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AN OPERATIONAL CALCULUS FOR PROBABILITY DISTRIBUTIONS VIA LAPLACE TRANSFORMS

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Abstract

In this paper we investigate operators that map one or more probability distributions on the positive real line into another via their Laplace–Stieltjes transforms. Our goal is to make it easier to construct new transforms by manipulating known transforms. We envision the results here assisting modelling in conjunction with numerical transform inversion software. We primarily focus on operators related to infinitely divisible distributions and Lévy processes, drawing upon Feller (1971). We give many concrete examples of infinitely divisible distributions. We consider a cumulant-moment-transfer operator that allows us to relate the cumulants of one distribution to the moments of another. We consider a power-mixture operator corresponding to an independently stopped Lévy process. The special case of exponential power mixtures is a continuous analog of geometric random sums. We introduce a further special case which is remarkably tractable, exponential mixtures of inverse Gaussian distributions (EMIGs). EMIGs arise naturally as approximations for busy periods in queues. We show that the steady-state waiting time in an M/G/1 queue is the difference of two EMIGs when the service-time distribution is an EMIG. We consider several transforms related to first-passage times, e.g. for the M/M/1 queue, reflected Brownian motion and Lévy processes. Some of the associated probability density functions involve Bessel functions and theta functions. We describe properties of the operators, including how they transform moments.

UNIMODAL DISTRIBUTIONS; INFINITELY DIVISIBLE DISTRIBUTIONS; LÉVY PROCESSES; COMPLETE MONOTONICITY; CUMULANTS; MOMENTS; RANDOM SUMS; INVERSE GAUSSIAN DISTRIBUTIONS; RENEWAL PROCESSES; SUBORDINATION; FIRST-PASSAGE TIMES; BESSEL FUNCTIONS; THETA FUNCTIONS; *M/M/*1 QUEUE; RANDOMIZED RANDOM WALK; BROWNIAN MOTION; POLLACZEK-KHINTCHINE FORMULA

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1. Introduction and summary

Our purpose in this paper is to aid and abet applications of Laplace transforms in applied probability. Laplace transforms currently seem to be somewhat in disfavor, but they have proven their worth in many studies, especially in queueing theory and related subjects such as risk theory and inventory theory. Moreover, we have found that these transforms can often be effectively inverted numerically; see Abate and Whitt (1992), (1995), Choudhury and Lucantoni (1995) and Choudhury *et al.* (1994).

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Given that we are ready to embrace Laplace transforms, we want to be able to work with them. For example, given that we can calculate performance measures for a large class of single-server queues once we have transforms characterizing the arrival process and the service times, we want to be able to find appropriate transforms to use for the arrival process and the service times. (Algorithms for $GI/G/1$ and $BMAP/G/1$ queues based on numerical transform inversion are discussed in Abate *et al.* (1993), (1994), (1995), Choudhury *et al.* (1994), (1996) and Lucantoni *et al.* (1994). Other algorithms for random walks and the $GI/G/1$ queue appear in Ackroyd (1980), Grübel (1991), Grübel and Pitts (1992), Keener (1994) and Konheim (1975).) Transforms for arrival processes and service times can be found by fitting transforms to data, for which we would suggest working with empirical transforms, as in Gaver and Jacobs (1988) and Abate *et al.* (1994), but that is not our focus here.

Here we develop an *operational calculus* for manipulating Laplace transforms (LTs) of probability density functions (p.d.f.'s) and, more generally, Laplace–Stieltjes transforms (LSTs) of cumulative distribution functions (c.d.f.'s), i.e.

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}\, dF(t), \tag{1.1}$$

where F is a c.d.f. and f is its p.d.f. if well defined (if F is absolutely continuous), assumed to be defined on the positive half line. We consider various operators that map one or more LSTs into another and discuss their properties. For instance, we indicate how the moments of the corresponding c.d.f.'s are transformed into new moments by the operator.

We hasten to say that others have already emphasized the importance of transforms. For example, the textbook by Giffin (1975) is certainly much in this spirit. Indeed, as is so often the case, much of what we do is in the classic book by Feller (1971), but we approach the subject in a different way. To see material only hinted at in Feller (1971), and to see how the different operators can be applied, see Section 13 on theta distributions.

We envision the LSTs and operators discussed here being used in conjunction with software to compute c.d.f.'s, p.d.f.'s, moments and asymptotic parameters via numerical inversion, as in Abate and Whitt (1992), (1995) and Choudhury and Lucantoni (1995). Here we contribute toward building an LST toolkit. Once many LSTs are readily available, it becomes possible to model interactively on the computer. Given simple expressions for the LSTs, we can specify them quickly and modify them easily. With the numerical inversion algorithms, the computer can generate plots of p.d.f.'s and c.d.f.'s and tables of moments, possibly together with data for fitting. Thus, our goal is to obtain simple expressions for LSTs.

The most familiar operators are mixture, convolution and compound operators based on these two, such as are obtained by considering random sums of i.i.d.

TABLE 1
The most familiar operators: mixtures, convolutions and compound operators based on these

Operator	Transform	Probability density function	kth moment or cumulant
simple mixture ($\mathcal{M}$)	$p\hat{f}(s) + (1-p)\hat{h}(s)$	$pf(t) + (1-p)h(t)$	$pm_k(f) + (1-p)m_k(h)$
unmixing ($\mathcal{M}^{-1}$)	$\frac{\hat{g}(s) - (1-p)\hat{h}(s)}{p}$		
general mixture ($\mathcal{M}$)	$\int_\Omega \hat{f}_x(s)\, dP(x)$	$\int_\Omega f_x(t)\, dP(x)$	$\int_\Omega m_k(f_x)\, dP(x)$
convolution ($\mathcal{C}$)	$\hat{f}(s)\hat{h}(s)$	$\int_0^t f(t-x)\, dH(x)$	$c_k(f) + c_k(h)$
deconvolution ($\mathcal{C}^{-1}$)	$\frac{\hat{g}(s)}{\hat{h}(s)}$		
random sum ($\mathcal{RS}$) ($\mathcal{C} + \mathcal{M}$)	$\hat{G}(\hat{f}(s))$	$\sum_{k=0}^{\infty} p_k f^{*k}(t)$	
geometric sum ($\mathcal{GS}$) (special case of ($\mathcal{RS}$)	$\frac{1-p}{1-p\hat{f}(s)}$	$(1-p)\sum_{k=0}^{\infty} p^k f^{*k}(t)$	
Poisson sum ($\mathcal{PS}$) (special case of $\mathcal{RS}$)	$e^{-\lambda(1-\hat{f}(s))}$	$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k f^{*k}(t)}{k!}$	

non-negative random variables. The most important random sums are no doubt the geometric random sum and the Poisson random sum (the compound Poisson distribution). These familiar operators are summarized in Table 1. In Table 1, p and p_k are probabilities, $\hat{G}(z) \equiv \sum_{k=0}^{\infty} p_k z^k$ is a probability generating function, $m_k \equiv m_k(f)$ is the kth moment, $c_k \equiv c_k(f)$ is the kth cumulant (semi-invariant or reduced correlation function) and $f^{k*}(t)$ is the k-fold convolution of f. In Table 1 we display both the operators and the inverse operators. The operators are always well defined, but the inverse operators are not. For example, the question of deconvolution is equivalent to factorizing the transform; see Lukacs (1970). We will devote considerable attention to inverse operators, but we will not focus on the elementary operators in Table 1.

Other basic operators that we will consider are displayed in Tables 2 and 3. In Table 2 we display the *exponential damping* (or attenuation) *operator,* which was used in Abate *et al.* (1994), and several familiar operators associated with renewal processes. Exponential damping shifts the dominant singularity. Exponential damping is important for relating long-tail distributions to distributions without an exponential tail that are dominated by an exponential; see Section 9. Again, the damping operator $\mathcal{D}$ is always defined, but its inverse is not. The inverse is defined whenever $\hat{f}(-a)$ is finite.

TABLE 2
More operators: exponential damping and familiar renewal-process operators

Operator	Transform	Probability density function	kth moment
exponential damping ($\mathscr{D}$)	$\frac{\hat{f}(s+a)}{\hat{f}(a)}$	$\frac{e^{-at}f(t)}{\int_0^\infty e^{-at}f(t)\,dt}$	
$\mathscr{D}^{-1}$	$\frac{\hat{f}(s-a)}{\hat{f}(-a)}$	$\frac{e^{at}f(t)}{\int_0^\infty e^{at}f(t)\,dt}$	
stationary-excess ($\mathscr{E}$)	$\frac{1-\hat{f}(s)}{sm_1}$	$\frac{1}{m_1}\int_t^\infty f(x)\,dx$	$\frac{m_{k+1}}{(k+1)m_1}$
$\mathscr{E}^{-1}$	$1-\frac{s\hat{g}(s)}{g(0)}$	$-\frac{g'(t)}{g(0)}$	$(k-1)m_{k-1},\ k \geqq 2$
stationary-lifetime ($\mathscr{L}$)	$-\frac{\hat{f}'(s)}{m_1}$	$\frac{tf(t)}{m_1}$	$\frac{m_{k+1}}{m_1}$
$\mathscr{L}^{-1}$	$\frac{1}{m_{-1}}\int_s^\infty \hat{g}(z)\,dz$	$\frac{g(t)}{tm_{-1}}$	$\frac{m_{k-1}}{m_{-1}}$
renewal-excess ($\mathscr{RE}$)	$\left(\frac{2m_1^2}{m_2-2m_1^2}\right)\left(\frac{\hat{f}(s)}{1-f(s)}-\frac{1}{m_1 s}\right)$		

TABLE 3
Probabilistic-structure operators

Operator	Transform	Probability density function	kth moment
unimodal ($\mathscr{U}$)	$\frac{1}{s}\int_0^s \hat{f}(z)\,dz = \int_0^1 \hat{f}(ts)\,dt$	$\int_t^\infty x^{-1}f(x)\,dx$	$\frac{m_k}{k+1}$
$\mathscr{U}^{-1}$	$\hat{g}(s)+s\hat{g}'(s)$	$-tg'(t)$	$(k+1)m_k$
cumulant-moment transfer ($\mathscr{T}$)	$\frac{-\log\hat{f}(s)}{m_1 s}$	$(-tg'(t))*f(t)=\frac{tf(t)}{m_1(f)}$	$\frac{c_{k+1}}{(k+1)m_1}$
$\mathscr{T}^{-1}$	$\exp(-as\hat{g}(s))$		(3.5)
infinitely divisible ($\mathscr{I}$) ($\mathscr{T}^{-1}\circ\mathscr{U}$)	$\exp\left(-a\int_0^s \hat{f}(z)\,dz\right)$	$f(t)*g(t)=\frac{tg(t)}{m_1(g)}$	(4.10)
$\mathscr{I}^{-1}$	$\frac{-1}{m_1(g)}\frac{d}{ds}\log\hat{g}(s) = -\frac{\hat{g}'(s)}{\hat{g}(s)m_1(g)}$		$\frac{c_{k+1}}{m_1}$
power mixture ($\mathscr{PM}$)	$\hat{h}(-\log\hat{f}(s))$		(6.3) and (6.4)
special case ($\mathscr{PM}\circ\mathscr{T}^{-1}$)	$\hat{h}(s\hat{f}(s))$		
exponential mixture ($\mathscr{EM}$)	$\frac{1}{1+s\hat{f}(s)}$	$f(t)*g(t)=\frac{1}{m_1(g)}\int_t^\infty g(x)\,dx$	(7.4)
$\mathscr{EM}^{-1}$	$\frac{1}{s}\left(\frac{1}{\hat{g}(s)}-1\right)$		

The renewal-excess operator maps the interrenewal time c.d.f. F into $U(t) - t/m_1(F)$ plus a normalization to make the limit 1, where $U(t)$ is the renewal function. The renewal-excess operator does not always yield a bona fide c.d.f.; it does whenever the c.d.f. F has the *increasing mean residual life* (IMRL) property, because then $U(t)$ is monotone. The renewal-excess operator and the stationary-excess operator play an important role in the $M/M/1$ queue; see Section 9 of Abate and Whitt (1988a), Section 4 of Abate and Whitt (1988b) and Section 7 of Abate and Whitt (1988d). The stationary-excess operator is discussed in Whitt (1985) and references therein.

