_t + M$, then

$$v(t) = \tilde{\Gamma}^{-1}(\int_0^t \lambda(y)e^{\theta y}dy) - t,$$

for $0 \le t < \tilde{t}$, where $\tilde{\Gamma}(t) \equiv \int_0^t b(y,0)e^{\theta y}dy$, $\tilde{\Gamma}^{-1}(x) \equiv \inf\{y > 0 : \tilde{\Gamma}(y) = x\}$, and $\tilde{t} \equiv \inf\{t>0 : \tilde{\Gamma}(t) = \int_{t_1}^t \lambda(y)e^{\theta y}dy\}$.

Proof of Theorem 4: finiteness of PWT v.

Proof Recalling the definition of $\sigma(t)$ in (9), and using Assumption 12, we obtain

$$\sigma(t) = \int_0^\infty b(t,x)h_G(x)\,dx \ge \int_0^\infty b(t,x)h_{G,L}\,dx = B(t)h_{G,L}.$$

However, in the overloaded interval, $B(t) = s(t)$ and $s(t) \ge s_{lbd}$ by Assumption 11. Hence we have the claimed lower bound on $\sigma(t)$. We use that lower bound to bound $E(t+u)-E(t)$ below. Note that

$$E(t+u) - E(t) = \int_t^{t+u} b(v,0)\,dv = \int_t^{t+u} (s'(v)+\sigma(v))\,dv \ge s(t+u) - s(t) + s_L h_{G,L} u.$$

By Assumption 11, $s(t+u) \geq s_L$. Starting from the definition (36), we apply the inequalities above to obtain

$$\begin{aligned} v(t) &\equiv \inf\{u \geq 0 : E(t+u) - E(t) + A_t(u) \geq Q(t)\} \\ &\leq \inf\{u \geq 0 : E(t+u) - E(t) \geq Q(t)\} \\ &\leq \inf\{u \geq 0 : (s_L h_{G,L} u - s(t) + s_L)^+ \geq Q(t)\} \leq \frac{Q(t) + s(t) - s_L}{s_L h_{G,L}} < \infty, \end{aligned}$$

where $Q(t) \leq Q(0) + \Lambda(t) < \infty$ for all t.

E Structure of the Boundary Waiting Time w.

Theorem 3 requires the positivity $\lambda_{inf} > 0$ in Assumption 10. We now consider cases in which $\lambda(t) = 0$ for some $t \geq 0$. That leads to more complicated behavior for the BWT function w.

E.1 The Zero Set of $\lambda(\cdot)$ Has Zero Lebesgue Measure.

First, suppose that $\lambda(t_0) = 0$ for some $t_0 > 0$ but the zero set of $\lambda(\cdot)$ has zero Lebesgue measure, i.e., $\int_0^T 1_{\{}\lambda(t) = 0\}dt = 0$, see Figure 8(a). Again we assume that both $b(t,0)$ and $\lambda(t)$ are continuous for $0 \leq t \leq T$.

We only consider the overloaded case (the underloaded case is not interesting since $w(t) = 0$). For simplicity, suppose the system is initially critically loaded, i.e., $B(0) = S(0)$, $w(0) = 0$, $Q(0) = 0$, and $\lambda(0) > \sigma(0)$, then the system becomes overloaded in the next moment.

We give a vivid example. Let the system be initially critically loaded and suppose $b(t,0) = 1$ as long as the system is overloaded. For instance, this can be achieved if $S(t) = 1$ and the service-time distribution is exponential with rate 1. Let the arrival-rate function $\lambda(t) = t^2 - 3t + 9/4$ and the abandon-time distribution be exponential with rate 0.5, i.e., $\bar{F}(x) = 0.5 \cdot e^{-0.5x}$ for $x \geq 0$.

We can see from Figure 8(a) that $\lambda(3/2) = 0$ and $\int_0^T 1_{\{\lambda(t)=0\}}dt = 0$ for all $T > 0$. Because $\lambda(0) = 9/4 > b(0) = 1$ the system becomes overloaded after time 0. We plot in Figure 8(b) the boundary waiting time $w(t), 0 \leq t \leq T$ with $T = 3$. One can see that the derivative of $w(t)$ reaches $-\infty$ once, and this corresponds to the fact that $\lambda(t)$ touches 0 once but does not stay at 0.

E.2 The Zero Set of $\lambda(\cdot)$ Has Positive Lebesgue Measure.

In a more general setup of the arrival process, $\lambda(t)$ can stay at 0 for a while meaning that the arrival process is turned off. For instance, it is natural that the arrival process may look like the first picture in Figure 9.

Intuition tells us in this case $w(t)$ cannot be continuous for all $t \geq 0$, it will jump at some times. But when will $w(t)$ jump? What will be the heights of the jumps? To answer these questions, we simply assume that $\lambda(t) = 0$ for $0 < \hat{t}_1 \leq t < \hat{t}_2 < \infty$. The case that $\lambda(t) = 0$ for t in finite disjoint intervals can be easily generalized. Note that $\lambda(t)$ being left-continuous or right-continous does not matter because it is just a rate function.

Again, we consider a vivid example. Suppose the system is initially overloaded with $w(0) = 2$ and $q(0,x) = e^{0.5x} 1_{\{0 \leq x \leq w(0)\}}$. We choose $\lambda(t)$ large enough such that the system stays overloaded for $t \geq 0$ and fix $b(t,0) = 0.5$. Let $\lambda(t) = (9t - 3t^2) \cdot 1_{\{0 \leq t < 3\}} + 3 \cdot 1_{\{t \geq 3.5\}}$. In other words, $\lambda(t)$ is quadratic for $t \in [0,3)$, stays at 0 for $t \in [3, 3.5)$, and is constant 3 for $t \geq 3.5$, see Figure 9(a). Let the abandon-time distribution be exponential with rate 0.5.

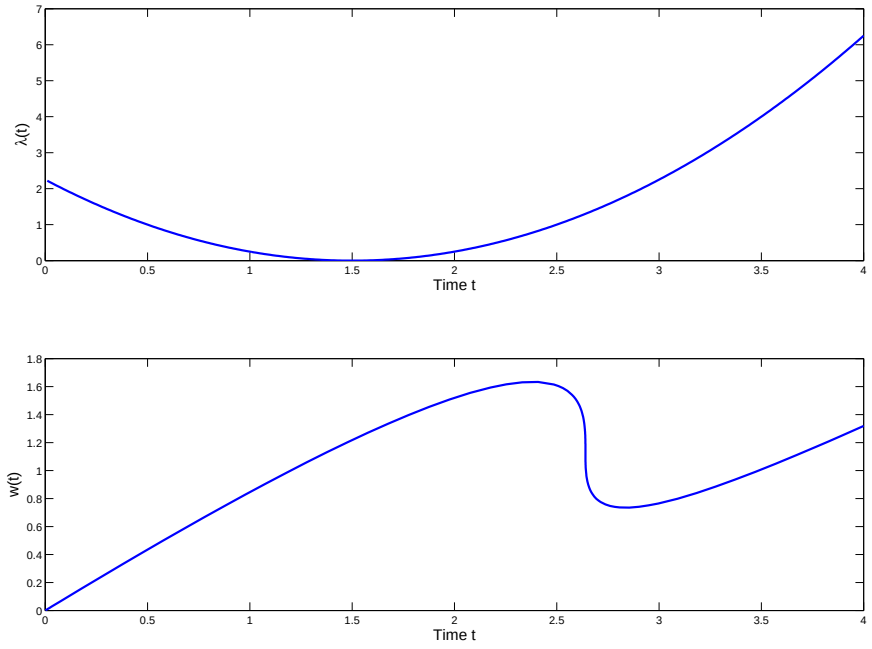

Fig. 8 An example of boundary waiting time $w(t)$ with $\lambda(t) = 0$ once.

In Figure 9(b), the red line is $q(t, x)$ at $t = 0$, which is a function of x. The blue line on the negative half-line is the arrival-rate function $\lambda(t)$ reflected with respect to the y axis. Imagine that with the origin fixed, the blue line moves to the right at rate 1, because new fluid keeps arriving to the system after time 0. The right boundary of the red line is the boundary waiting time $w(t)$ at each t, which is being controlled by the ratio between $b(t, 0) = 1$ and $q(t, w(t))$. So one can see that the right boundary of the red line is moving at rate $1 - b(t, 0)/q(t, w(t))$ since fluid at the front of the queue is being transported into service (eaten away) by $b(t, 0)$.

As time evolves, for the part of the reflected arrival-rate function that exceeds the origin (that is pushed onto the positive half-line), the height decreases with time because of abandonment. In Figure 9(c), all fluid that was in queue at time 0 is just gone at time t_1, and $w(t_1) = t_1$ because the blue line travelled by t_1 to the right. At time t_1, $q(t_1, x) = \lambda(t_1 - x) \cdot e^{-0.5} \cdot 1_{\{0 \le x \le t_1\}}$ which is the red line, and $q(t_1, w(t_1)) = q(t_1, t_1) = 0$ implies that $w'(t_1) = -\infty$, see Figure 10. Although $w'(t)$ has a discoutinuity at t_1, $w(t)$ itself is continuous at t_1.

At time t_2- which is the moment right before the quadratic part of $\lambda(t)$ is eaten away, the boundary waiting time $w(t_2-) = t_2 - 3$, where 3 is the length of the quadratic part of $\lambda(t)$. Then at time t_2+, $w(t)$ jumps from $w(t_2-) = t_2 - 3$ to $w(t_2+) = t_2 - 3.5$, because there is an interval of length 0.5 in which $\lambda(t) = 0$, see Figure 10. At t_2 the left derivative $w(t_2-) = \infty$ because $q(t_2-, w(t_2-)) = 0$.

This example shows that discontinuities of λ yield discontinuities of w', and λ staying at 0 over in interval yields discontinuities of w.

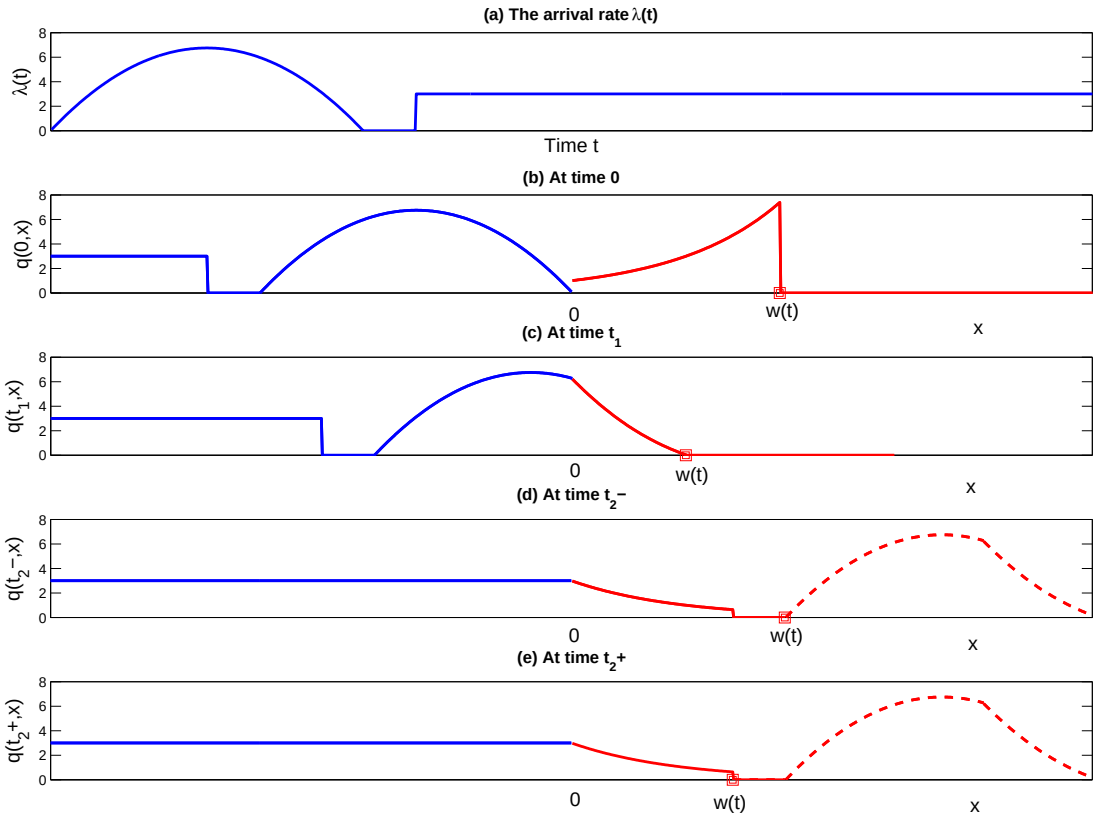

Fig. 9 The dynamics of $q(t,x)$ of an example with $\lambda(t) = 0$ for $0 < t_1 \leq t < t_2 < \infty$.

F More on the Flows

We next discuss the departure function S in (9) and the abandonment function A in (7). These flows are performance measures of interest in their own right, but they are also important because they enable us to extend the model treated here directly to open networks of fluid queues, in which the departing fluid or abandoning fluid from one queue become input to another queue; see [**?**].

F.1 Main Results

We show that the flows S and A inherit the structure of the original input Λ, so that the results in this paper extend to open networks of fluid queues. The following results are elementary. The proofs and other properties are given in the following subsection.

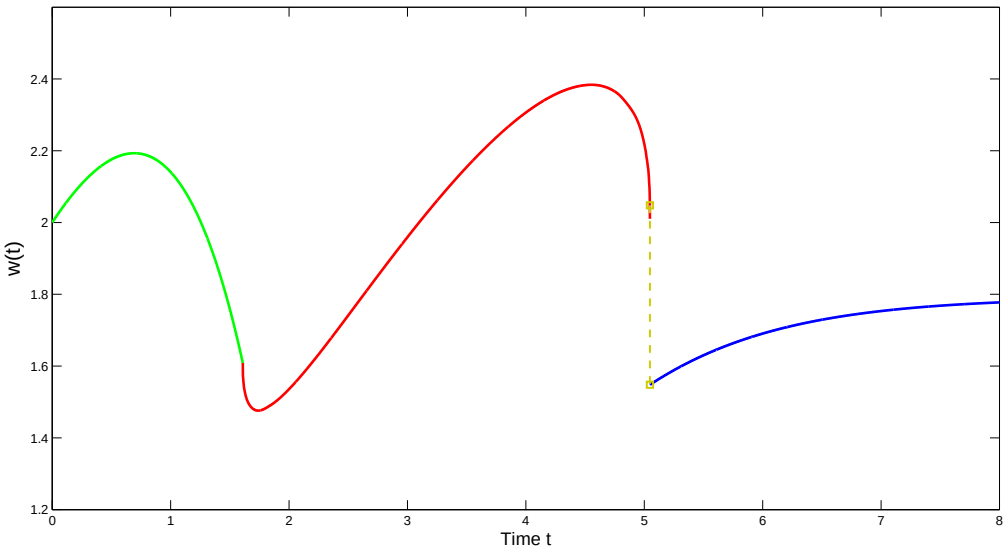

Fig. 10 An example of the boundary waiting time $w(t)$ with $\lambda(t) = 0$ for $0 < t_1 \leq t < t_2 < \infty$.