To see how the operators combine, note that the steady-state waiting-time LST in the $M/G/1$ queue is obtained from the service-time LST $\hat{f}$ (assumed to have mean 1) by the composition of the stationary-excess operator and the geometric-sum operator with $p = \rho$, the traffic intensity:

$$\mathscr{PK}(\hat{f}) = (\mathscr{GS} \circ \mathscr{E})(\hat{f}). \tag{1.2}$$

We call $\mathscr{PK}$ in (1.2) the *Pollaczek–Khintchine operator.* In this context, we provide a large collection of candidate service-time LSTs, from which the steady-state waiting-time c.d.f. can easily be computed by numerical inversion using (1.2). As in Abate and Whitt (1992), we would typically compute the complementary c.d.f. $G^c(t) \equiv 1 - G(t)$ after eliminating the known atom at the origin, i.e. $G^c(t)/G^c(0)$ with LT $(1 - \hat{g}(s))/sG^c(0)$. We consider generalizations of the operator $\mathscr{PK}$ in Section 12.

We call the operators in Table 3 *probabilistic-structure operators.* The operators arise when we study probabilistic structure such as infinite divisibility and subordination, but they are not customarily considered as tools for constructing new LSTs the way the operators in Table 1 are. However, the operators in Table 3 are discussed in Feller (1971).

This paper has a fairly honest origin. In Abate *et al.* (1995) we studied exponential approximations for tail probabilities of steady-state waiting times in queueing models based on asymptotics. In Abate *et al.* (1993), (1994) we investigated the different behavior that occurs when the service-time distribution has a long tail. For this purpose, we used our numerical inversion algorithms. However, these numerical inversion algorithms require that we be able to compute the values of the transform of the service-time distribution. When we considered the familiar long-tail (or subexponential) service-time distributions such as lognormal, Pareto or Weibull with shape parameter smaller than 1, as in Johnson and Kotz (1970a, b), we found that explicit expressions for the transforms are unavailable. Hence, we created a new family of long-tail distributions, a *Pareto mixture of exponentials* (PMEs), for which we could readily compute transform values. In particular, the PME density is

$$g_r(x) = \int_{(r-1)/r}^{\infty} f_r(y) y^{-1} e^{-x/y}\, dy, \tag{1.3}$$

where $f_r(x)$ is the Pareto density, i.e.

$$f_r(x) = r\left(\frac{r-1}{r}\right)^r x^{-(r+1)}, \qquad x \geqq (r-1)/r. \tag{1.4}$$

For $r = n$ or $r = n + \frac{1}{2}$ for n integer, closed-form expressions for the LTs of the PME density g_r in (1.3) are given in (2.18)–(2.21) of Abate *et al.* (1994).

Upon reading our paper, Donald Gaver pointed out that there are other ways to construct transforms. In particular, he pointed out that, for any LST $\hat{f}(s)$, $\hat{f}(\log(1 + s))$ is a new LST, as can be seen by considering mixtures of gamma distributions with different shape parameters:

$$\hat{g}(s) = \int_0^\infty (1+s)^{-x}\, dF(x) = \int_0^\infty \mathrm{e}^{-\log(1+s)x}\, dF(x) = \hat{f}(\log(1+s)). \tag{1.5}$$

Gaver's comment made us try to think about this issue more systematically; the present paper is the result. The operation (1.5) is a special case of the power-mixture operator in Table 3, which we discuss in Section 6.

Surprisingly, it does not seem that such a systematic operational calculus for LSTs has been presented before; for instance, the material here is not discussed in Johnson and Kotz (1970a, b). However, not only are a multitude of transform relations in Feller (1971) but the basic ideas are there as well. In additon, there is important material in the substantial literature on infinitely divisible distributions since Feller (1971) as can be seen from Bondesson (1988), (1992).

In particular, a key idea seems to be Bernstein's (1928) theorem (Feller (1971), p. 439).

Proposition 1.1 (Bernstein). *A function $\hat{f}(s)$ is the Laplace–Stieltjes transform of a bona fide c.d.f. F if and only if it is infinitely differentiable for* $\mathrm{Re}(s) > 0$ *with*

$$(-1)^n \hat{f}^{(n)}(s) \geqq 0 \text{ for all positive real } s \text{ and } n \geqq 0, \tag{1.6}$$

and $\hat{f}(0) = 1$.

A real-valued function satisfying (1.6) is said to be *completely monotone* (CM). Two easily verified criteria for a function to be CM are given on p. 441 of Feller (1971).

The Product Criterion. If ϕ and ψ are CM, *then so is the product $\phi\psi$.*

The Composition Criterion. If ϕ is CM *and ψ is a positive function with a* CM *derivative, then $\phi(\psi)$ is* CM.

Examples of CM functions are: $a\mathrm{e}^{-bx}$ for $a, b > 0$, $\phi(a + bx)$ for $a \geqq 0$, $b > 0$ when ϕ is CM, $a/(b + cx)^d$ for $a, b, c, d > 0$, and $-\phi'$ when ϕ is CM. Examples of positive functions with CM derivatives are: $\int_0^x \phi(u)\, du$ when ϕ is CM, $a \log(c + bx)$ for $a, b > 0$ and $c \geqq 1$, $(a + bx)^p$ for $a, b > 0$ and $0 < p < 1$, $a - \phi(bx)$ for $b > 0$, $a \geqq \phi(0)$ when ϕ is CM. Most of Tables 1–3 can be deduced from the two criteria above with these examples. In the rest of this paper we discuss some of the interesting cases in more detail.

The CM property is important not only in characterizing an LST, but also in characterizing infinitely divisible LSTs, as we will see in Section 4. Moreover, the CM property is also important for p.d.f.'s; see for example p. 63 of Keilson (1979). A p.d.f. is CM if and only if it is a mixture of exponential p.d.f.'s. Thus the PME p.d.f. is (1.3) is CM. It is significant that any CM p.d.f. is infinitely divisible (Feller (1971), p. 452).

We discuss quite a few classes of LSTs, but there are important classes that we do not discuss. One obvious omission is *phase-type distributions.* We omit them, not because they are not important, but because they are already well discussed in Chapter 2 of Neuts (1981). Much can be done with only phase-type distributions, as is evident from their being dense in the set of all probability distributions (in the standard topology of weak convergence). Physically, phase-type distributions can be represented as first-passage-time distributions in finite-state continuous-time Markov chains. However, all phase-type distributions have a pure exponential tail. It may be attractive to have alternatives with different tail behavior and/or fewer parameters. This is especially true for first-passage-time distributions, which have extensive application in risk theory and, more generally, in insurance mathematics; see for example Embrechts and Villaseñor (1988) and Embrechts and Klüppelberg (1993).

We also can obtain new LSTs from existing ones by using *location and scale parameters*; i.e. given an LST $\hat{f}$ for a random variable X, $e^{-sa}\hat{f}(bs)$ is the LST of the random variable $a + bX$ for each pair of non-negative real numbers (a, b). We focus on non-location-scale transformations.

For the special case of discrete distributions, the LSTs become *probability generating functions* upon making the change of variables $z = e^{-s}$. Generating functions arise directly when we consider the *moment generating function* constructed from the power series representation of the LST i.e.

$$\hat{f}(s) = \sum_{k=0}^{\infty} m_k \frac{(-s)^k}{k!} \tag{1.7}$$

with $m_0 = 1$. In the literature generating functions have received more attention than Laplace transforms, primarily because of their prominent role in combinatorics; see Graham *et al.* (1989), Odlyzko (1993), Riordan (1958), (1968) and Wilf (1994).

2. The unimodal operator

Suppose that we seek the LT $\hat{g}$ of a p.d.f. g on $[0, \infty)$ that is non-increasing or, more generally, suppose that we seek the LST $\hat{g}$ of a c.d.f. G on $[0, \infty)$ that is *unimodal* with mode at 0. (A c.d.f. on $[0, \infty)$ is unimodal with mode at 0 if it is concave on $[0, \infty)$.) One way that we may be able to obtain such a transform $\hat{g}$ is by applying Khintchine's (1938) representation of a unimodal distribution; see

Chapter 1 of Dharmadhikari and Joag-Dev (1988) and Feller (1971), pp. 158, 527. The general result is for c.d.f.'s on the entire real line; we consider the restriction to the positive half line.

The *unimodal operator* $\mathcal{U}$ maps any LST $\hat{f}$ into an LST $\hat{g}$ of a unimodal c.d.f. G with mode at 0 by

$$\mathcal{U}(\hat{f}) \equiv \hat{g}(s) = s^{-1} \int_0^s \hat{f}(z)\, dz = \int_0^1 \hat{f}(st)\, dt. \tag{2.1}$$

Note that in (2.1) t is a real variable, while z and s are complex variables with $\mathrm{Re}\,(s) > 0$ and $\mathrm{Re}\,(z) > 0$.

From the last integral representation in (2.1), it is clear that indeed $\hat{g}$ is an LST of a bona fide c.d.f. whenever $\hat{f}$ is an LST of a bona fide c.d.f., by virtue of Bernstein's theorem: the uncountable mixture obviously inherits the CM property. Khintchine's representation theorem states that a c.d.f. G on $[0, \infty)$ is unimodal with mode 0 *if and only if* it satisfies (2.1) and that there is a *one-to-one correspondence* between the c.d.f.'s F in (2.1) and the unimodal c.d.f.'s G. Hence, the inverse operation of (2.1) is defined for unimodal c.d.f.'s and no others.

Shepp (1962) developed a nice probabilistic representation for (2.1), showing that if X is a random variable distributed as F, then the random variable UX, where U is a uniform random variable on $[0, 1]$ independent of X, has the associated unimodal c.d.f. G. Starting from the Shepp representation, we can see that (2.1) can be obtained from the Parseval relation; Feller (1971), pp. 463, 619.

Examples of pairs $(\hat{f}, \mathcal{U}(\hat{f}))$ appear in Table 4 below. One $\mathcal{U}(\hat{f})$ is the LST of the time-dependent *first-moment* c.d.f. H_1 of reflected Brownian motion (RBM) in (1.4) and (1.10) of Abate and Whitt (1987). Another $\mathcal{U}(\hat{f})$ is the LT of what we call the

TABLE 4
Examples of pairs $(\hat{f}, \mathcal{U}(\hat{f}))$

Initial transform $\hat{f}$	Transform $\mathcal{U}(\hat{f})$ of unimodal G
deterministic, D, e^{-sb}	$\dfrac{1-e^{-sb}}{sb}$, uniform, U
Erlang, E_2, $\dfrac{1}{(1+s)^2}$	$\dfrac{1}{1+s}$, exponential, M
gamma $(\frac{1}{2})$, $\Gamma_{\frac{1}{2}}$, $\dfrac{1}{\sqrt{1+2s}}$	$\dfrac{2}{1+\sqrt{1+2s}}$, RBM, h_1
exponential, M, $\dfrac{1}{1+s}$	$\dfrac{\log(1+s)}{s}$, exponential-integral p.d.f.

exponential-integral p.d.f.

(2.2) $$g(t) \equiv E_1(t) = \int_t^{\infty} x^{-1} e^{-x}\, dx = \int_0^1 \mu^{-1} e^{-t/\mu}\, d\mu;$$

see 5.1.1 of Abramowitz and Stegun (1972) and Example 7.1 here.

Note that the unimodal operator $\mathcal{U}$ in (2.1) is *linear* for the LSTs, so that we can see what happens by considering deterministic c.d.f.'s. As noted in Table 4, $\mathcal{U}$ maps the LST of a point mass at b into the LT of the uniform p.d.f. on $[0, b]$. By the linearity, any LST of a discrete c.d.f. with masses p_k on points x_k is mapped into the LT of the corresponding mixture of uniform c.d.f.'s on $[0, x_k]$. The Shepp representation is the generalization to arbitrary c.d.f.'s F. This description provides a good physical interpretation of what the operator $\mathcal{U}$ is doing.

The Shepp representation UX allows us to easily make stochastic comparisons. Let $\leqq_{st}$ denote stochastic order. First, we see that $F >_{st} G$. Second, we see that if $F_1 \leqq_{st} F_2$ and F_i is transformed into G_i by the unimodality transformation, then $G_1 \leqq_{st} G_2$. Similar inheritance holds for other orderings (but only one way).

We now define the *inverse unimodal operator* $\mathcal{U}^{-1}$ as

(2.3) $$\mathcal{U}^{-1}(\hat{g}) \equiv \hat{f}(s) = \hat{g}(s) + s\hat{g}'(s).$$

We can apply the Khintchine representation and Bernstein's theorem to justify (2.3) A direct derivation does not seem easy.

Proposition 2.1. *The function $\hat{g}(s) + s\hat{g}'(s)$ is* CM, *and thus the* LST *of a bona fide c.d.f., for an* LST *$\hat{g}$ if and only if the c.d.f. G associated with $\hat{g}$ is unimodal with a mode at* 0.

Proof. If G is unimodal with a mode at 0, then $s\hat{g}(s) = \int_0^s \hat{f}(z)\, dz$ for some LST $\hat{f}$ by (2.1). By this representation, $s\hat{g}(s)$ is differentiable with derivative $s\hat{g}'(s) + \hat{g}(s) = \hat{f}(s)$. By Bernstein's theorem, $\hat{f}(s)$ must thus be CM. Moreover, since $\hat{g}$ is an LST, $\hat{f}(0) = 1$. On the other hand if $s\hat{g}'(s) + \hat{g}(s)$ is CM, then its integral is $s\hat{g}(s) = \int_0^s f(z)\, dz$, where $\hat{f}(s)$ is CM with $\hat{f}(0) = \hat{g}(0) = 1$, so that by (2.1), $\hat{g}$ must be the LST of a c.d.f. G which is unimodal with mode at 0.

The c.d.f. analog of (2.1) is $G(t) = \int_0^1 F(t/x)\, dx$. In general, G is absolutely continuous with p.d.f.

(2.4) $$g(t) = \int_t^{\infty} x^{-1}\, dF(x);$$

see Lemma 4.5.2 of Lukacs (1970).

From (2.4), we see that $g(0) < \infty$ if and only if $m_{-1}(F) < \infty$, where $m_{-k}(F)$ is the kth *negative moment*, i.e.

(2.5) $$m_{-k}(F) = \int_0^{\infty} x^{-k}\, dF(x).$$

Moreover, from (2.4), we see that if F is absolutely continuous with p.d.f. f, then g itself is absolutely continuous with density g', and

$$\frac{f(t)}{t} = -g'(t), \qquad t \geqq 0, \tag{2.6}$$

$$F^c(t) = G^c(t) + tg(t), \qquad t \geqq 0. \tag{2.7}$$

From (2.7), we also see that $F \geqq_{st} G$. From (2.6), we can read off the impact on asymptotics. For example, if $f(t) \sim \alpha t^{\beta} e^{-\eta t}$ as $t \to \infty$, then $g'(t) \sim \alpha t^{\beta-1} e^{-\eta t}$ and $g(t) \sim \alpha \eta t^{\beta-1} e^{-\eta t}$ as $t \to \infty$.

From above (e.g. the Shepp representation), we can easily determine the impact on moments. Let $m_k(F)$ be the kth moment of c.d.f. F and let $c^2(F)$ be its squared coefficient of variation. Then

$$m_k(G) = m_k(F)/(k+1) \quad \text{and} \quad c^2(G) = c^2(F) + \tfrac{1}{3}(c^2(F) + 1). \tag{2.8}$$

We can easily scale G so that it has the same mean as F. Letting X have c.d.f. F, we can let G^* have the c.d.f. of $2UX$. This new c.d.f. G^* has the same mean as F but is more variable than F, i.e. is larger than F in *convex stochastic order*: $E\phi(X) \leqq E\phi(2UX)$ for all real-valued convex functions ϕ for which the expectations are finite. (Apply Jensen's inequality after conditioning on X.) As a consequence, we see that $\mathscr{U}$ does not have a non-trivial fixed point, whether or not we rescale to keep the mean fixed.

We conclude by mentioning that by no means is the operator $\mathscr{U}$ the only operator yielding unimodal distributions. For example, the stationary-excess operator $\mathscr{E}$ does as well. Note that $\mathscr{U}$ is related to $\mathscr{E}$ via

$$\mathscr{U}(\hat{f}) = (\mathscr{E} \circ \mathscr{L}^{-1})(\hat{f}), \tag{2.9}$$

where $\mathscr{L}^{-1}$ is the inverse-stationary-lifetime operator, provided that the first negative moment $m_{-1}(F)$ is finite; see Section 5.

3. The cumulant-moment-transfer operator

In many applied probability settings we are interested in moments and cumulants (semi-invariants or reduced correlation functions). The moments m_k and the cumulants c_k can be identified as the coefficients in the Taylor series expansions (1.7) and

$$\log \hat{f}(s) = \sum_{k=1}^{\infty} c_k \frac{(-s)^k}{k!}. \tag{3.1}$$

The ordinary moments are related to the cumulants via

$$m_{n+1} = \sum_{k=0}^{n} \binom{n}{k} c_{k+1} m_{n-k}, \qquad n \geqq 0; \tag{3.2}$$

see p. 113 of Kendall and Stuart (1987). One reason the cumulants are useful is that the kth cumulant of a convolution (sum of independent random variables) is the sum of the kth cumulants; this property for convolution appears in Table 1.

To help identify moments and cumulants, it is useful to have a way to relate the cumulants of one c.d.f to the moments of another. From (3.1), we see that if the cumulants are all non-negative and we divide $\log f(s)$ by $-sm_1$, then it will look like the power series of an LST. This suggests the *cumulant-moment-transfer operator*

$$\mathcal{T}(\hat{f}) \equiv \hat{g}(s) = -\frac{\log \hat{f}(s)}{m_1 s}. \tag{3.3}$$

Combining (3.1) and (3.3), we see that

$$m_k(G) = \frac{c_{k+1}(F)}{(k+1)m_1(F)} \tag{3.4}$$

as indicated in Table 3 and

$$\frac{m_{n+1}(F)}{m_1(F)} = \sum_{k=0}^{n} (k+1)\binom{n}{k} m_k(G) m_{n-k}(F). \tag{3.5}$$

From (3.3), it is immediate that the inverse of $\mathcal{T}$ is

$$\mathcal{T}_a^{-1}(\hat{g}) \equiv \mathcal{T}^{-1}(\hat{g}) = \exp(-as\hat{g}(s)) \qquad \text{for } a > 0. \tag{3.6}$$

It remains to determine when the operators $\mathcal{T}$ and $\mathcal{T}^{-1}$ in (3.3) and (3.6) are well defined. Note that the cumulants $c_k(F)$ are finite whenever the moments $m_k(f)$ are finite. Obviously, for (3.4) to be valid we need the cumulants of F to be non-negative, which is not always the case (e.g. $c_4 < 0$ for the two-point distribution with probability $\frac{1}{2}$ given to 0 and 2). Moreover, the numbers $m_k(G)$ in (3.4) must satisfy growth conditions to be bona fide moments of a c.d.f. (Feller (1971), pp. 224–228). (For example, the odd cumulants of a uniform distribution on $[0, x]$ are 0, p. 59 of Johnson and Kotz (1970b), so that $\mathcal{T}(\hat{f})$ is not the LST of a bona fide c.d.f. when $\hat{f}$ is the uniform LST.)

We characterize the domain of $\mathcal{T}$ in terms of infinitely divisible distributions. By Feller (1971), pp. 176, 449, a c.d.f. G with LST $\hat{g}$ is *infinitely divisible* (ID) if $\hat{g}^{1/n}$ is an LST of a bona fide c.d.f. for all positive integers n. We establish the following result at the end of the next section, where we consider ID distributions in more detail.

Proposition 3.1. *The operator $\mathcal{T}$ is well defined if and only if $\hat{f}$ is the* LST *of an* ID *distribution with finite mean, in which case $\mathcal{T}(\hat{f})$ is the* LST *of a unimodal c.d.f. with mode at* 0. *The inverse operator $\mathcal{T}^{-1}$ in* (3.6) *is well defined for all $a > 0$ if and only if G is unimodal with a mode at* 0, *in which case $\mathcal{T}^{-1}(\hat{g})$ is the* LST *of an* ID *c.d.f. with finite mean.*

Now note that $\mathcal{T}(\hat{f}_{a,b}(s)) = \mathcal{T}(\hat{f}_{1,1})(bs)$ for positive real numbers a and b when

$\hat{f}_{a,b}(s) = \hat{f}(bs)^a$. Hence, even for specified scale parameter b, the operator $\mathscr{T}$ is *not one-to-one*. However, the operator $\mathscr{T}$ can be regarded as one-to-one on infinitely divisible equivalence classes: we say that two LSTs $\hat{f}$ and $\hat{g}$ are ID *equivalent* if $\hat{g}(s) = f(s)^a$ for some $a > 0$, and we write $\hat{f} \sim_{\mathrm{ID}} \hat{g}$. Note that $\mathscr{T}(\hat{f}) = \mathscr{T}(\hat{g})$ whenever $\hat{f} \sim_{\mathrm{ID}} \hat{g}$. Since $\mathscr{T}_a^{-1} \circ \mathscr{T}(\hat{f})(s) = \hat{f}(s)^{a/m_1}$, we see that $\mathscr{T}$ is one-to-one on ID equivalence classes.