Theorem 12 (the departure rate) *Assume that the conditions in Theorem* 11 *hold. For* $t \geq 0$,

$$\sigma(t) = \int_0^t b(t-x,0)g(x)\,dx + \int_0^\infty \frac{b(0,y)g(t+y)}{\bar{G}(y)}\,dy,$$

where $b(t,0) = \lambda(t-u)$ *in an underloaded interval, but is the solution to the fixed point equation in Theorem 10 during an overloaded interval. As a consequence,* $\sigma \in \mathbb{C}_p$.

Theorem 13 (abandonment rate) *Assume that the conditions in Theorem 4.1 of* [**?**] *hold, so that the BWT* w *is well defined and continuous. For* $t \geq 0$,

$$\alpha(t) = \left(\int_0^{w(t)} \lambda(t-x)f(x)\,dx\right) 1_{\{w(t)\leq t\}} + \left(\int_0^t \lambda(t-x)f(x)\,dx + \int_0^{w(t)-t} \frac{q(0,y)f(t+y)}{\bar{F}(y)}\,dy\right) 1_{\{w(t)>t\}}.$$

As a consequence, $\alpha \in \mathbb{C}_p$.

F.2 Elaboration on the Flows

We now elaborate on the discussion about the flows in the previous subsection; i.e., we discuss the departure process S in (9) and the abandonment process A in (7). Make the same assumptions as above including the conditions in Theorem 11 and Assumption 12.

Theorem 14 (departure rate)

1. *For $t \geq 0$,*

$$\sigma(t) = \int_0^\infty b(t,x)h_G(x)\,dx = \int_0^t b(t-x,0)g(x)\,dx + \int_0^\infty \frac{b(0,y)g(t+y)}{\bar{G}(y)}\,dy, \quad (80)$$

where $b(t,0) = \lambda(t-u)$ in an underloaded interval, but is the solution to the fixed point equation in Theorem 10 during an overloaded interval.
2. *$\sigma \in \mathbb{C}_p$, as assumed for λ in Assumption 2.*
3. *$\sigma(t) \geq B(t)h_{G,L} > 0$ for all $t \geq 0$, so that σ satisfies the requirement for λ in Assumption 10 over the interval $[\epsilon, t]$ for each $\epsilon > 0$.*
4. *If there exists a constant $h_{G,U}$ such that $h_G(x) \leq h_{G,U} < \infty$ for all $x \geq 0$, then $\sigma(t) \leq B(t)h_{G,U} \leq s(t)h_{G,U}$ for all $t \geq 0$.*
5. *If $b(t,0)$ is absolutely continuous with derivative $b'(u,0)$ in $\mathbb{C}_p$ on the interval $[0,t]$ (as occurs in the case of exponential service) and if*

$$\tau_2(b,g,t) \equiv \sup_{0 \leq s \leq t} \int_0^\infty \frac{b(0,y)|g'(s+y)|}{\bar{G}(y)}\,dy < \infty, \quad (81)$$

then σ is absolutely continuous with derivative (a.e.)

$$\sigma'(t) = b(0,0)g(t) + \int_0^t b'(u,0)g(x)\,dx + \int_0^\infty \frac{b(0,y)g'(s+y)}{\bar{G}(y)}\,dy. \quad (82)$$

Proof. We prove the properties in turn:

(i) (representation (80)) Apply (9) and Assumption 6.

(ii)($\sigma \in \mathbb{C}_p$) By the finiteness of the initial conditions, Assumption 8 and the continuity of $b(\cdot,0)$ from Theorem 11, $\sigma(t) < \infty$. By Theorem 11, $b(\cdot,0)$ is in $\mathbb{C}_p$. By the Lebesgue dominated convergence theorem, the continuity of $b(t,0)$ and $g(t+y)$ in the integrands of (80) is inherited by σ, so $\sigma \in \mathbb{C}_p$, as claimed.

(iii) (lower bound) By the initial relation in (80), we have $\sigma(t) \geq B(t)h_{G,L}$. Since $s(u) \geq s_L > 0$ for $0 \leq u \leq t$, $\lambda(t) \geq \lambda_{inf}(t) > 0$ and $\bar{G}(x) > 0$ for all x, we have $B(t) \geq t\lambda_{inf}(t)\bar{G}(t) \wedge s_L$ for all $t \geq 0$, which implies that there exist constants $\epsilon > 0$ and $\sigma_{\{inf,\eta,\epsilon\}}$ such that $\sigma(u) > \sigma_{\{inf,\eta,\epsilon\}} > 0$ for $0 < \epsilon \leq u \leq t$.

(iv) (upper bound) By the initial relation in (80), we have $\sigma(t) \leq B(t)h_{G,U}$, but we always have $B(t) \leq s(t)$.

(v)(derivative) We differentiate under the integral in (80) using Leibniz integral formula for differentiation under the integral, for which we require the finiteness of τ_2 in (81). ∎

The abandonment rate is somewhat more difficult. First, the abandonment is only positive during the overloaded intervals, so we assume that we are focusing on a single overloaded interval. Second, the abandonment depends on q, which in turn depends on w, which also is more complicated, requiring more conditions.

Theorem 15 (abandonment rate) *Assume that the conditions in Theorem 3 hold, so that the BWT w is well defined and continuous.*

1. *For $t \geq 0$,*

$$\alpha(t) = \left(\int_0^{w(t)} \lambda(t-x)f(x)\,dx\right) 1_{\{w(t) \leq t\}} + \left(\int_0^t \lambda(t-x)f(x)\,dx + \int_0^{w(t)-t} \frac{q(0,y)f(t+y)}{\bar{F}(y)}\,dy\right) 1_{\{w(t)>t\}}. \quad (83)$$

2. *$\alpha \in \mathbb{C}_p$, as assumed for λ in Assumption 2.*
3. *If Assumption 12 holds, then $\alpha(t) \geq Q(t)h_{G,L}$ for all $t \geq 0$.*
4. *If there exists a constant $h_{G,U}$ such that $h_G(x) \leq h_{G,U} < \infty$ for all $x \geq 0$, then $\sigma(t) \leq Q(t)h_{G,U}$, which is bounded over finite intervals, because Q is continuous.*

5. *If $b(t,0) > 0$ a.e., then α is absolutely continuous with derivative (a.e.)*

$$\begin{aligned}\alpha'(t) = &\left(\lambda(t-w(t))f(w(t))w'(t) + \int_0^{w(t)} \lambda'(t-x)f(x)\,dx\right) 1_{\{w(t)\le t\}} \\ &+ \left(\lambda(0)f(w(t)) + \int_0^t \lambda'(t-x)f(x)\,dx + \left(\frac{q(0,w(t)-t)f(w(t))}{\bar{F}(w(t)-t)}\right)(w'(t)-1)\right) \\ &+ \int_0^{w(t)-t} \frac{q(0,y)f'(s+y)}{\bar{F}(y)}\,dy\Bigg) 1_{\{w(t)>t\}}. \end{aligned} \tag{84}$$

Proof. We prove the properties in turn:

(i) (representation) Applying definition (7) and Assumption 6, we have

$$\alpha(t) = \int_0^\infty q(t,x)h_F(x)\,dx = \int_0^t q(t-x,0)f(x)\,dx + \int_0^\infty \frac{q(0,y)f(t+y)}{\bar{F}(y)}\,dy, \tag{85}$$

from which (83) follows.

(ii) ($\alpha \in \mathbb{C}_p$) Note that $\lambda, q(0,\cdot) \in \mathbb{C}_p$ by Assumption 2, $q(\cdot,0) \in \mathbb{C}_p$ by Theorem 3 and Corollary 3 and w is continuous by Theorem 3. Hence, by the Lebesgue dominated convergence theorem, the continuity of $\lambda(t,0)$ and $f(t+y)$ as a function of t in the integrands of (80) is inherited by σ, so $\sigma \in \mathbb{C}_p$, as claimed.

(iii) (lower bound) By the initial relation in (83), we have $\alpha(t) \ge Q(t)h_{F,L}$.

(iv) (upper bound) By the initial relation in (83), we have $\alpha(t) \le Q(t)h_{F,U}$.

(v) (derivative) We differentiate under the integral in (80) using Leibniz integral formula for differentiation under the integral. Since the integrands are bounded over the finite intervals, the integrals are finite. ∎

G More on the Algorithm

G.1 The Algorithm for the $G_t/M/s_t + GI$ Model

We formally state the algorithm for the $G_t/M/s_t + GI$ fluid model. We assume the system does not alternate between overloaded and underloaded for infinitely many times in a finite time horizon. It might be possible to construct functions $\lambda(\cdot)$ and $s(\cdot)$ such that the lengths of intervals in which the system is overloaded and underloaded converge to 0, but we do not consider those cases.

For given model parameters F, G, $\lambda(t)$, $s(t)$ for $0 \le t \le T$, and initial condition $q(0,x) = r(x)$ for $x \ge 0$, $Q(0) = \int_0^\infty r(x)dx$, $w(0) = w_0 \ge 0$ and $B(0) \le s(0)$ such that $Q(0)(s(0) - B(0)) = 0$, the algorithm is as follows:

- $n \leftarrow 1$, $t_0^u \leftarrow 0$, $t_0^d \leftarrow 0$, $F \leftarrow 1$
- IF $B(0) < s(0)$
  - $B_0 \leftarrow B(0)$
  - $t_n^u \leftarrow \inf\{t > t_{n-1}^d : B_0 e^{\mu t_{n-1}^d} + \int_{t_{n-1}}^t \lambda(y)e^{\mu y}dy \ge s(t)e^{\mu t}\}$
  - IF $t_n^u \ge T$, THEN $N \leftarrow n-1$, $U \leftarrow 1$. END.
  - ELSE
    - $t^* \leftarrow t_n^u$, $w_0 \leftarrow 0$
    - $w(t)$ solves ODE (32) for $t \ge t^*$ with $w(t^*) = w_0$, $t_n^d \leftarrow \inf\{t > t_n^u : w(t) = 0\}$
    - IF $s'(t) + s(t)\mu < 0$ for some $t^* \le t \le t_n^d$, THEN $F \leftarrow 0$, END.
    - IF $t_n^d \ge T$, THEN $N \leftarrow n$, $U \leftarrow 0$. END.
    - ELSE
      - $B_0 \leftarrow s(t_n^d)$, $n \leftarrow n+1$, go to line 4.
- ELSEIF $B(0) = s(0)$ and $Q(0) > 0$
  - $w(t)$ solves ODE (32) with $w(0) = w_0$, $\hat{t} \leftarrow \inf\{t > 0 : w(t) = t\}$

  - IF $s'(t) + s(t)\mu < 0$ for some $0 \le t \le \hat{t}$, THEN $F \leftarrow 0$, END.
  - $t^* \leftarrow \hat{t}$, $w_0 \leftarrow \hat{t}$, go to line 8.
- ELSE ($B(0) = s(0)$ and $Q(0) = 0$)
  - IF $\lambda(0) > \sigma(0)$, THEN $t^* \leftarrow 0$, $w_0 = 0$, go to line 8.
  - ELSE ($\lambda(0) > \sigma(0)$), $B_0 = s(0)$, go to line 4.

The output of this algorithm are integer variable N, indicator variables U and F, and a sequence of time points $\{t_n^u, t_n^d, n = 1, 2, \ldots, N\}$. $F = 0$ implies that the given service-capacity function $s(\cdot)$ is infeasible and the algorithm will end as soon as it detects this infeasibility. The service-capacity function $s(t)$ is feasible if the algorithm ends with $F = 1$. The value of indicator variable U indicates whether the system is underloaded ($U = 1$) or overloaded ($U = 0$) at time T. For instance, if the system is underloaded at time 0 and $U = 1$, then we know that the system is underloaded for $t \in [0, t_1^u] \cup [t_1^d, t_2^u] \cup \cdots \cup [t_{N-1}^d, t_N^u] \cup [t_N^d, T] \equiv T^U$, and the system is overload for $t \in [t_1^u, t_1^d] \cup [t_2^u, t_2^d] \cup \cdots \cup [t_N^u, t_N^d]$. Moreover, we have that $w(t)$ solves ODE (32) for $t \in [t_n^u, t_n^d]$ with $w(t_n^u) = 0$ for $1 \le n \le N$, and $w(t) = 0$ for all $t \in T^U$. Other cases are similar.

G.2 A Fluid Algorithm with Infeasible s.

Our main algorithm for the $G_t/GI/s_t + GI$ fluid model assumes that the staffing function s is feasible. That algorithm is designed to stop whenever the given staffing function s is detected to be infeasible. Now we want to apply the results in §9 to find the minimum feasible staffing function.

We illustrate how to do so for the $G_t/M/s_t + GI$ model; §9 shows how to do the same for more general GI service. In the context of the $G_t/M/s_t + GI$ model, a sufficient condition for feasibility over $[0, T]$ is

$$s(t) + s'(t) \ge 0, \quad 0 \le t \le T. \tag{86}$$

Here we want to generalize our algorithm. Suppose the target staffing function s is not feasible for all t. Instead of stopping the algorithm, we want (i) to produce a 'best' modified capacity function $s_f(t)$ and (ii) to finish the algorithm with our new target $s_f(t)$.

We only need to modify our initial algorithm when the system is in the overloaded regime. Flow conservation of the service facility says that $b(t, 0) = B'(t) + \mu\, B(t)$ which is equal to $s'(t) + s(t)$ if $s(t)$ were feasible. However, if we want to make $B(t)$ decrease as fast as possible, the best we can do is to set $b(t, 0) = 0$ and let fluid deplete with only its service completion. Therefore, when s becomes infeasible at t_1, i.e., $s'(t_1+) + s(t_1+)$ becomes negative, $B(t)$ will satisfy ODE $B'(t) = -B(t)$ for $t \in [t_1, t_1 + \delta]$ with $B(t_1) = s(t_1)$, which implies that $B(t) = s(t_1)e^{-(t-t_1)}$.

We let $t_2 \equiv \inf\{t_1 < t \le T : s(t) = B(t)\} \wedge T = \inf\{t_1 < t \le T : s(t) = s(t_1)e^{-(t-t_1)}\} \wedge T$. Note that $b(t, 0) = 0$ for $t_1 \le t \le t_2$ guarantees that the queue does not empty out before t_2 so that the system does not switch from overloaded to underloaded regime before t_2. This is so because with $b(t, 0) = 0$, abandonment becomes the only source that deplete the queue, and the abandonment rate $\alpha(t)$ goes to 0 as $Q(t)$ goes to 0. For instance, if the abandonment distribution is exponential with rate θ, then $\alpha(t) = \theta\, Q(t)$.

If $t_2 = T$, the system stays overloaded until T and we are done. Otherwise, we let $t_3 \equiv \inf\{t_2 < t \le T : s'(t) + S(t) < 0\} \wedge T$, $b(t, 0) = s'(t) + \mu\, s(t)$ for $t_2 \le t \le t_3$. Just as in the original algorithm, we solve ODE (32) with $w(t_2) = 0$ for $t_2 \le t \le t_3$. If $t_U \equiv \{t > t_2 : w(t) = 0\} < t_3$, then the system switches from overloaded to underloaded regime and we continue with the old algorithm in the main paper; otherwise, s becomes infeasible once again at t_3 while the system is overloaded, and we shall repeat the above argument, and as before, we run the algorithm dynamically until we proceed to time T.

It is not hard to see that under the above construction, we successfully obtain the interval I_{inf} in which s is infeasible and a modified service-capacity function $s_f(t) = B(t)\, 1_{t \in I_{inf}} + s(t)\, 1_{t \in [0,T]/I_{inf}}$. Also, $s_f(t)$ is the closest feasible function to the given target $s(t)$.

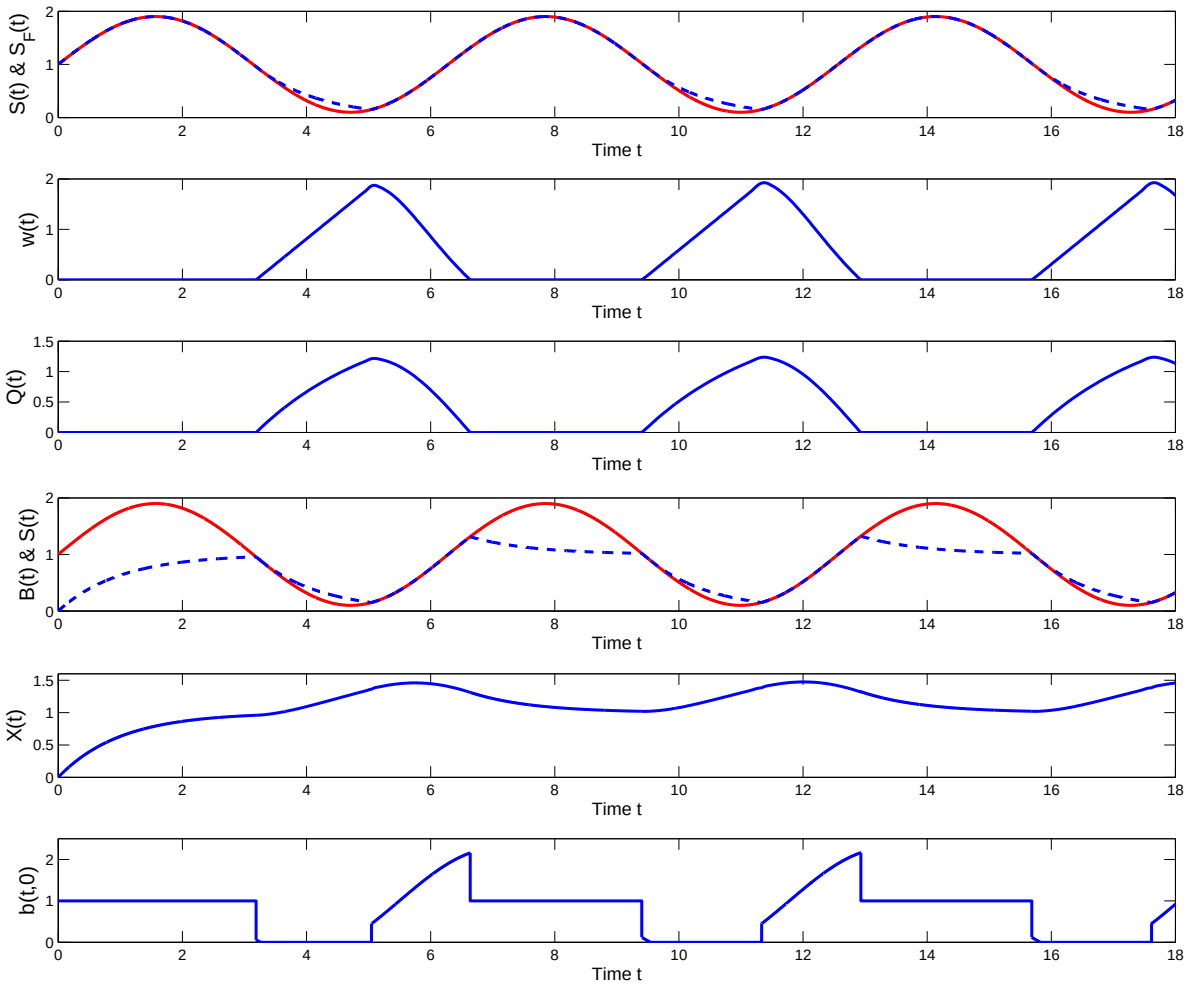

Fig. 11 The $M/M/s_t + M$ fluid model with infeasible s.

Example of the Algorithm. To evaluate the performance of the modified algorithm, we use the example in §I.2.1, i.e., we consider the Markovian $M/M/s_t + M$ model that has a Poisson arrival process with a constant rate λ, exponential service and abandonment distributions with rates μ and θ respectively, and a sinusoidal capacity function

$$s(t) \equiv \lambda + \bar{\lambda} \cdot \sin(c \cdot t). \tag{87}$$

We still let $\lambda = 1$, $c = 1$, $\mu = 1$, $\theta = 0.5$. To make s infeasible, we let $\bar{\lambda} = 0.9\lambda = 0.9$ instead of $0.6\lambda = 0.6$ in §I.2.1. Now s has greater fluctuation and it is easy to see that condition (86) is no longer satisfied.

We plot the performance measures of the fluid model in Figure 11. Compared with Figure 21, we see that $I_{inf} \equiv [3.27, 5.05] \cup [9.55, 11.33] \cup [15.84, 17.62]$ is the interval in which s becomes infeasible. For $t \in I_{inf}$, $s_f(t)$ (the blue dashed curve) is different from (above) s (the red solid curve), and $B(t)$ follows s_f instead of s since $B(t)$ cannot decrease

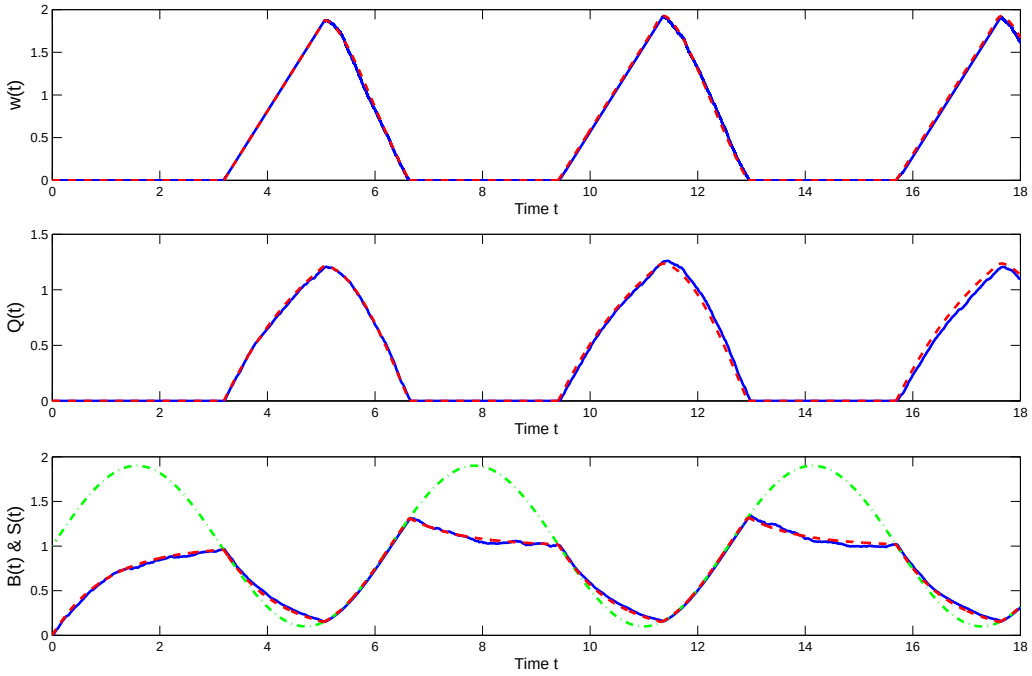

Fig. 12 The $M/M/s_t + M$ fluid model with infeasible s compared with simulation.

as fast as $s(t)$. Moreover, since $b(t, 0) = 0$ for $t \in I_{inf}$, $w(t)$ increases with slope 1. In other words, since the system stops transporting fluid from the queue into service, whatever is waiting at the head of the queue keeps waiting there. However, $Q(t)$ does not increase with rate 1 because abandonment still occurs.

Figure 12 shows that $w(t)$, $Q(t)$ and $B(t)$ obtained from our modified algorithm (the red dashed curves) agrees with single sample paths of simulation estimates of $w_n(t)$, $\hat{Q}_n(t)$ and $\hat{B}_n(t)$ (the blue solid curves), where we still set the fluid scaling factor $n = 1000$. Both $B(t)$ and $\hat{B}_n(t)$ are distinct from the given service-capacity function s (the dashed green curve) in I_{inf}.

H Stabilizing Delays with General Initial Conditions

Is §10 we showed how to choose a staffing function to stabilize the PWT v at any desired target v^*. However, Theorem 8 considered a special initial condition: the system is initially empty. We generalize Theorem 8 to arbitrary initial conditions in the next theorem.

Theorem 16 *Consider the $G_t/GI/s_t+GI$ fluid model with a general arrival-rate function λ and initial conditions $w(0-) \equiv w_0 \geq 0$, $b(0-, x) \equiv \psi(x) \geq 0$ for $x \geq 0$, $q(0-, x) \equiv \phi(x) \geq 0$ for $0 \leq x \leq w_0$, $Q(0-) = \int_0^{w_0} q(0-, x)dx$, $s(0-) = B(0-) = \int_0^{\infty} b(0-, x)dx$. For any given $v^* \geq 0$, we can make the system overloaded such that the PWT is fixed at v^*, i.e.,*

$v(t) = v^*$ for all $t \geq 0$, by letting the service-capacity function be

$$\begin{aligned} s(t) = & \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{G}(t)\int_{v^*}^{w_0 \vee v^*} \phi(x)dx \\ & + \bar{F}(v^*)\left(\int_{(t-v^*)^+}^{t-(v^*-w_0)^+} \frac{\phi(w_0 \wedge v^* - t + x)\,\bar{G}(x)}{\bar{F}(w_0 \wedge v^* - t + x)}dx\right) \cdot 1_{\{t \geq (v^*-w_0)^+\}} \\ & + \bar{F}(v^*)\left(\int_0^{t-v^*} \lambda(t-x-v^*)\bar{G}(x)dx\right) \cdot 1_{\{t \geq v^*\}}. \end{aligned} \tag{88}$$

If we do so, then

$$\begin{aligned} w(t) &= v^* \cdot 1_{\{t \geq (v^*-w_0)^+\}}, \\ b(t,0) &= \delta_0(t)\int_{v^*}^{w_0 \vee v^*} \phi(x)dx + \frac{\phi(w_0 \wedge v^* - t)\,\bar{F}(v^*)}{\bar{F}(w_0 \wedge v^* - t)} \cdot 1_{\{(v^*-w_0)^+ \leq t < v^*\}} + \lambda(t-v^*)\bar{F}(v^*) \cdot 1_{\{t \geq v^*\}}, \\ B(t) &= s(t), \\ \sigma(t) &= \int_t^\infty \psi(x-t)\frac{g(x)}{\bar{G}(x-t)}dx + g(t)\int_{v^*}^{w_0 \vee v^*} \phi(x)dx \\ &\quad + \bar{F}(v^*)\left(\int_{(t-v^*)^+}^{t-(v^*-w_0)^+} \frac{\phi(w_0 \wedge v^* - t + x)\,g(x)}{\bar{F}(w_0 \wedge v^* - t + x)}dx\right) \cdot 1_{\{t \geq (v^*-w_0)^+\}} \\ &\quad + \bar{F}(v^*)\left(\int_0^{t-v^*} \lambda(t-x-v^*)g(x)dx\right) \cdot 1_{\{t \geq v^*\}}, \\ Q(t) &= \left(\int_t^{w_0 \wedge v^*} \frac{\phi(x-t)\,\bar{F}(x)}{\bar{F}(x-t)}dx + \int_0^t \lambda(t-x)\,\bar{F}(x)dx\right) \cdot 1_{\{0 \leq t \leq (v^*-w_0)^+\}} \\ &\quad + \left(\int_0^t \lambda(t-x)\,\bar{F}(x)dx + \int_t^{v^*} \frac{\phi(x-t)\,\bar{F}(x)}{\bar{F}(x-t)}dx\right) \cdot 1_{\{(v^*-w_0)^+ < t < v^*\}} \\ &\quad + \left(\int_0^{v^*} \lambda(t-x)\,\bar{F}(x)dx\right) \cdot 1_{\{t \geq v^*\}}, \\ \alpha(t) &= \left(\int_t^{w_0 \wedge v^*} \frac{\phi(x-t)\,f(x)}{\bar{F}(x-t)}dx + \int_0^t \lambda(t-x)\,f(x)dx\right) \cdot 1_{\{0 \leq t \leq (v^*-w_0)^+\}} \\ &\quad + \left(\int_0^t \lambda(t-x)\,f(x)dx + \int_t^{v^*} \frac{\phi(x-t)\,f(x)}{\bar{F}(x-t)}dx\right) \cdot 1_{\{(v^*-w_0)^+ < t < v^*\}} \\ &\quad + \left(\int_0^{v^*} \lambda(t-x)\,f(x)dx\right) \cdot 1_{\{t \geq v^*\}} \end{aligned}$$

where $\delta_y(t)$ is the direct-delta function at y, i.e., $\delta_y(t) = 0$ for $t \neq y$, $\int_a^b \delta_y(t)dt = 1$ if $a \leq y \leq b$.