The operator $\mathscr{T}$ is continuous in the sense that $\mathscr{T}(\hat{f}_n) \to \mathscr{T}(\hat{f})$ provided that $F_n(t) \to F(t)$ at all continuity points t of a bona fide (proper) c.d.f. F and $m_1(F_n) \to m_1(F) < \infty$, by virtue of the continuity theorem for LSTs (Feller (1971), p. 431). Moreover, if $\mathscr{T}(\hat{f}_n)$ is an LST of a bona fide c.d.f. for all n, then so is $\mathscr{T}(\hat{f})$, again by the continuity theorem for LSTs.

Now suppose that we start with a proper p.d.f. f, so that F has no atom at 0. We can rewrite (3.3) as

$$s\hat{g}(s) = \frac{1}{m_1(f)} \log\left(\frac{1}{\hat{f}(s)}\right). \tag{3.7}$$

Then $\hat{f}(s) \to 0$ as $s \to \infty$. From (3.7), we see that $s\hat{g}(s) \to \infty$ as $s \to \infty$, which implies that, if G has a density, then $g(0) = \infty$. On the other hand, if f has an atom $F(0) > 0$ at 0, then $\hat{f}(s) \to F(0)$ as $s \to \infty$, so that $s\hat{g}(s) \to -\log(F(0))/m_1$ as $s \to \infty$. Hence, F and G tend to have opposite behavior at the origin.

By Proposition 3.1, $\mathscr{T}^{-1}$ provides a way to construct ID distributions starting with unimodal distributions. For example, if we apply the inverse operator $\mathscr{T}^{-1}$ to a simple exponential $\hat{g}(s) = (1+s)^{-1}$, then we get

$$\hat{f}(s) = \mathscr{T}^{-1}(\hat{g})(s) = e^{-s/(1+s)}, \tag{3.8}$$

which has an atom at 0. This c.d.f. is known to chemical engineers as the *percolation concentration function.* Note that $\hat{f}$ in (3.8) has an essential singularity at $s = -1$. If we convolve F in (3.8) with a gamma c.d.f. then we get a c.d.f. discussed several times in Feller (1971), pp. 58, 349, 438. We return to this example in Section 11.

As another example, suppose that we apply $\mathscr{T}^{-1}$ to the uniform c.d.f. on $[0, 1]$. Then we get

$$\hat{f}(s) = \exp(e^{-s} - 1). \tag{3.9}$$

We call this the *Bell distribution* because the moments are the Bell numbers; see p. 23 of Wilf (1994).

It is interesting to consider the fixed point of operators $\mathscr{T}$ and $\mathscr{T}^{-1}$.

Proposition 3.2. *The operator $\mathscr{T}$ has a fixed point among* ID *equivalence classes with mean* 1, *which is given by the solution to the equation.*

$$\hat{f}(s) = e^{-s\hat{f}(s)}. \tag{3.10}$$

The distribution is characterized by its moments, which are

$$m_n = (n+1)^{n-1}, \qquad n \geqq 1. \tag{3.11}$$

Proof. Differentiate (3.3) with $\hat{g} = \hat{f}$ to obtain an expression for $\hat{f}'$ in terms of $\hat{f}$. Fixing $\hat{f}(s)$ for some s with $\mathrm{Re}\,(s) > 0$, thus determines the derivatives $\hat{f}^{(k)}(s)$ for all k. Since $\hat{f}$ is analytic for $\mathrm{Re}\,(s) > 0$, $\hat{f}$ is determined by the coefficients of its power series about s. The parameter a in $\hat{f}(s)^a$ allows us to fix the initial value of $\hat{f}(s)$ at any desired value.

From the moment relation (3.5), we see that the moments of the fixed point c.d.f. in (3.10) are

$$m_{n+1} = \sum_{k=0}^{n} (k+1)\binom{n}{k} m_k m_{n-k}, \tag{3.12}$$

where we have set $m^1 = 1$ to fix the free parameter. From (3.12), we can deduce (3.11). By the Carleman moment growth criterion (Feller (1971), p. 228), the moments in (3.11) uniquely determine the distribution.

It turns out that the moment generating function associated with (3.10) is classic, but the analytic form of $\hat{f}$ is evidently *not* known; see for example (45) on p. 128 of Riordan (1958), p. 202 of Graham *et al.* (1989) and 5.1.4 on p. 139 of Wilf (1994). We thus call this fixed point the *Caley–Eisenstein–Pólya* (*CEP*) *LST*. From (3.11) we can readily deduce that the radius of convergence of the moment generating function is e^{-1}. Moreover, by Theorem 5.3 of Abate *et al.* (1995), assuming that $f(t) \sim \alpha t^{\beta} e^{-\eta t}$ as $t \to \infty$ for some constants α, β and η, we see that

$$f(t) \sim \frac{e^{\frac{3}{2}} e^{-t/e}}{\sqrt{2\pi t^3}} \qquad \text{as } t \to \infty. \tag{3.13}$$

We now give a probabilistic interpretation for the operator $\mathcal{T}^{-1}$.

Proposition 3.3. *If G is unimodal with mode at* 0, *then*

$$\hat{f}(s)\mathcal{U}^{-1}(\mathcal{T}^{-1}(\hat{g}))(s) = \mathcal{L}(\hat{f})(s), \tag{3.14}$$

where $\mathcal{L}$ is the stationary-lifetime operator in Table 2 *and $\mathcal{U}^{-1}$ is the inverse unimodal operator in Section* 2, *which for p.d.f.s becomes*

$$(-tg'(t)) * f(t) = \frac{tf(t)}{m_1(f)}, \qquad t > 0. \tag{3.15}$$

Proof. Multiply both sides by s and then differentiate in (3.3) to get

$$s\hat{g}'(s) + \hat{g}(s) = \frac{-\hat{f}'(s)}{m_1(f)\hat{f}(s)}. \tag{3.16}$$

Then multiply both sides of (3.16) by $\hat{f}(s)$ to get (3.14). Finally, apply (2.5) to get (3.15).

Note that (3.14) and (3.15) say that F convolved with $\mathcal{U}^{-1}(\dot{\mathcal{T}}^{-1}(G))$ coincides with $\mathcal{L}(F)$. (We use transform operators applied to c.d.f.s to mean the c.d.f.s of the operators applied to the associated transforms.)

4. The infinite-divisibility operator

The topic receiving the greatest attention in Feller (1971) may be infinitely divisible (ID) distributions. Classically, interest in ID distributions was due primarily to their role as all possible marginal distributions for Lévy stochastic processes (processes with stationary and independent increments that satisfy some additional conditions) and all possible limit distributions for row sums of triangular arrays of i.i.d. random variables (Feller (1971), p. 303), but we believe that they can also be important in direct probability modelling.

By the product criterion in Section 1, the family of ID distributions is closed under convolution. Every mixture of exponential distributions and every mixture of geometric distributions is ID (Feller (1971), p. 452). The Poisson distribution and Poisson random sums are ID; indeed, every ID distribution is the limit of a sequence of Poisson random sums (Feller (1971), p. 303). From Tables 1 and 2 and (3.6), we see that the Poisson sum operator can be expressed in terms of the operators $\mathcal{T}^{-1}$ and $\mathcal{E}$ via $\mathcal{PS} = \mathcal{T}^{-1} \circ \mathcal{E}$. ID distributions are usually (but not always) unimodal; see Chapter 5 of Dharmadhikari and Joag-Dev (1988). For characterizations of discreteness, singularity and absolute continuity of ID c.d.f.s, see pp. 124–126 of Lukacs (1970). Nevertheless, despite all these results and more recent ones, ID distributions remain somewhat elusive. We will be constructing many examples of ID distributions.

By Theorem 1, p. 450 of Feller (1971), a c.d.f. G on $[0, \infty)$ is ID if and only if $\hat{g}(s) = e^{-\psi(s)}$ where $\psi(0) = 0$ and ψ' is CM. The exponent function ψ can be represented as

$$\psi(s) = \int_0^\infty \frac{1 - e^{-sx}}{x} \, dP(x) \tag{4.1}$$

for a measure P satisfying $\int_1^\infty x^{-1} \, dP(x) < \infty$.

Formula (4.1) provides a general way to construct LSTs, but it is not directly an operator on LSTs. An interesting special case arises when the derivative ψ' is actually a constant multiple of an LST. Then we can write

$$\hat{g}(s) \equiv \mathcal{I}_a(\hat{f}) \equiv \mathcal{I}(\hat{f}) = \exp\left(-a \int_0^s \hat{f}(z) \, dz\right), \tag{4.2}$$

where $\hat{f}$ is an LST of a bona fide c.d.f. F and $a > 0$. We call $\mathcal{I}$ in (4.2) the *infinitely divisible* (ID) *operator.* From (4.2) we see that $m_1(G) = a$.

Note that $\mathcal{I}$ can be related to operator $\mathcal{U}$ by

$$\mathcal{I}(\hat{f}) = \exp\left(-as\mathcal{U}(\hat{f})(s)\right), \tag{4.3}$$

i.e. $\mathcal{I} = \mathcal{T}^{-1} \circ \mathcal{U}$. Since $\frac{d}{ds}(s\hat{g}) = \hat{g} + s\hat{g}'$, we could apply Proposition 2.1 to deduce that $s\mathcal{U}(\hat{f})(s)$ has a CM derivative.

We remark that ID distributions are often represented in other forms; see Feller (1971). For example, the LST of a *compound Poisson distribution* where a is the mean of the Poisson distribution and $\hat{h}$ is the LST of the summands is

$$\hat{g}(s) = \exp(-a(1 - \hat{h}(s))), \tag{4.4}$$

but we can rewrite the exponent in (4.4) as

$$a(\hat{1} - h(s)) = -a \int_0^s \hat{h}'(z)\, dz = am_1(h) \int_0^s -\frac{\hat{h}'(z)}{m_1(h)}\, dz, \tag{4.5}$$

where $-\hat{h}'(s)/m_1(h)$ is an LST of a bona fide c.d.f., indeed, $\mathcal{L}(\hat{h})$; see Table 2. Hence (4.4) is a special case of (4.2).

However, not all ID distributions can be represented in the form (4.2). For example, the *stable laws* on $[0, \infty)$ with exponent α, $0 < \alpha < 1$ have LSTs

$$g(s) = \exp(-s^{\alpha}); \tag{4.6}$$

see Feller (1971), p. 448. Thus, $\psi(s) = s^{\alpha}$ and $\psi'(s) = \alpha s^{-(1-\alpha)}$. Note that ψ' is CM, but $\psi'(0) = \infty$, so that ψ' cannot be an LST of a bona fide c.d.f. Indeed, we see that (4.2) is the general representation for the LST of an ID c.d.f. with finite mean $m_1(G) = \psi'(0)$. When $-\psi'(0) < \infty$, $\psi'(s)/\psi'(0)$ is the LST of a bona fide c.d.f.