Proof. (i) If the system is initially underloaded, i.e., $w(0-) = w_0 = 0$, $q(0-,x) = \phi(x) = 0$, $Q(0-) = 0$, $B(0-) \leq s(0-)$. This case is similar to Theorem 8 where the system is initially empty. Note the only difference is that there is fluid in the service facility. Let $B^o(t)$ be the fluid in service that has been in service at $0-$. Then we have

$$B^o(t) = \int_t^\infty b(t,x)dx = \int_t^\infty b(0-,x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx.$$

Again, we do not allow any input to enter service until time $t = v^*$, we can let the staffing function be

$$\begin{aligned} s(t) &= B^o(t) + s^*(t) \\ &= \int_t^\infty \frac{\psi(x-t)\,\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{F}(v^*)\int_0^{t-v^*} \bar{G}(x)\lambda(t-v^*-x)dx \cdot 1_{\{t>v^*\}}, \end{aligned}$$

where $s^*(t)$ is defined in (48). It is obvious that this expression coincides with (88) when $w_0 = q(0-,x) = \psi(x) = 0$. When we do this, the input rate to the service $b(t,0)$ is the same as in Theorem 8. The proof of other performance measures are similar.

(ii) If the system is initially overloaded, i.e., $w(0-) = w_0 > 0$, $q(0-,x) = \phi(x) \geq 0$, $Q(0-) = \int_0^{w_0}\phi(x)dx > 0$, $s(0-) = B(0-)$. There are two cases (a) $w_0 \geq v^*$, (b) $w_0 < v^*$.

(ii.a) If $w_0 > v^*$, then in order for $v(t) = v^*$. We let all fluid that has been in queue for $x > v^*$ enter service immediately at time 0. The quantity of fluid that enters service at 0 is $\int_{v^*}^{w_0} q(0-,x)dx = \int_{v^*}^{w_0}\phi(x)dx$. However, this will make $B(t)$ have an atom at 0. Similar argument to Theorem 8 implies that it suffices to match $b(t,0)$ with $q(t,v^*)$ for all $t \geq 0$. If $t \leq v^*$, $q(t,v^*) = q(0-,v^*-t)\bar{F}(v^*)/\bar{F}(v^*-t)$. If $t > v^*$, then all fluid that has been in queue at $0-$ has entered service, which implies that $q(t,v^*) = q(t-v^*,0)\bar{F}(v^*) = \lambda(t-v^*)\bar{F}(v^*)$. Therefore, we have

$$\begin{aligned} b(t,0) &= \delta_0(t)\int_{v^*}^{w_0}\phi(x)dx + q(t,v^*) \\ &= \delta_0(t)\int_{v^*}^{w_0}\phi(x)dx + \frac{\phi(v^*-t)\bar{F}(v^*)}{\bar{F}(v^*-t)}\cdot 1_{\{0\leq t<v^*\}} + \lambda(t-v^*)\bar{F}(v^*)\cdot 1_{\{t\geq v^*\}}. \end{aligned}$$

The service capacity and fluid content in service are

$$s(t) = B(t) = B^o(t) + \int_0^t b(t-x,0)\bar{G}(x)dx.$$

If $0 \leq t < v^*$, we have

$$\begin{aligned} s(t) &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \int_{v^*}^{w_0}\phi(x)dx\int_0^t \delta_0(t-x)\bar{G}(x)dx + \bar{F}(v^*)\int_0^t \frac{\phi(v^*-t+x)\bar{G}(x)}{\bar{F}(v^*-t+x)}dx, \\ &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{G}(t)\int_{v^*}^{w_0}\phi(x)dx + \bar{F}(v^*)\int_0^t \frac{\phi(v^*-t+x)\bar{G}(x)}{\bar{F}(v^*-t+x)}dx. \end{aligned}$$

If $t \geq v^*$, we have

$$\begin{aligned} s(t) &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{G}(t)\int_{v^*}^{w_0}\phi(x)dx \\ &\quad + \int_0^t \left(\frac{\phi(v^*-t+x)\bar{F}(v^*)}{\bar{F}(v^*-t+x)}\cdot 1_{\{0\leq t-x<v^*\}} + \lambda(t-x-v^*)\bar{F}(v^*)\cdot 1_{\{t-x\geq v^*\}}\right)\bar{G}(x)dx \\ &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{G}(t)\int_{v^*}^{w_0}\phi(x)dx \\ &\quad + \bar{F}(v^*)\left(\int_{t-v^*}^t \frac{\phi(v^*-t+x)\bar{G}(x)}{\bar{F}(v^*-t+x)}dx + \int_0^{t-v^*}\lambda(t-x-v^*)\bar{G}(x)dx\right). \end{aligned}$$

It is easy to see that this expression coincides with (88).

(ii.b) If $w_0 \leq v^*$, then we do not allow any input to enter service until time $v^* - w_0$, which implies

$$b(t,0) = \frac{\phi(w_0-t)\,\bar{F}(v^*)}{\bar{F}(w_0-t)}\cdot 1_{\{v^*-w_0\leq t<v^*\}} + \lambda(t-v^*)\bar{F}(v^*)\cdot 1_{\{t\geq v^*\}}.$$

Therefore, if $0 \le t \le v^* - w_0$, no new fluid enters service,

$$s(t) = B^o(t) = \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx.$$

If $v^* - w_0 < t < v^*$,

$$\begin{aligned} s(t) &= B^o(t) + \int_0^t \frac{\phi(w_0 - t + x)\,\bar{F}(v^*)}{\bar{F}(w_0 - t + x)} \cdot 1_{\{v^* - w_0 \le t - x < v^*\}} \bar{G}(x)dx \\ &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{F}(v^*)\int_0^{t-(v^*-w_0)} \frac{\phi(w_0 - t + x)\,\bar{G}(x)}{\bar{F}(w_0 - t + x)}dx. \end{aligned}$$

If $t \ge v^*$,

$$\begin{aligned} s(t) &= B^o(t) + \int_0^t \left(\frac{\phi(w_0 - t + x)\,\bar{F}(v^*)}{\bar{F}(w_0 - t + x)} \cdot 1_{\{v^* - w_0 \le t - x < v^*\}} + \lambda(t - x - v^*)\bar{F}(v^*) \cdot 1_{\{t-x \ge v^*\}} \right) \bar{G}(x)dx \\ &= \int_t^\infty \psi(x-t)\frac{\bar{G}(x)}{\bar{G}(x-t)}dx + \bar{F}(v^*)\left(\int_{t-v^*}^{t-(v^*-w_0)} \frac{\phi(w_0 - t + x)\,\bar{G}(x)}{\bar{F}(w_0 - t + x)}dx + \int_0^{t-v^*} \lambda(t - x - v^*)\bar{G}(x)dx \right). \end{aligned}$$

It is easy to see that this expression coincides with (88). The proof of other performance measures is similar.

I Comparisons with Simulation

In this section we present additional results evaluating the fluid model approximations by comparing them to simulation results for large-scale queueing models. These results complement those for the $M_t/H_2/s + E_2$ example in §2.

We start by applying our algorithm to the special "base" case of an $M_t/M/s+M$ model, having only a time-varying arrival rate function. For this special case, we could also have applied [24–26]. In §I.2 we present additional simulation results for allowing the alternative features: (i) time-varying staffing function, (ii) non-exponential abandonment-time cdf, and (iii) non-Poisson arrival process. (The fluid model does not change when we change the arrival process from M_t, to G_t, but the queueing system does.)

In §2 we already considered the $M_t/H_2/s + E_2$ model, which has both time-varying arrival rate and non-exponential service and patience distributions. We consider other examples in §I.3.

I.1 A Base Example

We start by applying our algorithm to the base case of an $M_t/M/s+M$ model, having only a time-varying arrival rate function.

For the initial $M_t/M/s + M$ model fluid example, we consider constant staffing s. We let the arrival rate function λ be sinusoidal, i.e.,

$$\lambda(t) \equiv a + b \cdot \sin(c \cdot t), \quad t \ge 0, \tag{89}$$

where we let $b \equiv 0.6a$, $c \equiv 1$ and $a \equiv s$. By making the average input rate a coincide with the fixed staffing level s, we ensure that the system will alternate between overloaded and underloaded. We let the service rate be $\mu \equiv 1$ and the abandonment rate $\theta \equiv 0.5$; i.e., $G(x) \equiv 1 - e^{-x}$ and $F(x) = 1 - e^{-\theta x} = 1 - e^{-0.5x}$ for $x \ge 0$. Without loss of generality, for the fluid model we let $s \equiv 1$.

Figure 13 shows key fluid performance functions of this $M_t/M/s+M$ example. In Figure 13, we plot key fluid performance measures for $0 \le t \le T$, where $T = 16$. It is easy to see

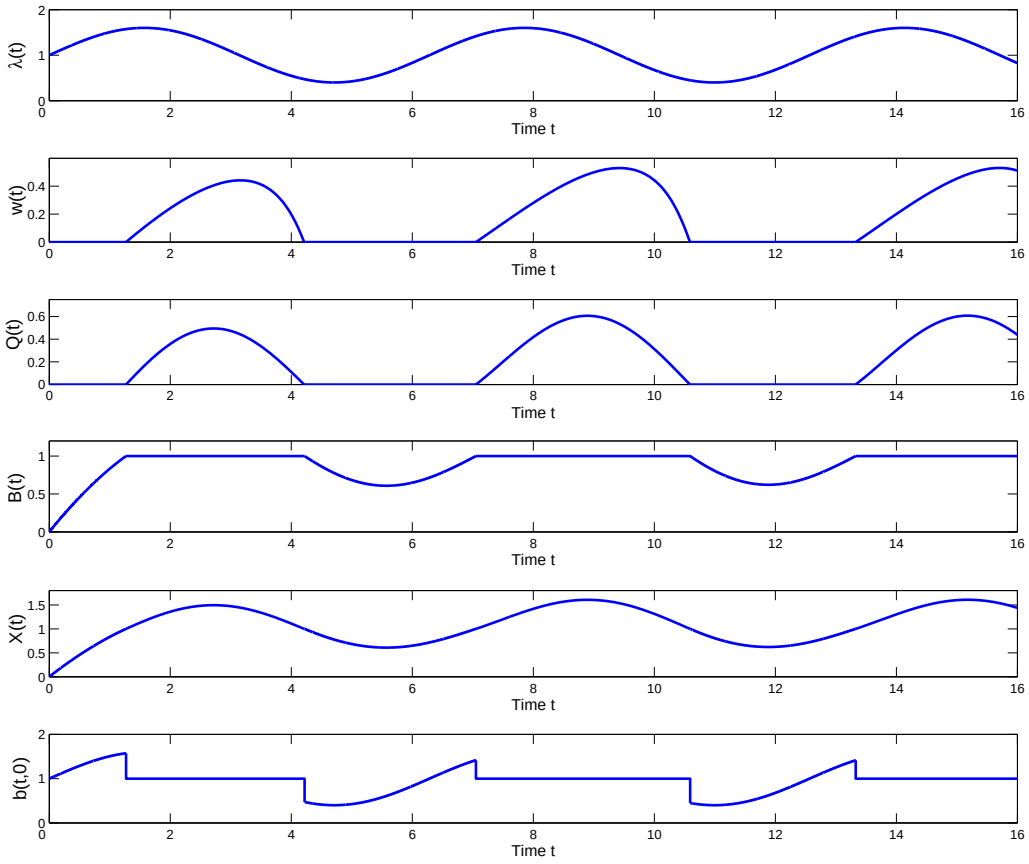

Fig. 13 The performance functions of the $M_t/M/s+M$ fluid model with sinusoidal arrival-rate function: (i) arrival rate $\lambda(t)$; (ii) waiting time $w(t)$; (iii) fluid in buffer $Q(t)$; (iv) fluid in service $B(t)$; (v) total fluid $X(t)$; (vi) rate into service $b(t,0)$.

that the system alternates between underloaded (when $Q(t) = 0$ and $B(t) < s(t) = 1$) and overloaded (when $Q(t) > 0$ and $B(t) = s(t) = 1$) intervals.

In Figure 14 and 15, we plot the two-parameter functions $q(t,x)$ and $b(t,x)$ for $0 \le t \le 4$. The decays on the x dimension in Figure 14 and 15 are due to customer abandonment and service completion, the shape changes on the time dimension are according to the sinusoidal arrival rate. The discontinuity at $t_1 = 1.66$ of $b(t,x)$ is consistent with the discontinuity of $b(t,0)$ in Figure 13, $q(t,x) = 0$ for $0 \le t \le t_1$ since the system is underloaded.

As discussed in §2, it is important that the fluid model provide useful approximations for stochastic queueing models. We apply simulation to show that the fluid approximation indeed is effective for that purpose. For very large queueing systems, the stochastic system behaves like the fluid model, having relatively small stochastic fluctuations. That is illustrated for the same example for a queueing system with 1000 servers in Figure 16. (In the plot, the queueing content processes are scaled by dividing by $n = 1000$, so that s remains at 1.)

We did not plot the abandonment rate α and the service-completion rate σ, because in the exponential case they are simple functions of the performance measures shown:

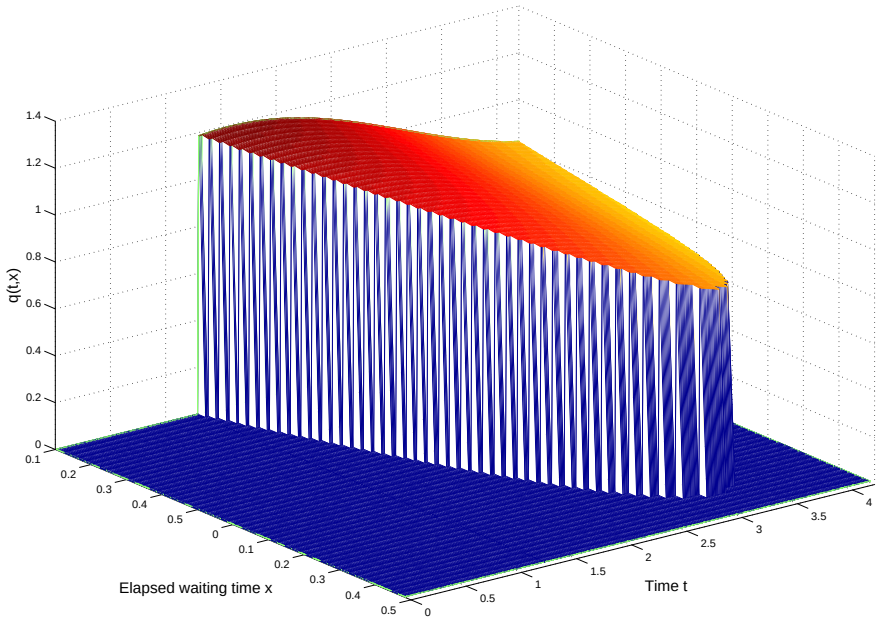

Fig. 14 The two-parameter density function q for the $G_t/M/s+M$ fluid model with sinusoidal arrival-rate function.

$\alpha(t) = \theta Q(t) = 0.5Q(t)$ and $\sigma(t) = \mu B(t) = B(t)$. All performance functions are continuous except for the transportation-rate function $b(\cdot, 0)$, which has discontinuities when the system alternates between underloaded and overloaded: $b(t, 0) = \lambda(t)$ when the system is underloaded; $b(t, 0) = s = 1$ when the system is overloaded.

With the MSHT scaling, we let $n \equiv 1000$. Since, $s = 1$, that makes $s_n = a_n = 1000$, which of course is very large. The other parameters of the queueing model are the same as for the fluid model, e.g., $b_n = 0.6a_n = 600$. In Figure 16 we compare the simulation results for the queueing performance functions W_n, $\bar{Q}_n$ and $\bar{B}_n$ from a single simulation run to the associated fluid model counterparts w, Q and B. The blue solid lines represent the queueing model performance, while the red dashed lines represent the corresponding fluid performance. Since n is so large, we get close agreement for individual sample paths; we are not displaying averages over multiple simulation runs.

Of course, most service systems have fewer servers. It is thus important that the fluid approximation can still be useful with fewer servers. With fewer servers, the stochastic fluctuations in the queueing stochastic processes play an important role. In that case, the fluid model can still be very useful by providing a good approximation for the *mean values* of the queueing stochastic processes. That is illustrated from the plot of the average of the scaled performance measures of 200 independent sample paths when there are only 20 servers in Figure 17.

In Figure 18 below we plot the analog of Figure 16 for the case of one sample path of the simulation with $n = 100$, for the same fluid model. In Figure 19 below we plot the average of 10 sample paths. We see that the fluid approximation provides only a rough approximation for a single sample path, but it is remarkably accurate for the average over 10 sample paths.

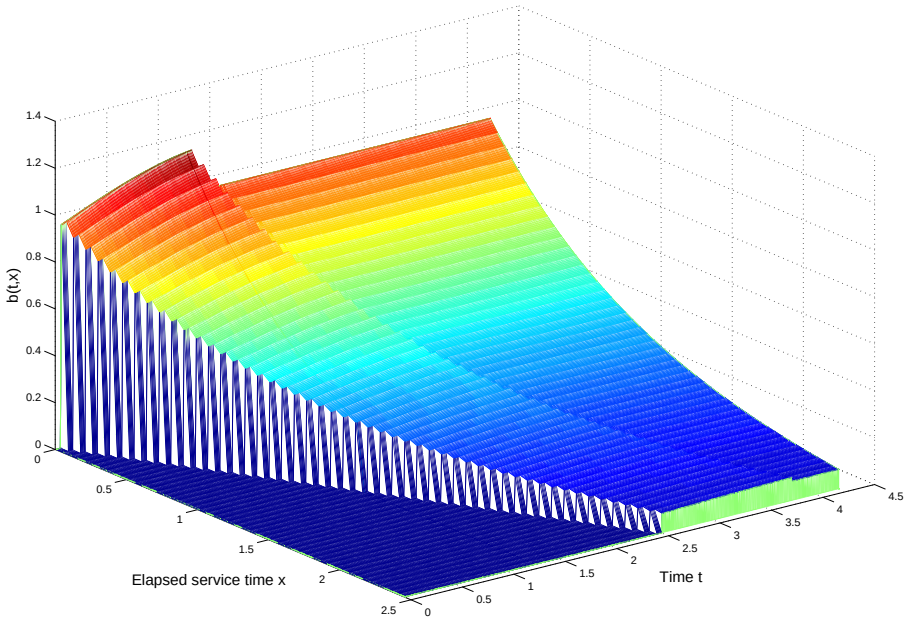

Fig. 15 The two-parameter density function b for the $G_t/M/s + M$ fluid model with sinusoidal arrival-rate function.

The accuracy is especially high in this example, because the extent of the overloads and underloads are quite large.

The quality of the approximation does degrade as n decreases, for the given fluid model. To illustrate, we plot a single sample path for $n = 20$ in Figure 20 and the average over 200 sample paths in Figure 17. (The latter appears in the main paper.) The stochastic fluctuations are so much greater for a single sample path that we need to average over more sample paths to get a good estimate. For $n = 20$, the fluid model clearly yields a good approximation only for the mean values, but the mean is remarkably well approximated for $n = 20$. The approximation for the mean values in Figure 17 are so good that it is evident that the fluid model approximations can provide useful approximations for the mean values for much smaller n (and thus the number of servers, s_n).

I.2 Variants of the Base Model

We now consider three variants of the base model in order to illustrate consider: (i) time-varying staffing, (ii) non-exponential abandonment and (iii) a non-Poisson arrival process.

I.2.1 Time-Varying Staffing Levels

We now consider a Markovian $M/M/s_t + M$ model that has a Poisson arrival process with a constant rate λ, exponential service and abandonment distributions with rates μ and θ

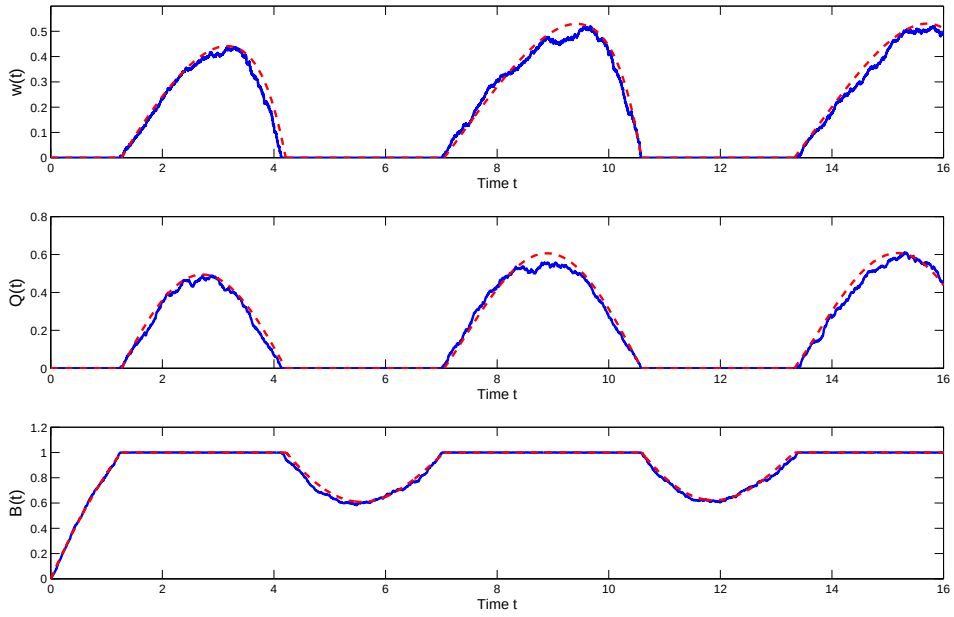

Fig. 16 Performance of the $M_t/M/s+M$ fluid model (dashed lines) compared with simulation results (solid lines): one sample path of the scaled queueing model for $n = 1000$.

respectively, and a sinusoidal capacity function

$$s(t) \equiv \lambda + \bar{\lambda} \cdot \sin(c \cdot t). \tag{90}$$

In the previous base example in §I.1, we fixed the capacity function and varied the arrival rate around it; now we fix the arrival rate λ and vary $s(t)$ around λ. We let $\lambda = 1$, $\bar{\lambda} = 0.6\lambda = 0.6$, $c = 1$, $\mu = 1$ and $\theta = 0.5$.

Before implementing the algorithm, we first verify that this capacity function s is feasible. With exponential service distribution, we know that a sufficient condition for the feasibility of s is

$$s'(t) \geq -\mu s(t), \quad t \geq 0. \tag{91}$$

In this example, we require $c\cos(ct) \geq -\mu\lambda - \mu\bar{\lambda}\sin(ct)$ which is equivalent to $\sin(ct+\bar{\theta}) \geq -(\mu/\sqrt{c^2+\mu^2})(\lambda/\bar{\lambda})$ where $\bar{\theta} \equiv \arctan(c/\mu)$. It is easy to check that this equality holds with $\lambda = 1$, $\bar{\lambda}$, $\mu = 1$ and $c = 1$.

We plot the performance measures of the $M/M/s_t + M$ fluid model in Figure 21 and compare them with simulation estimates in 22, analogs to Figure 13 and 16. In Figure 22, our simulations add real system constraints. First the staffing levels must be integer-valued, so they must be rounded. Second, when the staffing levels decrease, we do not remove servers until they complete the service in progress. As in §I, we let $n = 1000$ for the sequence of scaled queueing models in §**??**. Thus we have $\lambda_n = a_n = 1000$, $b_n = 600$, $c_n = 1$.

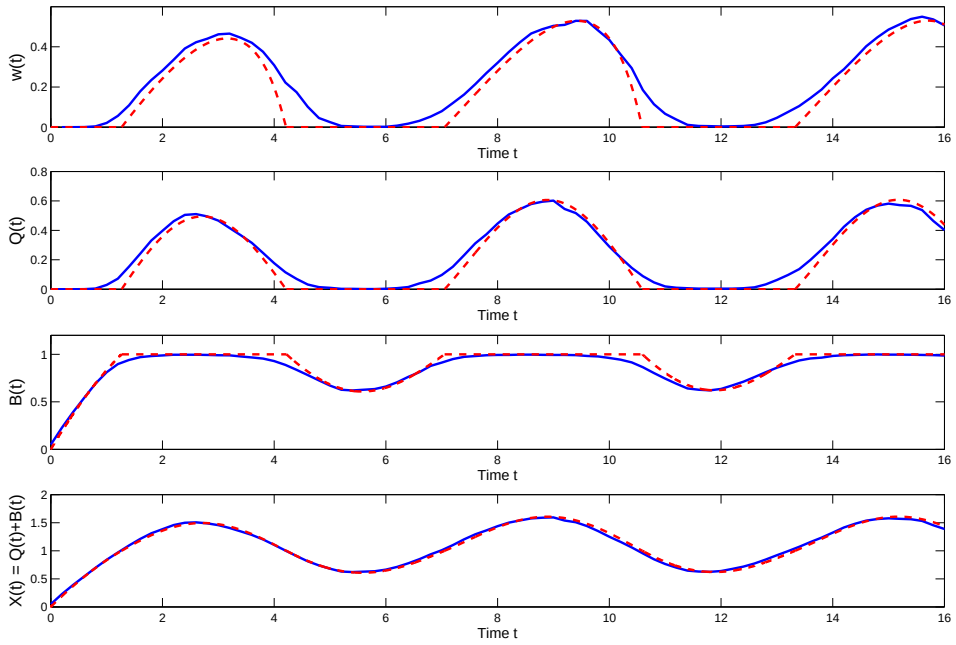

Fig. 17 Performance of the $M_t/M/s+M$ fluid model (dashed lines) compared with simulation results (solid lines): an average of 200 sample paths of the scaled queueing model based on $n = 20$.

I.2.2 Simulation Comparisons for the $M_t/M/s_t + GI$ Fluid Model.

For the general abandon-time distribution, we considered two cases: Erlang-2 (E2) and Hyperexponential-2 (H2). Let A be the generic abandonment time. A follows E2 implies that $A = X_1 + X_2$ in distribution, where X_1 and X_2 are two iid exponential random variables. Moreover, $f(x) = \gamma^2 x e^{-\gamma x}$, where γ is rate of X_1.

If A follows H2, then A is a composition of two exponential random variables, i.e., $f(x) = p \cdot \theta_1 e^{-\theta_1 x} + (1-p) \cdot \theta_2 e^{-\theta_2 x}$, where θ_1 and θ_2 are the rates of these two exponential random variables, and $0 < p < 1$ is the sampling probability.

If we fix the mean of A, i.e., let $E[A] = 1/\theta$, E2 has squared coefficient of variation (SCV) $C_{SCV} \equiv Var(A)/E[A]^2$ less than 1; H2 has C_{SCV} greater than 1 if p, θ_1 and θ_2 are appropriately chosen.

For E2, we let $f(x) \equiv 4\theta^2 x e^{-2\theta x}$ such that $C_{SCV} = 1/2$. For H2, we let $f(x) = p \cdot \theta_1 e^{-\theta_1 x} + (1-p) \cdot \theta_2 e^{-\theta_2 x}$ with $p = 0.5(1 - \sqrt{0.6})$, $\theta_1 = 2p\theta$, $\theta_2 = 2(1-p)\theta$, such that $C_{SCV} = 4$.

We still let the arrival-rate function λ be sinusoidal, as in (89). We let $a = 1$, $b = 0.6 * a = 0.6$, $c = 1$. We let the service-capacity function be constant $s = 1$. Let $\theta = 0.5$ and $\mu = 1$. We plot the dynamics of the $M_t/M/s + E2$ and $M_t/M/s + H2$ fluid models in Figure 23 and 25 respectively for $t \in [0, T]$ with $T = 16$. The performance measures shown in Figure 23 and 25 are the boundary waiting time $w(t)$, the fluid in queue $Q(t)$, the fluid in service $B(t)$, the total fluid in the system $X(t)$, the abandonment rate $\alpha(t)$, and the transportation rate $b(t, 0)$. We omit the departure rate $\sigma(t) = \mu B(t)$ because of the exponential service times.