From (4.2), it is immediate that the *inverse* ID *operator* is

$$\mathcal{I}^{-1}(\hat{g}) = \frac{-1}{m_1(g)} \frac{d}{ds} \log \hat{g}(s) = \frac{-\hat{g}'(s)}{m_1(g)\hat{g}(s)} = \frac{\mathcal{L}(\hat{g})(s)}{\hat{g}(s)}, \tag{4.7}$$

where $\mathcal{L}$ is the stationary-lifetime operator in Table 2. From the discussion above, the domain of the inverse operator $\mathcal{I}^{-1}$ is precisely the set of ID c.d.f.'s with finite mean. Since $\mathcal{I} = \mathcal{T}^{-1} \circ \mathcal{U}$, $\mathcal{I}^{-1} = \mathcal{U}^{-1} \circ \mathcal{T}$.

From the last expression in (4.7), we see that *the inverse operator* $\mathcal{I}^{-1}$ *has a relatively simple direct probabilistic interpretation*: $\mathcal{I}^{-1}(\hat{g})$ is obtained by deconvolving G from the lifetime distribution of G. We thus can write

$$\mathcal{I}^{-1}(\hat{g}) = C^{-1}(\mathcal{L}(\hat{g}), \hat{g}). \tag{4.8}$$

As indicated in the next section, $\mathcal{L}(\hat{g})$ is stochastically greater than $\hat{g}$, so that the deconvolution might make sense for any g with finite mean. As indicated above, $\hat{g}$ is a factor of $\mathcal{L}(\hat{g})$ if and only if $\hat{g}$ is ID with finite mean. The p.d.f. characterization of $\mathcal{I}^{-1}$ is

$$(g * f)(t) \equiv \int_0^t g(t - x)\, dF(x) = \frac{tg(t)}{m_1(g)}, \qquad t \geqq 0. \tag{4.9}$$

The characterization (4.9) was given earlier by Steutel (1973), Corollary 5.4.

Note that $\mathscr{I}^{-1} \circ \mathscr{I}_a(\hat{f}) = \hat{f}$, while $\mathscr{I}_a \circ \mathscr{I}^{-1}(\hat{g}) = \hat{g}^{a/m_1}$, so that $\mathscr{I}^{-1}$ is not one-to-one, but as with $\mathscr{T}$ in Section 3, $\mathscr{I}^{-1}$ is one-to-one on ID equivalence classes.

From (4.2), (4.7) or (4.8), it is easy to see that $\mathscr{I}$ and $\mathscr{I}^{-1}$ have the exponential distribution (with mean 1) as a fixed point. By essentially the same argument as for Proposition 3.2, we obtain the following result.

Proposition 4.1. *The operator $\mathscr{I}^{-1}$ has the exponential distribution with mean* 1 *as the unique fixed point among* ID *equivalence classes with mean* 1.

Note that the operator $\mathscr{I}$ applied to an exponential with mean b yields a gamma with mean 1 and shape parameter b^{-1}; i.e. $\mathscr{I}((1+bs)^{-1}) = (1+bs)^{-1/b}$.

Using (4.7) and the cumulant generating function (3.1), we see that the moments of F are related to the cumulants of G by

$$m_k(F) = \frac{c_{k+1}(G)}{c_1(G)} \tag{4.10}$$

or, using (3.2),

$$\frac{m_{k+1}(G)}{m_1(G)} = \sum_{k=0}^{n} \binom{n}{k} m_k(F) m_{n-k}(G). \tag{4.11}$$

For more on ID distributions, see Steutel (1973), Bondesson (1988), (1992) and references cited therein. Bondesson focuses on various subclasses of ID distributions, the main one being *generalized gamma convolutions* (GGC), which were first introduced by Thorin (1977a, b). Another is *generalized convolutions of mixtures of exponential distributions* (GCMED). Generalized convolutions of a class contain the class and are closed under convolutions and convergence in distribution. The class GGC is a proper subclass of GCMED. In general, the class of all ID LSTs $\mathscr{I}(\hat{f})$ is *not* contained in GCMED, since the c.d.f. F associated with the LST $\hat{f}$ need not be absolutely continuous and, even when F is absolutely continuous with density f, $f(t)/t$ need not be CM; see Chapter 9 of Bondesson (1992). However, when F is absolutely continuous and $\hat{g} = \mathscr{I}(\hat{f})$, g is GGC if and only if f is CM; see pp. 29–30 of Bondesson (1992). Since mixtures of exponentials constitute CM, both CM and GGC are contained in GCMED, but neither contains the other.

We give examples of pairs $(\hat{f}, \mathscr{I}(\hat{f}))$ in Section 10. Our constructive approach to obtaining ID transforms is similar to Bondesson's approach. We conclude this section by proving Proposition 3.1.

Proof of Proposition 3.1. By (4.2), if $\hat{f}$ is ID with finite mean, then $\mathscr{T}(\hat{f})(s) = as^{-1}\int_0^s \hat{h}(z)\,dz$ for an LST $\hat{h}$, which is the LST of a unimodal c.d.f. with mode at 0 by Section 2. By Proposition 2.1 and the composition criterion, $\mathscr{T}^{-1}(g)$ in (3.6) is the LST of a bona fide c.d.f. for all $a>0$ when $\hat{g}$ is an LST of a unimodal c.d.f. with mode at 0. Considering $\mathscr{T}^{-1}(\mathscr{T}(\hat{f}))$, we see that $\hat{f}(s)^{a/m_1(f)}$ is a bona fide c.d.f. for all $a>0$, which implies that $\hat{f}$ must be an LST of an ID c.d.f.

5. The stationary-lifetime operator

For any c.d.f. F with finite mean, the associated *stationary-lifetime* c.d.f. is

(5.1) $$G(t)=\frac{1}{m_1(F)}\int_0^t x\,dF(x), \qquad t\geqq 0.$$

If F is absolutely continuous with p.d.f. f, then G is absolutely continuous with p.d.f.

(5.2) $$g(t)=\frac{tf(t)}{m_1(f)}, \qquad t\geqq 0.$$

It is well known that for a renewal process with interrenewal times distributed according to F, the distance between the next point and the last point at time t in equilibrium has the c.d.f. G in (5.1). The c.d.f. G is stochastically larger than F. This is easy to see with p.d.f.s as in (5.2). With p.d.f.'s we have the stronger likelihood ratio ordering since $g(t)/f(t)=t/m_1(f)$.

Using transforms, (5.1) becomes

(5.3) $$\hat{g}(s)=-\frac{\hat{f}'(s)}{m_1(f)}.$$

Again, even if we did not have the time-domain formula (5.1), we could deduce that $\hat{g}$ in (5.3) is an LST of a bona fide c.d.f. whenever $\hat{f}$ is by applying Bernstein's theorem: $-\hat{f}'$ is CM whenever $\hat{f}$ is CM.

Even though the stationary-lifetime operator is quite well known, its inverse does not seem to be. With p.d.f.'s, the *inverse-stationary-lifetime operator* is defined simply by setting

(5.4) $$f(t)=\frac{g(t)}{tm_{-1}(g)}, \qquad t\geqq 0,$$

where $m_{-k}(g)$ is the kth negative moment of g as in (2.5). The inverse operator is well defined whenever $m_{-1}(g)<\infty$. In transforms, (5.4) becomes

(5.5) $$\hat{f}(s)=\frac{1}{m_{-1}(g)}\int_s^\infty \hat{g}(z)\,dz=1-\frac{1}{m_{-1}(g)}\int_0^s \hat{g}(z)\,dz.$$

We remark that, from Parseval's relation (Feller (1971), pp. 463, 619), we can also write

(5.6) $$m_{-1}(g)=\int_0^\infty \hat{g}(s)\,ds.$$

To get (5.6), write

(5.7) $$\int_0^\infty \hat{f}(\theta y)g(y)\,dy=\int_0^\infty \hat{g}(\theta y)f(y)\,dy$$

and let $\theta = 1$ and $f(y) = y^n$. Then $\hat{f}(s) = n!\, s^{-(n+1)}$ and

$$\int_0^\infty g(y) y^{-(n+1)}\, dy = \frac{1}{n!} \int_0^\infty y^n \hat{g}(y)\, dy. \tag{5.8}$$

For $n = 1$, (5.8) becomes (5.6).

From (5.5), (5.6) and Bernstein's theorem, we see that $\hat{f}$ is the LST of a bona fide c.d.f. whenever $\hat{g}$ is and $m_{-1}(g) < \infty$. (Note that $\hat{f}'(s) = -\hat{g}(s)/m_{-1}(g)$.)

From (5.1) and (5.5), we see that the moments are related by

$$m_k(g) = \frac{m_{k+1}(f)}{m_1(f)} \quad \text{and} \quad m_{k+1}(f) = \frac{m_k(g)}{m_{-1}(g)}. \tag{5.9}$$

Example 5.1. If $g(t) = 2(e^{-t} - e^{-2t})$, $t \geqq 0$, then $\hat{f}(s) = \log((2+s)/(1+s))/\log 2$ and $f(t) = (e^{-t} - e^{-2t})/t \log 2$, $t \geqq 0$.

6. The power-mixture operator

Suppose that $\hat{f}$ is an LST of an ID c.d.f., so that $\hat{f}(s)^t$ is an LST of a bona fide c.d.f. for each non-negative real number t. Then, analogous to random sums, we can construct an LST of a new c.d.f. by mixing with respect to an arbitrary c.d.f. H, i.e.

$$\mathcal{PM}(\hat{f}, H) \equiv \hat{g}(s) \equiv \int_0^\infty \hat{f}(s)^t\, dH(t) = \hat{h}(-\log \hat{f}(s)). \tag{6.1}$$

We call the operator $\mathcal{PM}$ mapping the pair $(\hat{f}, H)$ into $\hat{g}$ the *power-mixture operator*. We can think of the power-mixture operator applying to LSTs of *arbitrary* c.d.f.'s F if we consider the composition using (4.2); i.e. $\mathcal{O}(\hat{f}, H) = \mathcal{PM}(\mathcal{I}(f), H)$.

Note that the power-mixture operator $\mathcal{PM}$ in (6.1) is the special case of the general mixture operator $\mathcal{M}$ in Table 1 in which $\hat{f}_x(s) = \hat{f}(s)^x$ and the mixing p is done by the c.d.f. H on $[0, \infty)$. A mixture has a natural direct interpretation in the time domain, as indicated in Table 1. Roughly speaking, the power $\hat{f}(s)^x$ may be thought of as an x-fold convolution. Indeed, if H has support on the integers, then the power mixture is equivalent to a random sum, and (6.1) is defined for all $\hat{f}$. Note that Gaver's example in Section 1 is the special case of (6.1) in which $\hat{f}$ is exponential, i.e. $\hat{f}(s) = (1+s)^{-1}$.

We remark that the power-mixture operator arises naturally with Lévy processes; see Feller (1971), pp. 345, 451 and Prabhu (1980). The exponent t in $\hat{f}(s)^t$ arises natural in the stochastic process context as *time*; i.e. when $\hat{f}$ is an LST of an ID c.d.f, $\hat{f}(s)^t$ is the marginal distribution at time t of a Lévy process $\{X(t): t \geqq 0\}$. Then $\hat{h}(-\log \hat{f})$ is the LST of the *stopped Lévy process* $X(T)$ where T is a random time independent of $\{X(t): t \geqq 0\}$ that has c.d.f. h.