In Figure 24 and 26 we compare the fluid approximations with simulation experiments. The queueing model has a nonhomogeneous Poisson arrival process with sinusoidal rate

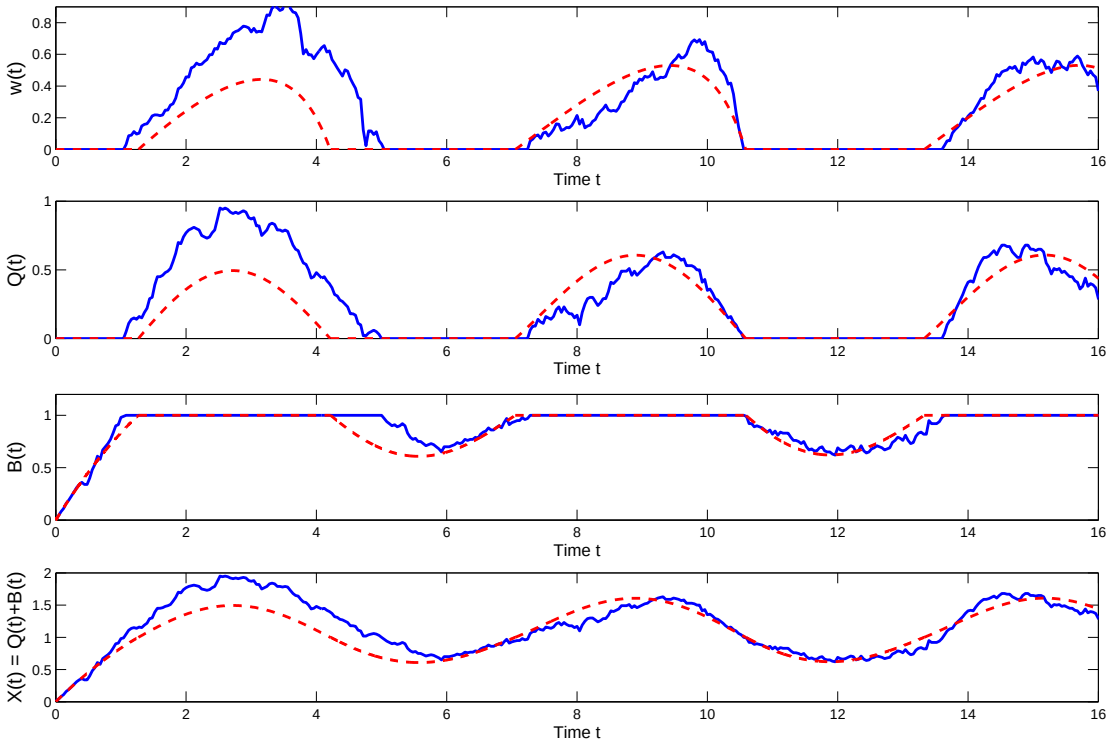

Fig. 18 Performance of the $M_t/M/s+M$ fluid model compared with simulation results: one sample path of the scaled queueing model for $n = 100$.

function as in (89), with $a = s = 2000$, $b = 0.6a = 1200$. In Figure 24 and 26, the blue solid lines of the simulation estimations of single sample paths applied with fluid scaling, and the red dashed lines are the fluid approximations. We conclude that the fluid approximation is remarkably accurate.

I.2.3 Simulation Comparisons for the $G_t/M/s_t+M$ Fluid Model.

We first explain how to construct a non-Poisson arrival process that has a well-defined rate function. We do so by performing a time transformation of a rate-1 stationary point process. The following construction applies to any stationary point process, but we apply it to an equilibrium renewal process with H_2 times between renewals.

Let $\mathbf{M} \equiv \{M(t) : t \geq 0\}$ be a delayed renewal process; i.e., let $X_1, X_2, X_3, \ldots$ be independent random variables with finite means, such that X_1 follows cdf H, X_n follows cdf G for $n \geq 2$. Let $S_n \equiv \sum_{k=1}^{n} X_k$ and define $M(t) \equiv \sup\{n \geq 0 : S_n \leq t\}$. In particular, if we let $H(x) = G_e(x) \equiv 1/m_X \int_0^t \bar{G}(u)du$ for $m_X \equiv E[X_2]$, which is the equilibrium distribution of G, then $\mathbf{M}$ becomes an equilibrium renewal process and we have $E[M(t)] = t/m_X$ for any $t \geq 0$. We call $\mathbf{M}$ standard equilibrium renewal process (SERP) if $m_X = 1$.

For a given rate function $\lambda(t)$, let $\Lambda(t) \equiv \int_0^t \lambda(u)du$. We assume that $\lambda(t) > 0$ for $t \geq 0$, hence $\Lambda(t)$ is a strictly increasing function. For a given SERP $\mathbf{M}$, we construct a process that has rate function $\lambda(t)$ by performing a change of time with respect to this function

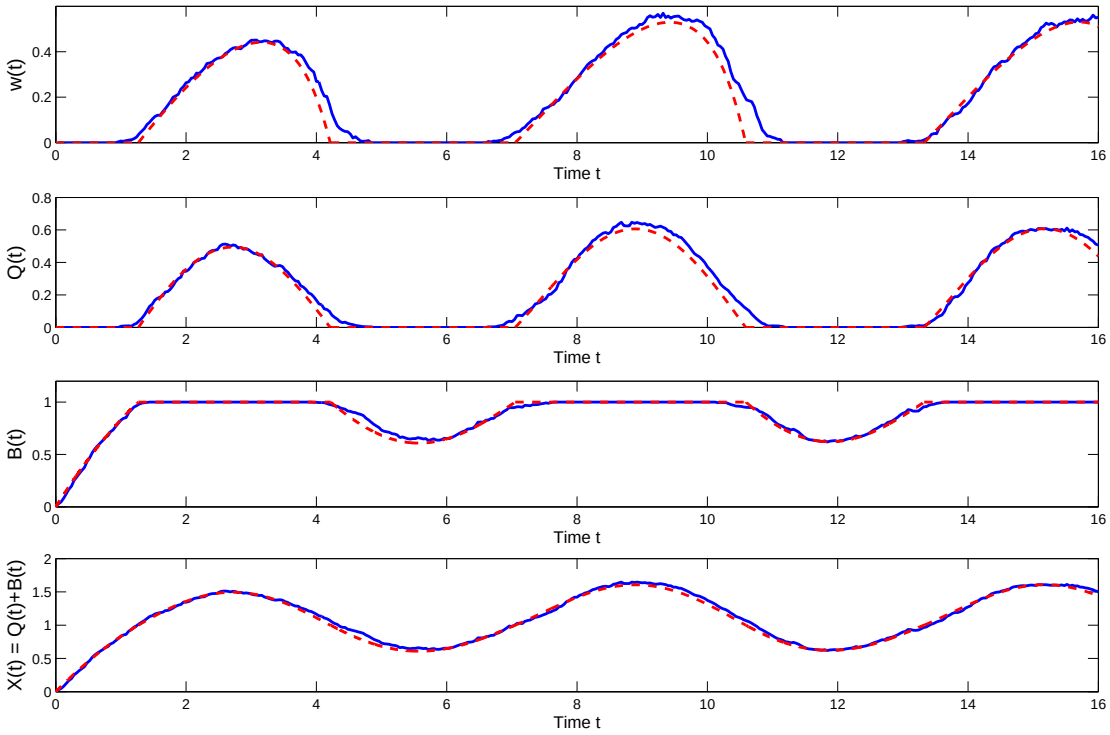

Fig. 19 Performance of the $M_t/M/s+M$ fluid model compared with simulation results: an average of 10 sample paths of the scaled queueing model based on $n = 100$.

$\Lambda(t)$. We define $\mathbf{N} \equiv \{N(t) \equiv M(\Lambda(t)) : t \geq 0\}$. Since $E[N(t)] = \Lambda(t)$ for $t \geq 0$, process $\mathbf{N}$ has the given deterministic arrival rate function $\lambda(t)$.

When the cdf G is not exponential, $\mathbf{N}$ is a non-Poisson arrival counting process that has time-dependent rate function $\lambda(t)$. Now we explain how to simulate the point process associated with $\mathbf{N}$, i.e., to simulate the times of arrivals of $\mathbf{M}$. For a given sample path of the SERP $\mathbf{M}$, let $S_n = s_n$ for $n \geq 0$, we want to determine the arrival times t_n's, where t_n is the time at which the nth arrival occurs. It is easy to see that $t_n = \Lambda^{-1}(s_n)$ for $n \geq 0$, where $\Lambda^{-1}(\cdot)$ is the inverse of $\Lambda(\cdot)$, which is unique, continuous and strictly increasing since $\Lambda(\cdot)$ is continuous and strictly increasing. Therefore, to obtain a sample path of $\mathbf{N}$, we simulate a sample path of $\mathbf{M}$ and do a change of time.

In Figure 27, we compare the fluid approximation with simulation experiments of the $G_t/M/s+M$ model based on an equilibrium H_2 arrival process and the same arrival rate function used in the M_t example. Here the only difference from Figure 16 is that the arrival process (G_t) is not Poisson; it has the same sinusoidal rate function as (89). In particular, the arrival process is created (by the deterministic time transformation) from a rate-1 equilibrium renewal process with H_2 interarrival times, having cdf $G(x) \equiv 1-p_1e^{-\nu_1 x}-p_2e^{-\nu_2 x}$ for $p_1+p_2 = 1$, $p_1 = 0.5(1-\sqrt{0.6})$, $\nu_1 = 2p_1\nu$ and $\nu_2 = 2p_2\nu$ with $1/\nu = 1$ being the mean interarrival time.

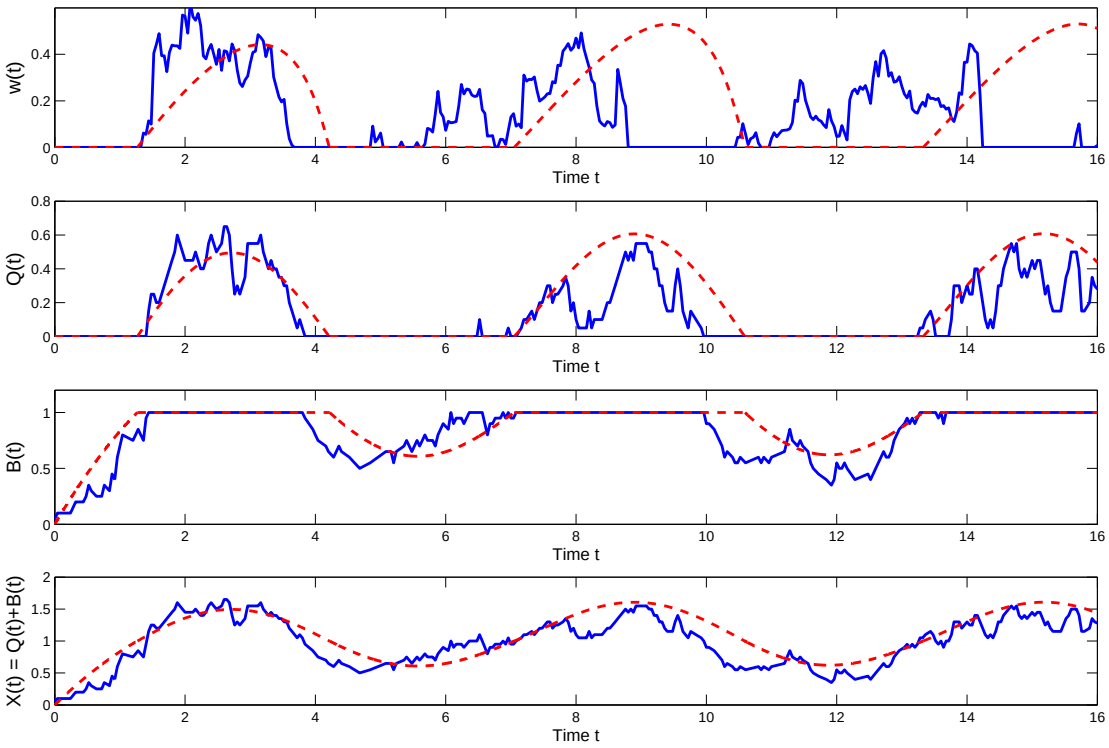

Fig. 20 Performance of the $M_t/M/s+M$ fluid model compared with simulation results: one sample path of the scaled queueing model for $n = 20$.

I.3 More Simulation Comparisons for the Example in §2 with GI Service

Here we consider the $M_t/H_2/s+E_2$ example in §I with smaller n. As shown in Figure 28, we plot the mean value functions, obtained by averaging the paths of 500 independent simulation runs, with $n = 15$. Although less accurate than the case $n = 30$, the fluid model serves as a much better approximation than the algorithm of M service.

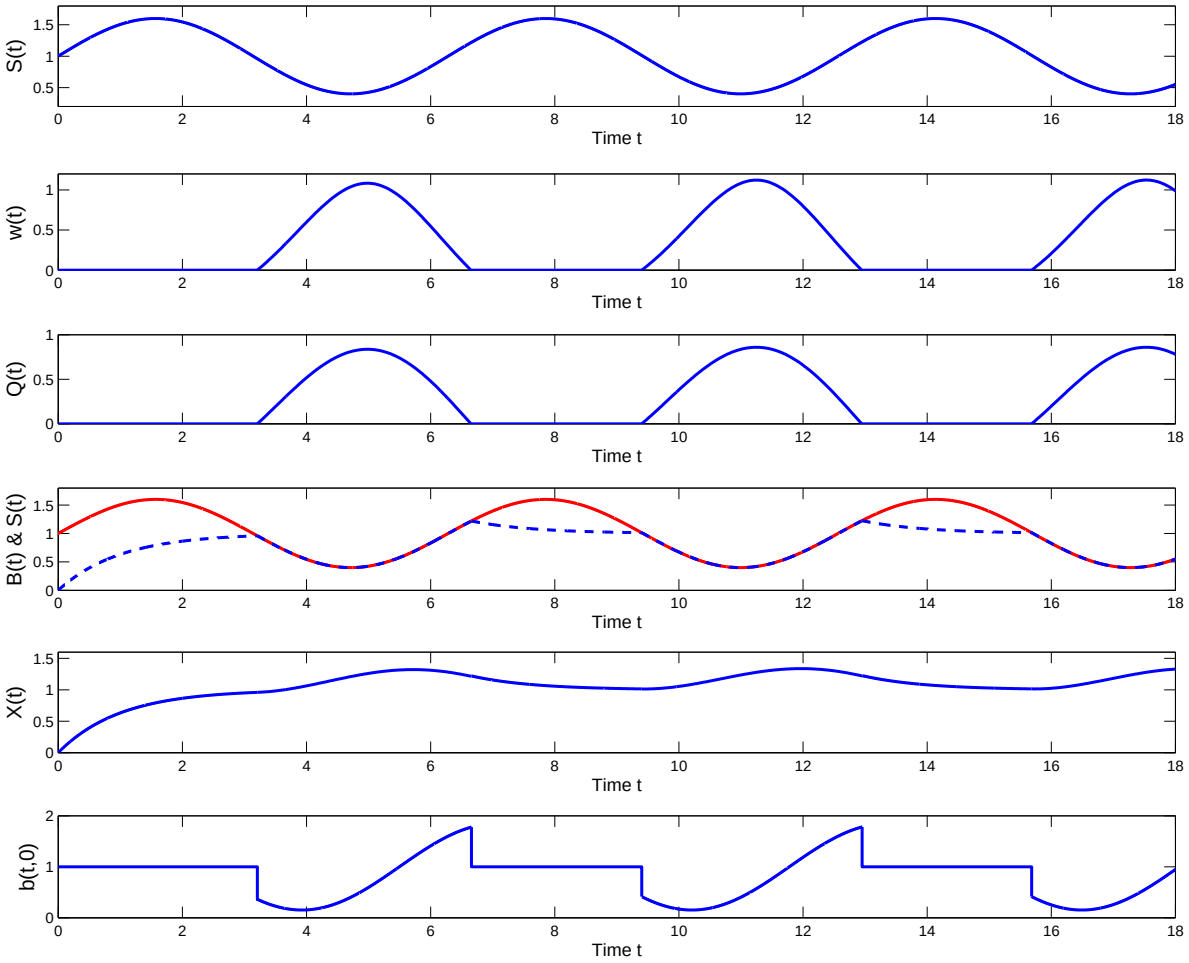

Fig. 21 The $M/M/s_t + M$ fluid model with sinusoidal service-capacity function.

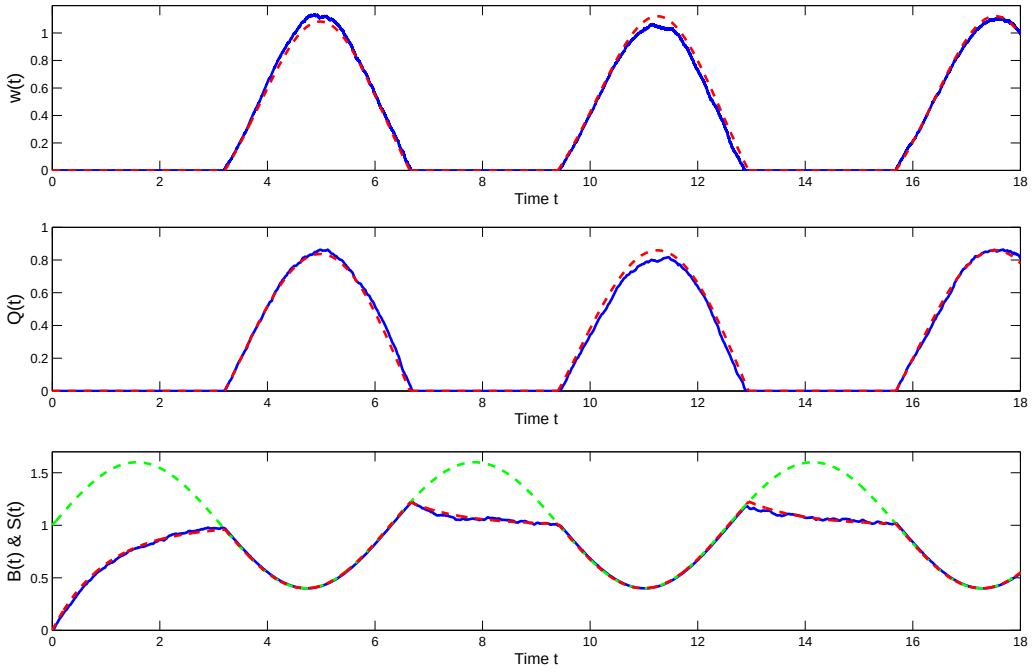

Fig. 22 The $M/M/s_t + M$ fluid model compared with simulations of the queueing system.

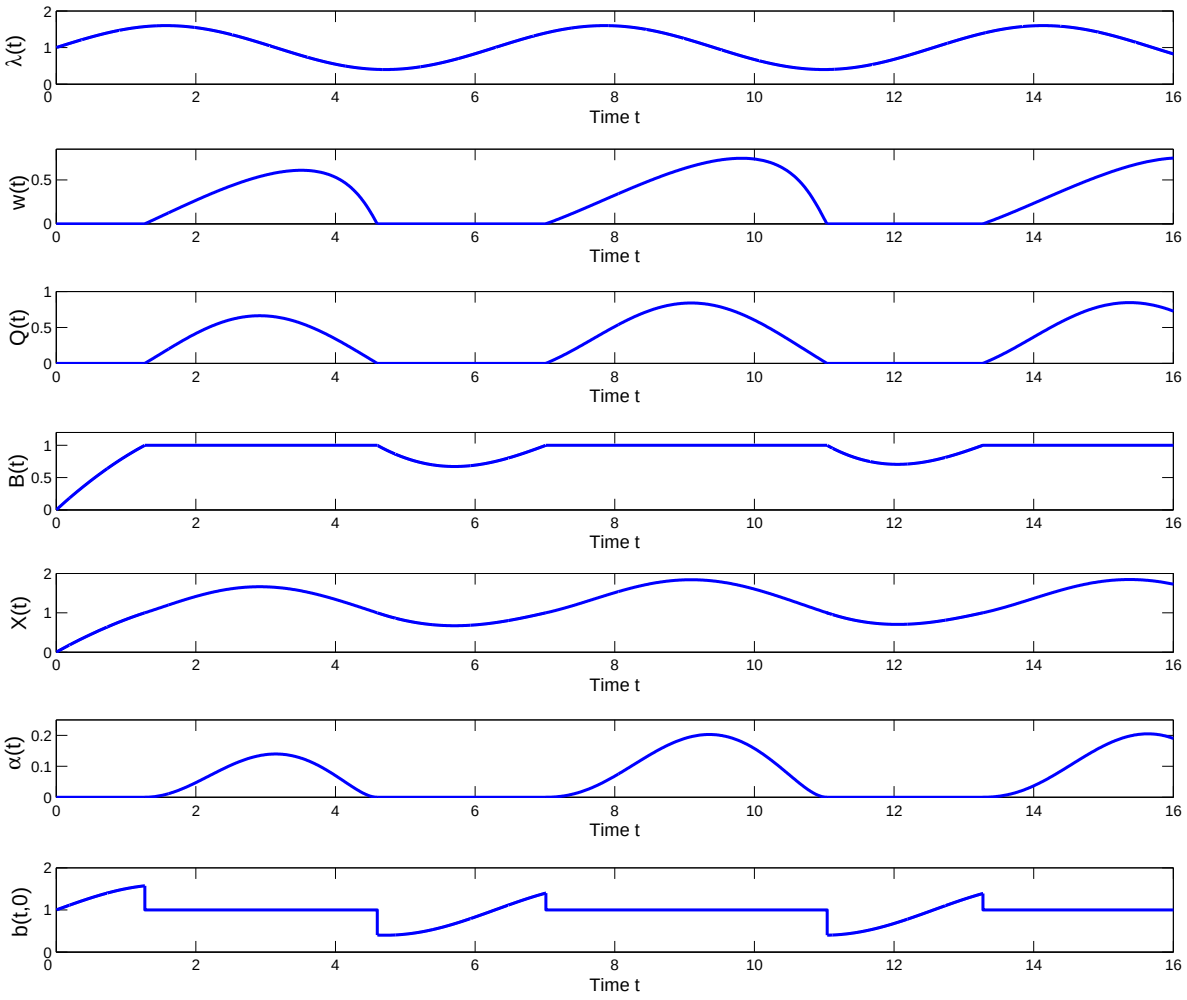

Fig. 23 The $M_t/M/s + E2$ fluid model with sinusoidal arrival-rate function.

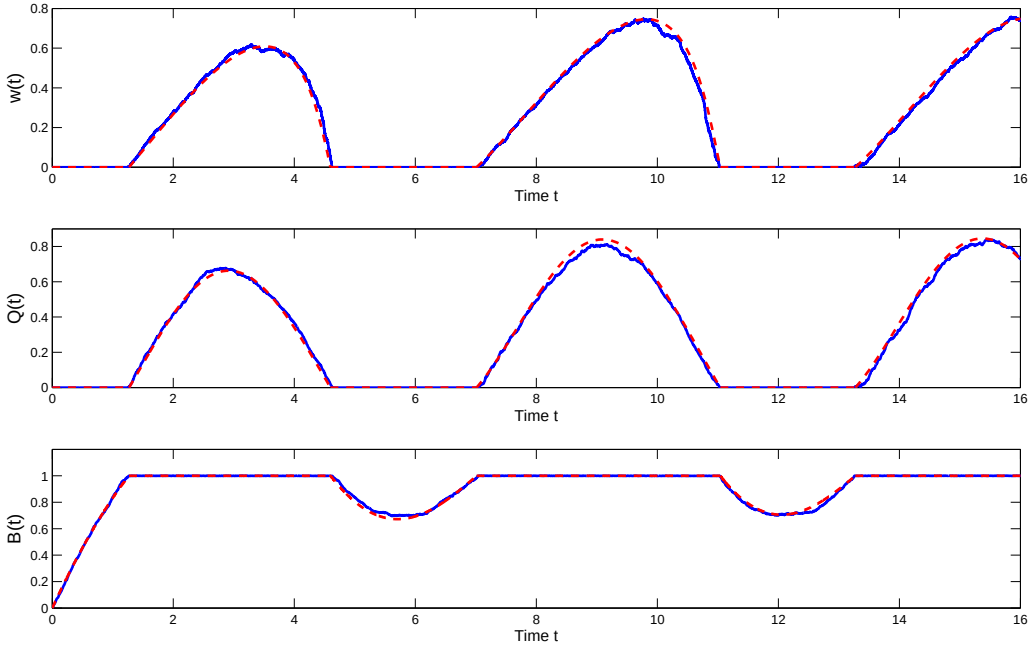

Fig. 24 The $M_t/M/s+E2$ fluid model compared with simulations of the queueing system.

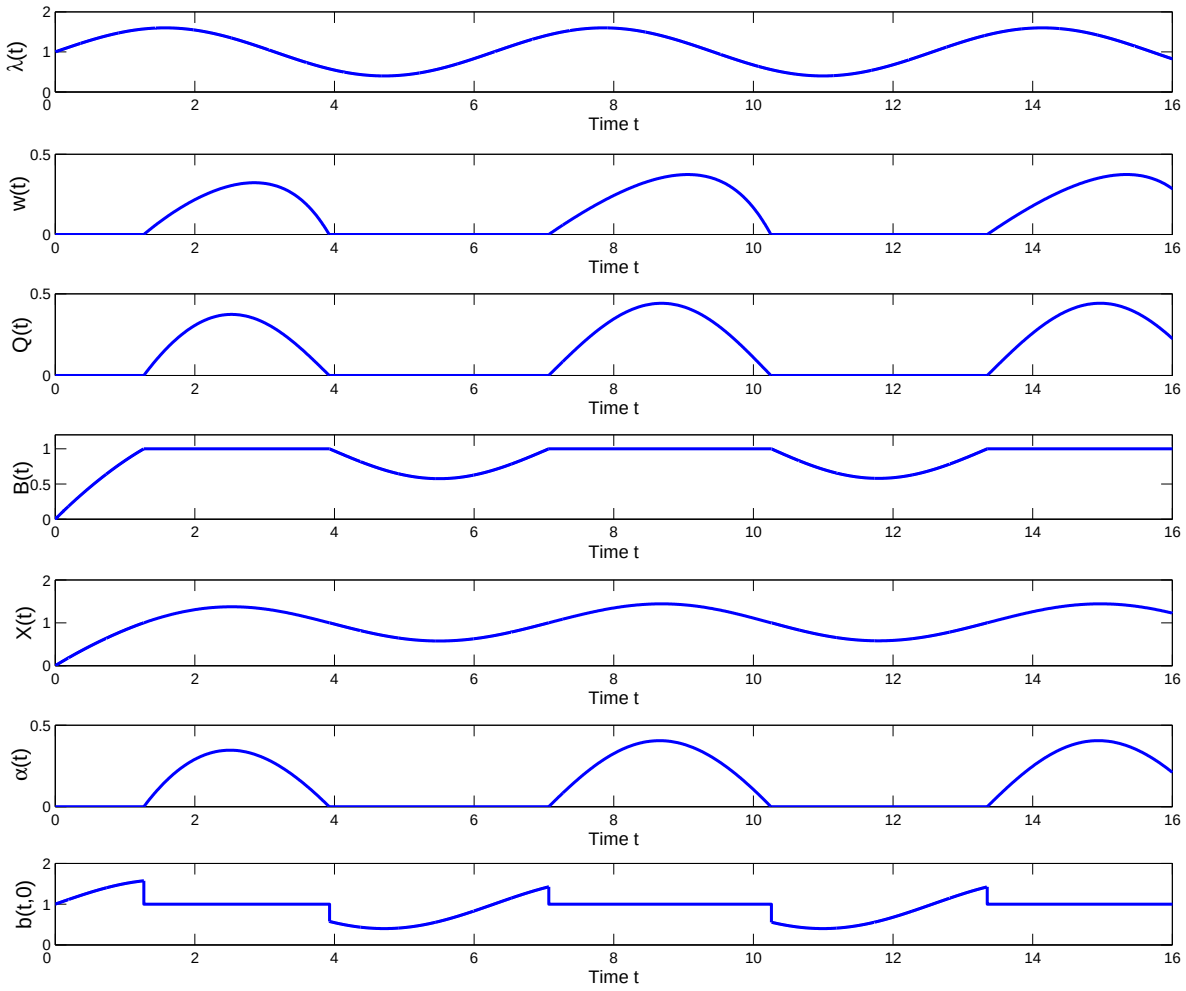

Fig. 25 The $M_t/M/s + H2$ fluid model with sinusoidal arrival-rate function.

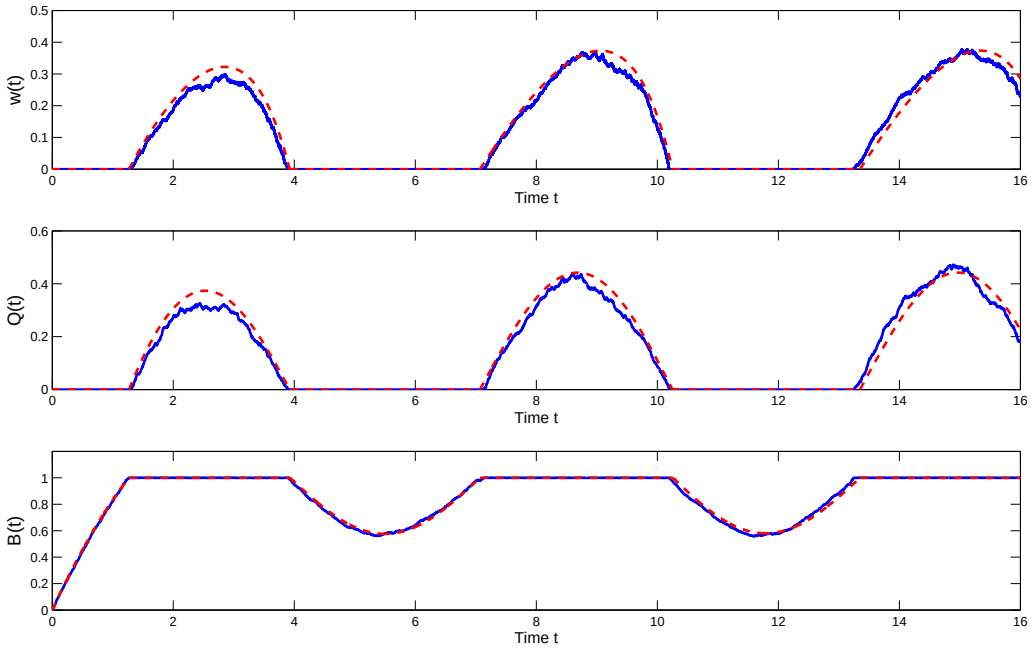

Fig. 26 The $M_t/M/s+H2$ fluid model compared with simulations of the queueing system.