If the random time t is replaced by a Lévy stochastic process $\{T(t): t \geqq 0\}$, then $\{X(T(t)): t \geqq 0\}$ is the new Lévy process. We then say $\{X(T(t))\}$ is *subordinate* to

$\{X(t)\}$ and $\{T(t)\}$ is the *directing process* or *random time change* (Feller (1971), pp. 345–349). However, in this paper we are more interested in constructing LSTs than in constructing stochastic processes. Hence, we are more interested in $X(T)$ for an arbitrary non-negative random variable T independent of $\{X(t)\}$.

From (6.1), we see that

$$m_1(G) = m_1(F)m_1(H). \tag{6.2}$$

More generally, we see that

$$m_k(G) = \int_0^\infty m_k(\hat{f}(s)^t)\, dH(t), \qquad k \geqq 1, \tag{6.3}$$

and

$$c_k(f(s)^t) = tc_k(f), \qquad k \geqq 1, \tag{6.4}$$

which can be computed using (3.2). For instance,

$$\begin{aligned} m_2(G) &= \int_0^\infty [c_2(\hat{f}(s)^t) + c_1(\hat{f}(s)^t)^2]\, dH(t) \\ &= c_2(\hat{f})m_1(H) + c_1^2(\hat{f})m_2(H) \\ &= m_2(\hat{f})m_1(H) - m_1(\hat{f})^2 m_1(H) + m_1^2(\hat{f})m_2(H). \end{aligned} \tag{6.5}$$

We can establish stochastic comparisons using the Laplace transform ordering, defined by $\hat{f}_1 \leqq_{\mathrm{LT}} \hat{f}_2$ if $\hat{f}_1(s) \geqq \hat{f}_2(s)$ for all positive real s; see for example p. 22 of Stoyan (1983). From (6.1), it is immediate that if $\hat{f}_1 \leqq_{\mathrm{LT}} \hat{f}_2$ and $H_1 \leqq_{\mathrm{st}} H_2$, then $\hat{g}_1 \leqq_{\mathrm{LT}} \hat{g}_2$. Hence, $\mathcal{PM}$ can be regarded as a monotone operator in this sense.

7. Exponential mixtures

If we compose the power-mixture operator with the inverse-moment-cumulant-transfer operator, then we get an operator that is an analog of the random-sum operator $\mathcal{RS}(\hat{f})(s) = \hat{G}(\hat{f}(s))$ in Table 1, namely,

$$(\mathcal{PM} \circ \mathcal{T}^{-1})(\hat{f}) \equiv \hat{h}(s\hat{f}(s)), \tag{7.1}$$

which is defined for all LSTs of c.d.f.'s that are unimodal with mode at 0.

As a further special case, we obtain the *exponential mixture operator* ($\mathcal{EM}$) if we make H exponential with mean 1, i.e.

$$\mathcal{EM}(\hat{f}) \equiv \frac{1}{1 + s\hat{f}(s)}. \tag{7.2}$$

The exponential mixture operator is an analog of the geometric-random-sum operator in Table 1.

From (7.2), we see that the *inverse-exponential-mixture operator* is

$$\hat{f} \equiv \mathscr{EM}^{-1}(\hat{f}) = \frac{1}{s}\left(\frac{1}{\hat{g}(s)} - 1\right) = \frac{\mathscr{E}(\hat{g})(s)}{\hat{g}(s)}. \tag{7.3}$$

In other words

$$\mathscr{C}(f, \mathscr{EM}(\hat{f})) = \mathscr{E}(\hat{f}) \tag{7.4}$$

or, with p.d.f.'s,

$$f(t) * g(t) = \frac{1 - G(t)}{m_1(G)}. \tag{7.5}$$

By Corollary 1.5.2 of Abate and Whitt (1987), the RBM first moment LST $\hat{h}_1$ in Table 4 is the unique fixed point of the operators $\mathscr{EM}$ and $\mathscr{EM}^{-1}$.

From (7.5), we get a recursive expression for the moments, namely,

$$\frac{m_{n+1}(G)}{n+1} = \sum_{k=0}^{n} \binom{n}{k} m_k(F) m_{n-k}(G). \tag{7.6}$$

For the geometric random sum we have the analog of (7.6),

$$m_n(G) = \frac{p}{(1-p)} \sum_{k=1}^{n} \binom{n}{k} m_k(F) m_{n-k}(G). \tag{7.7}$$

It remains to determine the domain of $\mathscr{EM}^{-1}$, or, equivalently, the range of $\mathscr{EM}$. By (7.4) or (7.5), we see that a necessary condition is to have $G \leqq_{\text{st}} \mathscr{E}(G)$, which means that G must be *new worst than used in expectation* (NWUE); see Theorem 3.1(iii) of Whitt (1985).

Example 7.1. Let

$$\hat{f}(s) = \frac{\log(1 + \mu s)}{\mu s}, \tag{7.8}$$

be the exponential-integral LST as in (2.2), which has moments

$$m_k(F) = \frac{k!\, \mu^k}{(k+1)}. \tag{7.9}$$

Then

$$\mathscr{EM}(\hat{f}) \equiv \hat{g}(s) = \frac{\mu}{\mu + \log(1 + \mu s)} \tag{7.10}$$

and

$$m_{n+1}(G)=\sum_{k=0}^{n}\frac{(n+1)!}{(k+1)(n-k)!}\mu^k m_{n-k}(G); \tag{7.11}$$

e.g. $m_1=1$, $m_2=\mu+2$, $m_3=2\mu^2+6\mu+6$ and $m_4=6\mu^3+22\mu^2+36\mu+24$.

8. Exponential mixtures of inverse Gaussian distributions

The *inverse Gaussian* (*IG*) *distribution* arises as the first-passage-time distribution for RBM; see p. 565 of Abate and Whitt (1987) and Section 6 of Abate and Whitt (1988c). We shall refer to the IG transform as

$$\hat{f}(s;a,v)=\exp(-r_2(s)/v), \tag{8.1}$$

where

$$r_2(s)=\sqrt{2vas+1}-1. \tag{8.2}$$

Note that $r_2(s)$ in (8.2) is a positive function with a completely monotone derivative for real s, so that $\hat{f}(s;a,v)$ is an LST of an ID distribution. For more on ID properties of first-passage times, see p. 129 of Steutel (1973) and p. 145 of Bondesson (1992).

In (8.1), the parameter a is a location (mean) parameter and v is a shape parameter. Thus $\hat{f}(s;1,v)$ has moments $m_1=1$, $m_2=v+1$, $m_3=3v^2+3v+1$ and $m_4=15v^3+15v^2+6v+1$. In general

$$m_{n+1}=\sum_{k=0}^{n}\frac{(r+k)!}{k!\,(r-k)!}\left(\frac{v}{2}\right)^k. \tag{8.3}$$

The associated IG p.d.f. is

$$f(t;1,v)=\frac{1}{\sqrt{2\pi vt^3}}\exp(-(t-1)^2/2vt), \qquad t\geqq 0. \tag{8.4}$$

We can generate the IG LST from other LSTs using the operators we have considered. First,

$$\hat{f}(s;1,v)=\mathscr{T}^{-1}(\hat{h}_1(s;v/2))=\exp(-s\hat{h}_1(s;v/2)), \tag{8.5}$$

where $\hat{h}_1(s;v/2)$ is the RBM first-moment LST with v as a scale parameter, i.e.

$$\hat{h}_1(s;v/2)=\frac{2}{1+\sqrt{2vs+1}}. \tag{8.6}$$

We can also generate the IG by exponentially damping the stable law with exponent $\frac{1}{2}$; i.e. we damp the transform $\exp(-\sqrt{2vas})$ by replacing s by $s + (2va)^{-1}$, then multiply the entire transform by the constant $\exp(v^{-1})$ to obtain (8.1).

Since the IG distribution has all its inverse moments, the inverse-stationary-lifetime operator $\mathcal{L}^{-1}$ can be applied repeatedly to yield the *generalized inverse Gaussian (GIG) distribution* in Jorgensen (1982) and Embrechts (1983). For instance, $\mathcal{L}^{-1}(\hat{f}(s; v+1, v))$ yields the p.d.f.

$$g(t) = \frac{f(t; v+1, v)}{t}, \qquad t \geqq 0, \tag{8.7}$$

which has moments $m_1 = 1$, $m_2 = 1 + v$, $m_3 = (1+v)^3$ and $m_4 = (1+v)^3(1 + 3v + 3v^2)$. The LT of $g(t)$ is

$$\hat{g}(s) = \frac{\hat{f}(s; v+1, v)}{\hat{p}(s; v, v+1)} \tag{8.8}$$

for $\hat{f}$ in (8.1) and $\hat{p}$ in (8.9) below. The last entry of Table 5 in Section 10 is the GIG transform for the case $v = 1$. The GIG distribution is GGC; see p. 59 of Bondesson (1992).

Now we consider *exponential mixtures of inverse Gaussian distributions* (EMIGs), which can be obtained by applying the power mixture operator $\mathscr{PM}$ in Section 6 to the IG LST $\hat{f}(s; 1, \mu)$ in (8.1) or the exponential mixture operator $\mathscr{EM}$ from Section 7 to the RBM LST $\hat{h}_1(s; \mu/2)$ in (8.6), i.e.

$$\begin{aligned} \hat{p}(s; 1, \mu) &= \mathscr{EM}(\hat{h}_1(s; \mu/2)) = \frac{1}{1 + s\hat{h}_1(s; \mu/2)} \\ &= \mathscr{PM}(\hat{f}(s; 1, \mu)) = \frac{1}{1 - \log(\hat{f}(s; 1, \mu))} \\ &= \frac{\mu}{\mu - 1 + \sqrt{2\mu s + 1}}. \end{aligned} \tag{8.9}$$

We originally became interested in EMIGs in our study of approximations of busy-period distribution; see Abate and Whitt (1988c). In a $GI/G/1$ queue, the busy period can be identified with the first passage to 0 of the workload process starting at an arrival epoch with a random initial level distributed according to a service time. Heyman (1974) suggested approximating the busy-period distribution by approximating the first passage time to 0 from any level in this representation by an IG. When the service-time distribution is exponential, the overall approximation for the busy-period distribution is thus an EMIG.

Proposition 8.1. *The* EMIG *is infinitely divisible.*

Proof. See the second row of Table 5 in Section 10.

The associated EMIG p.d.f. is an exponential mixture of IG p.d.f.'s as follows:

$$\rho(t; 1, \mu) = \int_0^\infty f(t; x, \mu/x)e^{-x}\, dx. \tag{8.10}$$

Its moments satisfy the recurrence

$$m_{n+1} = \sum_{k=0}^{n} \frac{(n+1-k)(n+k)!}{k!} \left(\frac{\mu}{2}\right)^k; \tag{8.11}$$

e.g. $m_1 = 1$, $m_2 = \mu + 2$, $m_3 = 3\mu^2 + 6\mu + 6$ and $m_4 = 15\mu^3 + 30\mu^2 + 36\mu + 24$.