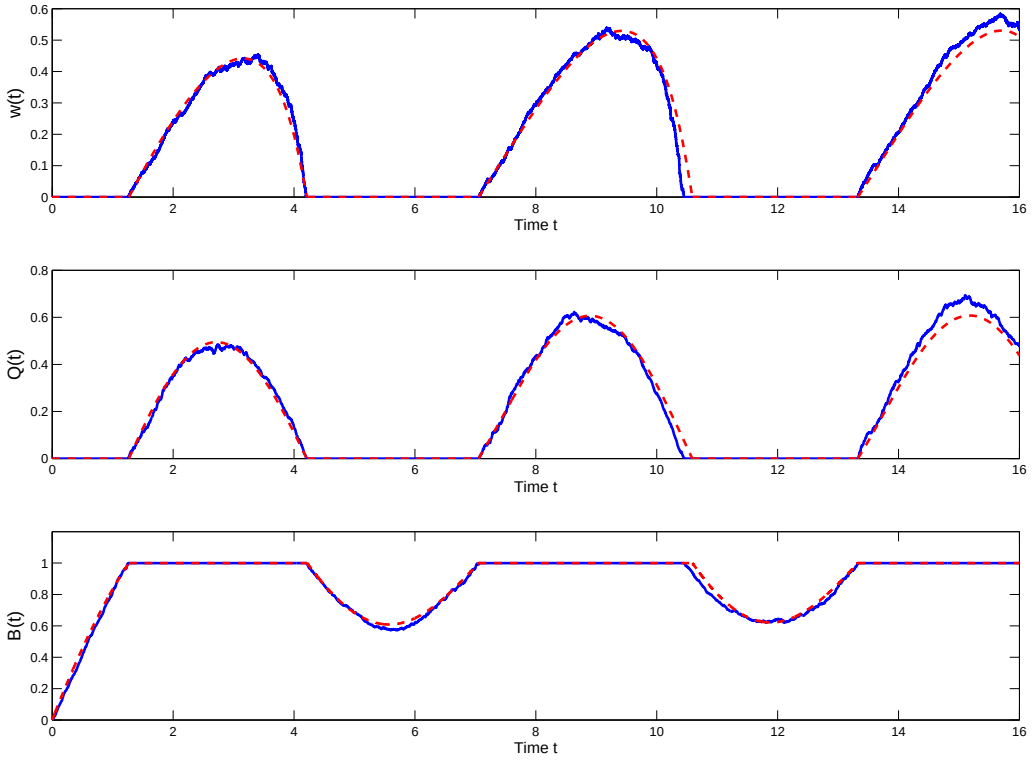

Fig. 27 The $G_t/M/s+M$ fluid model compared with simulations of the queueing system. The arrival process is a time-transformed equilibrium H_2 renewal process with $c_a^2 = 4$, as described in §I.2.3.

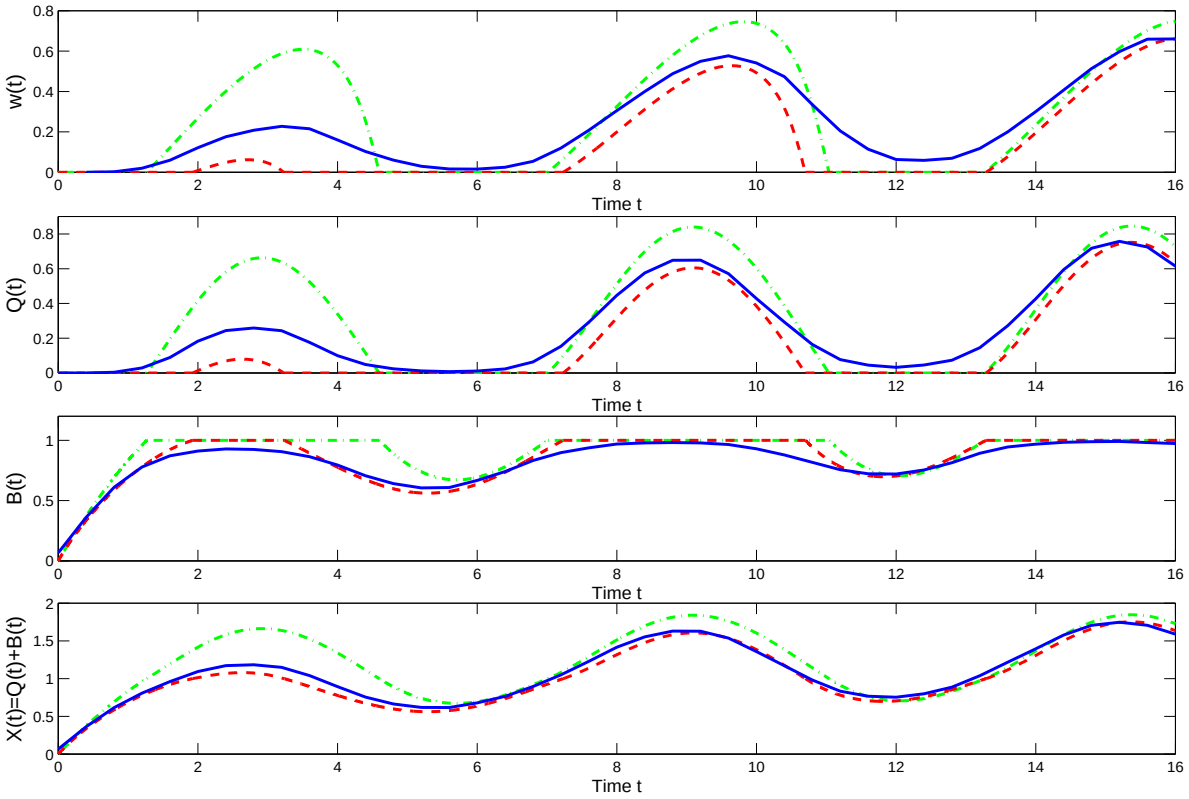

Fig. 28 Simulation comparison for the $M_t/H_2/s + E_2$ fluid model: (i) simulation estimates of an average of 500 sample paths of the scaled queueing model based on $n = 15$ (blue solid lines), (ii) fluid functions for H_2 service (red dashed lines) and (iii) fluid functions assuming M service (green dashed lines).

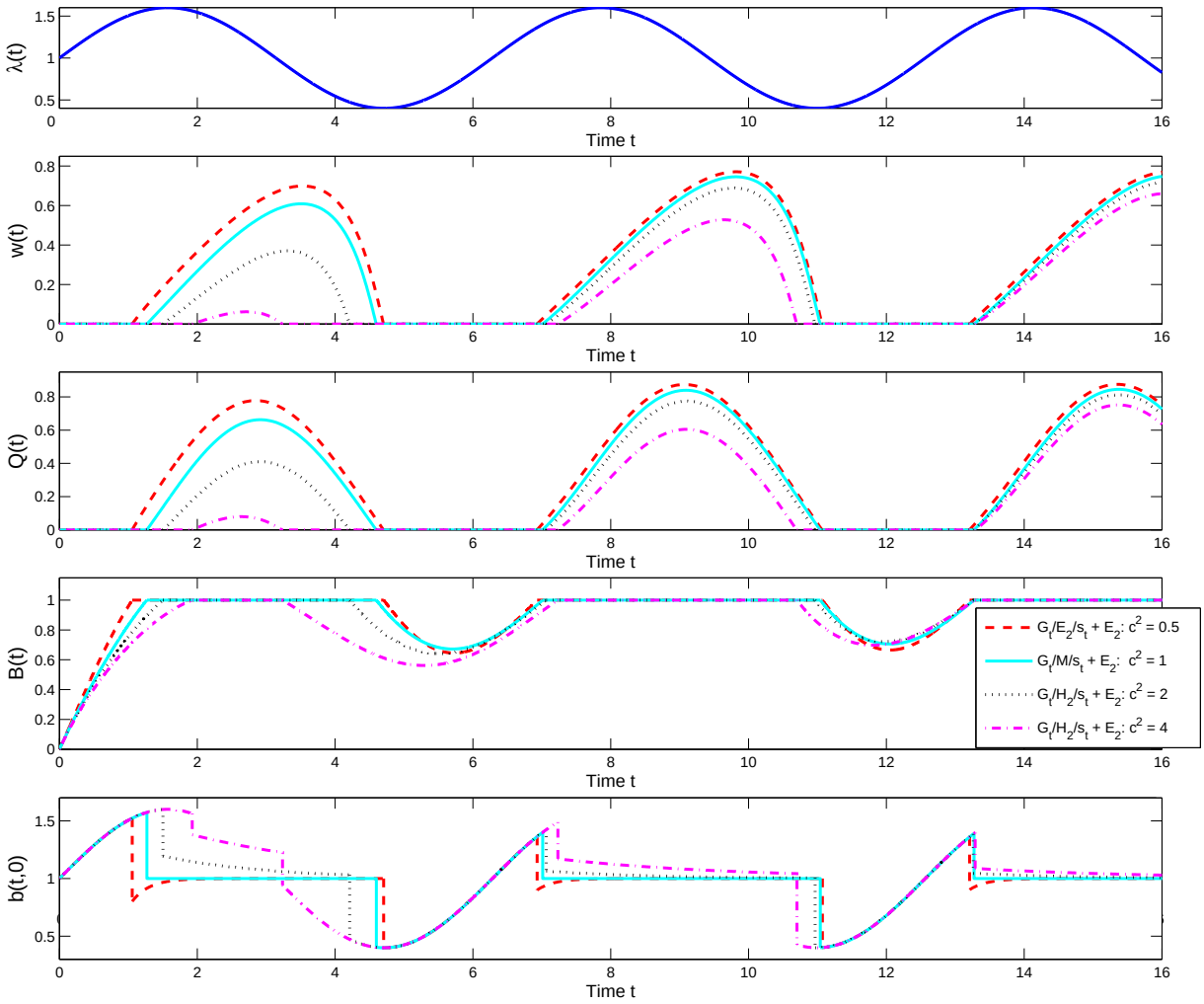

Fig. 29 Fluid dynamics of the $G_t/GI/s + E_2$ model with fixed mean service time and $E2$ patience distribution. The service distributions are: (i) $E2$ ($CVS = 0.5$); (ii) M ($CVS = 1$); (iii) $H2$ ($CVS = 2$) and (iv) $H2$ ($CVS = 4$).

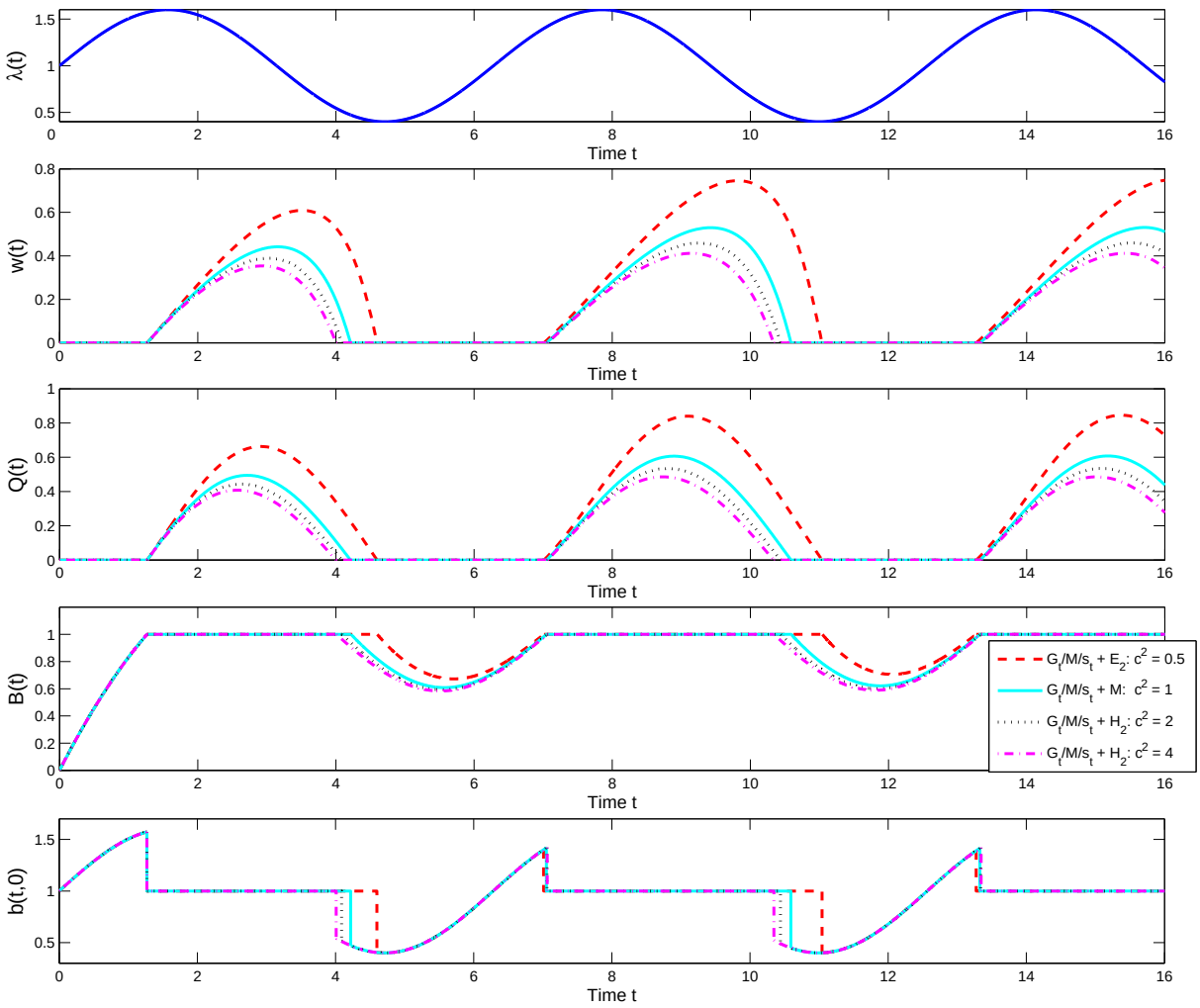

Fig. 30 Fluid dynamics of the $G_t/M/s + GI$ model with fixed mean patience time and M service distribution. The patience distributions are: (i) $E2$ ($CVS = 0.5$); (ii) M ($CVS = 1$); (iii) $H2$ ($CVS = 2$) and (iv) $H2$ ($CVS = 4$).