It is possible to derive many interesting relations for EMIGs. We give three below and discuss more in Section 10. First, the stationary-excess operator is easily applied:

$$\mathscr{E}(\hat{\rho}(s; a, \mu)) \equiv \hat{\rho}_e(s; a, \mu) = \hat{h}_1(s; a\mu)\hat{\rho}(s; a, \mu). \tag{8.12}$$

The next two relations are for the p.d.f.'s, namely,

$$\rho(t; a/\mu, \mu) * \rho(t; a/\nu, \nu) = \frac{\nu}{(\nu-\mu)}\rho(t; a/\mu, \mu) - \frac{\mu}{(\nu-\mu)}\rho(t; a/\nu, \nu) \tag{8.13}$$

and

$$\rho_e(t; a, \mu) = \frac{\mu}{(\mu-2)} h_1(t; a\mu) - \frac{2}{(\mu-2)}\rho(t; a, u). \tag{8.14}$$

We conclude this section by giving an explicit expression for the steady-state waiting-time c.d.f. when the service-time p.d.f. is an EMIG. The special case of a $\Gamma_{\frac{1}{2}}$ p.d.f. (an EMIG with $\nu = 1$) was given in (9.21) of Abate and Whitt (1992) without explanation. We provide the explanation here.

In terms of p.d.f.'s, we can write the Pollaczek–Khintchine operator in (1.2) as

$$\mathscr{P}\mathscr{K}(g)(t) = (1-\rho)\delta(t) + \rho w_\rho(t), \tag{8.15}$$

where g is the service-time p.d.f. and δ is a delta function; i.e. $w_\rho(t)$ is the p.d.f. of the conditional steady-state waiting time given that the server is busy upon arrival. (Here and below we use operators applied to p.d.f.'s instead of LSTs with the obvious meaning.)

Proposition 8.2. *Consider an $M/G/1$ queue with* EMIG *service-time p.d.f. $\rho(t; 1, \nu)$. Then the conditional waiting-time p.d.f. is*

$$\begin{aligned} w_\rho(t) &= \rho(t; \nu/\mu_1, \mu_1) * \rho(t; \nu/\mu_2, \mu_2) \\ &= \frac{\mu_2}{(\mu_2-\mu_1)}\rho(t; \nu/\mu_1, \mu_1) - \frac{\mu_1}{\mu_2-\mu_1}\rho(t; \nu/\mu_2, \mu_2), \end{aligned}$$

where $\mu_1(\mu_2) = (1+\nu/2) \pm \sqrt{(1+\nu/2)^2 - 2(1-\rho)\nu}$.

Proof. Do a little algebra, using

$$\hat{w}_\rho(s)=\frac{(1-\rho)\hat{g}_e(s)}{1-\rho\hat{g}_e(s)}=\frac{1-\rho}{\hat{g}_e(s)^{-1}-\rho}=\frac{1}{ar_2(s)^2+br_2(s)+1}$$

$$=\frac{1}{(1+cr_2(s))(1+dr_2(s))},$$

where $r_2(s)$ is as in (8.2) with $a=1/2(1-\rho)\nu$ and $b=a(2+\nu)$. Also use

$$\hat{g}_e(s)=\hat{p}_e(s)=\frac{1}{(1+r_2(s)/2)}\frac{1}{(1+r_2(s)/\nu)},$$

which follows from (8.2) and (8.11).

We remark that Asmussen (1992) has recently obtained a new characterization of the waiting time distribution when the service-time distribution is phase type in the $GI/G/1$ model.

9. Constructing distributions with non-exponential tails

We say that a p.d.f. f has an *exponential tail* if $f(t)\sim\alpha e^{-\eta t}$ as $t\to\infty$, where α and η are positive constants and $f(t)\sim g(t)$ as $t\to\infty$ means that $f(t)/g(t)\to 1$ as $t\to\infty$. As discussed in Abate *et al.* (1994), a p.d.f. may fail to have an exponential tail for two reasons. First, we may have $e^{\eta t}f(t)\to 0$ for all positive η, in which case we say that the p.d.f. has a *long tail* (also called subexponential); second, we may have $e^{\eta t}f(t)\to 0$ for some positive η but still f does not have an exponential tail. Of particular interest to us is the special case in which $f(t)\sim\alpha t^{-\beta}e^{-\eta t}$ as $t\to\infty$ for $\alpha,\beta,\eta>0$; in this case we say the density f has a *semi-exponential tail.*

As in Section 5 of Abate *et al.* (1994), we note that we can transform an LST of a p.d.f. with an exponential tail or with a tail asymptotic to $\alpha t^{\beta}e^{-\eta t}$ for $\beta>0$ into an LST of a p.d.f. with a semi-exponential tail by one or more applications of the inverse-stationary-lifetime operator $\mathscr{L}^{-1}$ provided that the negative moments are finite: given $g(t)\sim\alpha t^{\beta}e^{-\eta t}$ as $t\to\infty$, $\mathscr{L}^{-1}(g)(t)\sim\alpha t^{-(\beta+1)}e^{-\eta t}/m_{-1}(g)$ as $t\to\infty$.

We could also use the unimodal operator $\mathscr{U}$ for the same purpose: given $f(t)\sim\alpha t^{-\beta}e^{-\eta t}$ as $t\to\infty$, $\mathscr{U}(f)(t)\sim\alpha\eta^{-1}t^{-(\beta+1)}e^{-\eta t}$ as $t\to\infty$. It may be surprising, though, that the operator $\mathscr{U}$ need not change the rate of decay of a long-tail density.

Example 9.1. Let $f_2(t)$ be the Pareto density in (1.4) with parameter $r=2$, and let $u_a(t)=1/a$, $0\leqq t\leqq a$, i.e., the uniform density on $[0, a]$. Then, from (2.6), in terms of p.d.f.'s,

$$\mathscr{U}(f_2)(t)=\tfrac{1}{3}f_2(t)+\tfrac{2}{3}\mu_{\frac{1}{2}}(t),\qquad t\geqq 0. \tag{9.1}$$

Example 9.2. Consider the Weibull density $f(t) = (\frac{1}{2}\sqrt{t})\exp(-\sqrt{t})$, with parameter $\frac{1}{2}$. Then

$$\mathcal{U}(f)(t) = 2f(t) - E_1(\sqrt{t}), \qquad t \geqq 0, \tag{9.2}$$

where $E_1(t)$ is the exponential integral p.d.f. in (2.2), while

$$\mathcal{U}^{-1}(f)(t) = (f_e(t) + f(t))/2, \tag{9.3}$$

where $f_e \equiv \mathcal{E}(f)$ is the associated stationary-excess p.d.f. in Table 2. From (9.2) and (9.3), we see that

$$E_1(\sqrt{t}) = \mathcal{U}(f_e)(t), \qquad t \geqq 0. \tag{9.4}$$

Since $m_k(f) = (2k)!$, we have $m_k(E_1) = (2k+1)!/(k+1)$.

Moreover, it is known that the Weibull p.d.f. f here is a $\Gamma_{\frac{1}{2}}$-mixture of exponentials, i.e.

$$1 - F(t) = \mathrm{e}^{-\sqrt{t}} = \int_0^\infty \gamma(x;1)\mathrm{e}^{-t/2x}\,dx, \tag{9.5}$$

so that f is CM and thus ID. From Table 4,

$$\mathcal{U}(\gamma(t;1)) = h_1(t;\tfrac{1}{2}) = 2\gamma(t;1) - \gamma_e(t;1). \tag{9.6}$$

Hence, from (9.2) and (9.4),

$$E_1(\sqrt{t}) = \int_0^\infty \gamma_e(x;1)(2x)^{-1}\mathrm{e}^{-t/2x}\,dx, \tag{9.7}$$

where

$$\gamma_e(t;1) = 2(1 - \Phi(\sqrt{t})), \tag{9.8}$$

with $\Phi(t)$ being the standard (mean 0, variance 1) normal c.d.f. These manipulations were done with p.d.f.'s and c.d.f.'s, because f has the relatively complicated LST

$$\hat{f}(s) = \sqrt{\pi/s}\exp(\tfrac{1}{4}s)(1 - \Phi(1/\sqrt{2s})). \tag{9.9}$$

We remark that it is possible to expand $\hat{f}$ in (9.9) about $s = 0$ using (26.2.12) on p. 932 of Abramowitz and Stegun (1972).

Example 9.3. Consider the PME density $g_2(t)$ in (1.3) with parameter $r = 2$. We find that

$$\mathcal{U}(g_2)(t) = \tfrac{1}{3}g_2(t) + \tfrac{2}{3}(2E_1(2t)), \qquad t \geqq 0, \tag{9.10}$$

and

$$\mathcal{U}^{-1}(g_2)(t) = 3g_2(t) - 2(2e^{-2t}), \qquad t \geqq 0. \tag{9.11}$$

Hence, once again $\mathcal{U}(g_2)$ has essentially the same asymptotic tail behavior as g_2 itself.

Next, as discussed in Section 5 of Abate *et al.* (1994), we can convert p.d.f.'s with a long tail into one with a semi-exponential tail by using the exponential damping operator $\mathcal{D}$. For example, if $f(t) \sim \alpha t^{-\beta}$ as $t \to \infty$, then $\mathcal{D}_\eta(f)(t) \sim \alpha t^{-\beta} e^{-\eta t} / \hat{f}(\eta)$. Conversely, if $g(t) \sim \alpha t^{-\beta} e^{-\eta t}$, then $\mathcal{D}_\eta^{-1}(g)(t) \sim \alpha t^{-\beta} / \hat{g}(-\eta)$.

Example 9.4. Consider the EMIG LST $\hat{\rho}(s; 1, v)$ in (8.9). We know that $\rho(t; 1, v) \sim \alpha t^{-\frac{3}{2}} e^{-\eta t}$ as $t \to \infty$. If we want a density asymptotic to $\alpha' t^{-\frac{5}{2}} e^{-\eta t}$ as $t \to \infty$ for some constant α', then we can apply the unimodal operator $\mathcal{U}$;

$$\mathcal{U}(\rho(t; 1, v)) = v h_1(t; v/2) - (v-1)\mathcal{T}(\rho(t; 1, v)), \tag{9.12}$$

where

$$\mathcal{T}(\hat{\rho}(s; 1, v)) = \frac{1}{s} \log (1 + r_2(s)/v) \tag{9.13}$$

for $r_2(s)$ in (8.2). Thus,

$$\mathcal{U}(\rho(t; 1, v)) \sim \frac{2v}{t} \rho(t; 1, v) \qquad \text{as } t \to \infty. \tag{9.14}$$

In the special case $v = 2$,

$$\hat{\rho}(s; a, 2) = \hat{h}_1(s; a) = \frac{2}{1 + \sqrt{1 + 4as}} \tag{9.15}$$

and

$$\mathcal{U}(h_1(t; 1)) = 2h_1(t; 1) - \mathcal{T}(h_1(t; 1)). \tag{9.16}$$

Then

$$\mathcal{U}(\hat{h}_1(s; 1)) = 2\left(\frac{2}{1 + \sqrt{1 + 4s}} - \frac{1}{s} \log ((\sqrt{1 + 4s} + 1)/2)\right), \tag{9.17}$$

so that

$$\mathcal{U}(h_1(t; 1)) \sim \frac{8e^{-t/4}}{\sqrt{\pi t^5}} \left(1 - \frac{6}{t}\right) \qquad \text{as } t \to \infty \tag{9.18}$$

and $\mathcal{U}(h_1(t; 1))$ has moments

$$m_k = \frac{k!}{(k+1)^2} \binom{2k}{k}; \tag{9.19}$$

e.g. $m_1 = \frac{1}{2}$, $m_2 = \frac{4}{3}$, $m_3 = \frac{30}{4}$ and $m_4 = \frac{336}{5}$.

10. Feller's first Bessel distributions

In this section we consider probability distributions associated with first-passage times in RBM and the $M/M/1$ queue, extending the discussion in Abate and Whitt (1987), (1988a, b, c, d) and Abate *et al.* (1991). In particular, the distributions here arise in the associated unrestricted process, see Section 7 of Abate and Whitt (1988a). Feller (1966), (1971) calls this process a randomized random walk; see pp. 58, 65, 437, 482 of Feller (1971).

We will focus on the LST.

$$\hat{\omega}(s) \equiv \hat{\omega}(s;\mu) = \frac{1}{\mu - 1}(\sqrt{u^2 + 2\mu s} - \sqrt{1 + 2\mu s}) \tag{10.1}$$

for $\mu \geqq 1$, which has p.d.f.

$$\omega(t) = \sqrt{\frac{\mu}{2\pi t^3}} \exp(-t/2\mu)\left(\frac{1 - \exp(-(\mu^2 - 1)t/2\mu)}{\mu - 1}\right). \tag{10.2}$$

Feller (1966) proved that the LST $\hat{\omega}$ in (10.1) is ID. We call this distribution *Feller's first Bessel function distribution,* because $\hat{\omega}(s)^{2r}$ has p.d.f.

$$\omega_{2r}(t) = \left(\frac{\mu + 1}{\mu - 1}\right)^r \frac{r}{t} \exp(-(\mu^2 + 1)t/4\mu) I_r((\mu^2 - 1)t/4\mu), \tag{10.3}$$

where $I_r(t)$ is the Bessel function of order r (Feller (1971), p. 58). Feller focuses on the case without drift, but as he remarks the general case considered here is treated in the same way. Moreover, Feller (1966) gives a direct construction of $\omega_{2r}(t)$ as the marginal p.d.f. of a Lévy process.

We remark that another approach to infinite divisibility of ω is via complete monotonicity. In Abate and Whitt (1988b) it is shown that the busy period p.d.f. and many related p.d.f.'s are completely monotone (a mixture of exponentials). Since CM implies ID, the busy period p.d.f. $b(t)$ must be ID. We can then identify the LST $\hat{\omega}$ via $\hat{\omega}(s)^2 = \hat{b}(as)$ for appropriate scaling. We return to this at the end of this section.

Note that (10.3) is actually consistent with (10.2), because the Bessel function of fractional order simplifies, i.e.

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh z = \frac{e^z}{\sqrt{2\pi z}}(1 - e^{-2z}). \tag{10.4}$$

Feller exhibited the p.d.f. (10.3), but he did not consider the special case of $r = \frac{1}{2}$ in (10.2). The p.d.f. ω in (10.2) was used in Example 5 of Abate *et al.* (1995) to give an example of a service-time distribution with a finite moment generating function for which the $M/G/1$ steady-state waiting-time distribution does *not* have an exponential tail.

We will approach Feller's first Bessel p.d.f. ω via the gamma density with shape parameter $\frac{1}{2}$ $(\Gamma_{\frac{1}{2}})$, i.e.

$$\gamma(t;\mu)=\frac{1}{\sqrt{2\pi\mu t}}e^{-t/2\mu},\qquad t\geqq 0, \tag{10.5}$$

where $\mu\geqq 1$, which has transform $\hat{\gamma}(s;\mu)=(1+2\mu s)^{-\frac{1}{2}}$ and stationary-excess density

$$\mathscr{E}(\gamma)\equiv\gamma_e(t;\mu)=\frac{2}{\mu}(1-\Phi(\sqrt{t/\mu}), \tag{10.6}$$

where Φ is the standard (mean 0, variance 1) normal c.d.f.

For this purpose, we define two new p.d.f.'s,

$$\begin{aligned}\gamma_c(t;\mu)&=\gamma(t;\mu)*\gamma(t;1/\mu),\qquad t\geqq 0,\\&=\tfrac{1}{2}\exp\left(-(\mu^2+1)t/4\mu\right)I_0((\mu^2-1)t/4\mu)\end{aligned} \tag{10.7}$$

and

$$\begin{aligned}\gamma_d(t;\mu)&=\frac{\mu}{(\mu-1)}\gamma(t;\mu)-\frac{\mu^{-1}}{1-\mu^{-1}}\gamma(t;\mu^{-1}),\qquad t\geqq 0,\\&=\frac{\mu}{(\mu-1)}\frac{1}{\sqrt{2\pi\mu t}}(e^{-t/2\mu}-e^{-\mu t/2}),\qquad t\geqq 0.\end{aligned} \tag{10.8}$$

The associated LSTs are

$$\hat{\gamma}_c(s;\mu)=\frac{\mu}{\sqrt{\mu^2+2\mu s}\sqrt{1+2\mu s}} \tag{10.9}$$

and

$$\hat{\gamma}_d(s;\mu)=\frac{\mu}{\mu-1}\left(\frac{1}{\sqrt{1+2\mu s}}-\frac{1}{\sqrt{\mu^2+2\mu s}}\right). \tag{10.10}$$

From (10.1), (10.9) and (10.10), we immediately obtain the following.

Proposition 10.1. $\hat{\gamma}_d(s;\mu)=\hat{\gamma}_c(s;\mu)\hat{\omega}(s)$.

Note that γ_d has finite first negative moment. Hence we can also apply the inverse-stationary-lifetime operator $\mathscr{L}^{-1}$ to $\hat{\gamma}_d$ to get $\hat{\omega}$.

Proposition 10.2. $\hat{\omega}=\mathscr{L}^{-1}(\hat{\gamma}_d)$, *so that* $\omega(t)=t^{-1}\gamma_d(t;\mu)$ *and* $\omega(t)$ *is as given in* (10.2).

Note that $\omega(t;\mu)\to\gamma(t;1)$ as $\mu\to 1$. We will now determine the moments of ω.

Proposition 10.3. *The moments of* $\omega(t;\mu)$ *are*

$$m_{n+1} = \frac{(2n)!}{2^n n!} \sum_{k=-n}^{n} \mu^k; \tag{10.11}$$

e.g. $m_1 = 1$, $m_2 = (\mu + 1 + \mu^{-1})$, $c^2 = \mu + \mu^{-1}$, *and* $m_3 = 3(\mu^2 + \mu + 1 + \mu^{-1} + \mu^{-2})$.

To prove Proposition 10.3, we use the following proposition, which relates ω to the RBM first-moment p.d.f. h_1. It involves elementary algebra. From (4.4) of Abate and Whitt (1987),

$$\begin{aligned} h_1(t;\mu) &= 2\gamma(t;\mu) - \gamma_e(t;\mu) \\ &= 2\gamma(t;\mu) - \frac{2}{\mu}(1 - \Phi(\sqrt{t/\mu})). \end{aligned} \tag{10.12}$$

Proposition 10.4. *The stationary-excess distribution of* ω *is*

$$\mathscr{E}(\omega) \equiv \omega_e(t) = \frac{\mu}{\mu - 1} h_1(t;\mu) - \frac{1}{\mu - 1} h_1(t;\mu^{-1}), \tag{10.13}$$

where $\hat{h}_1$ *is as in* (8.6).

Proof of Proposition 10.3. We start with

$$m_k(\gamma) = \frac{(2k)!}{2^k k!}\mu^k, \tag{10.14}$$

from which we deduce that

$$m_k(h_1) = \frac{k!}{k+1}\binom{2k}{k}\left(\frac{\mu}{2}\right)^k, \tag{10.15}$$

from which (10.11) follows, using (10.13).

We now give two more relations. Since convolution preserves infinite divisibility, the following transform is ID:

$$\hat{\xi}(s) \equiv \hat{\gamma}(s;\mu)\hat{\omega}(s) = \frac{\gamma_d(s;\mu)}{\hat{\gamma}(s;\mu^{-1})} = \frac{1}{\mu - 1}\left(\sqrt{\frac{\mu^2 + 2\mu s}{1 + 2\mu s}} - 1\right). \tag{10.16}$$

The LST $\hat{\xi}$ has the p.d.f.

$$\xi(t) = \frac{(\mu + 1)}{4\mu}\exp(-(\mu^2 + 1)t/4\mu)(I_0((\mu^2 - 1)t/4\mu) + I_1((\mu^2 - 1)t/4\mu)). \tag{10.17}$$

Next, note that

$$\hat{\eta}(s) \equiv \frac{\hat{\omega}_e(s)}{\hat{\omega}(s)} = \frac{\mu}{(\mu+1)}\hat{h}_1(s;\mu) + \frac{1}{(\mu+1)}\hat{h}_1(s;\mu^{-1}). \tag{10.18}$$

Since the associated p.d.f. η is the mixture of two CM p.d.f.'s, η is CM and thus ID.

We now return to the busy-period LST. If we choose measuring units so that the busy period has mean 1 and $m_2 = c^2 + 1 = 2/(1-\rho)$, then

$$\hat{b}(s) = \frac{1}{c^2-1}(c^2 + s - \sqrt{1 + 2c^2 s + s^2}) \tag{10.19}$$

and

$$b(t) = \frac{c^2+1}{\sqrt{c^4-1}} \frac{\exp(-c^2 t)}{t} I_1(\sqrt{c^4-1}\, t), \qquad t \geqq 0. \tag{10.20}$$

Proposition 10.5. *With the scaling in* (10.1) *and* (10.19), $\hat{b}(s) = \hat{\omega}(s/2)^2$ *or*

$$b(t) = 4\omega(2t) * \omega(2t), \qquad t \geqq 0.$$

Proof. Apply (10.1) and (10.10) with $c^2 = (\mu + \mu^{-1})/2$ or $\mu = c^2 + \sqrt{c^4 - 1}$.

We conclude this section by giving examples of distributions in this section related by the ID operator $\mathscr{I}$ in Section 4. They appear in Table 5.

TABLE 5
Examples of LST pairs $(\hat{f}, \mathscr{I}(\hat{f}))$

Original LST $\hat{f}$	Infinitely divisible LST $\mathscr{I}(\hat{f})$
$\frac{\mu}{\mu+1}\left(\frac{1}{1+2\mu s}\right) + \frac{1}{\mu+1}\hat{\gamma}_c(s;\mu)$	$\hat{\gamma}(s;\mu)\hat{\omega}(s)$
$\hat{\gamma}(s;\nu)\hat{\rho}(s;1,\nu)$	$\hat{\rho}(s;1,\nu)$
$\hat{\rho}(s;1,\nu)$	$\frac{\exp(-r_2(s))}{\hat{f}(s)^{\nu-1}}$
$\frac{\mu^2}{\mu^2+1}\left(\frac{1}{1+2\mu s}\right) + \frac{1}{\mu^2+1}\left(\frac{1}{1+2s/\mu}\right)$	$\hat{\gamma}_c(s;\mu)$
$\hat{\gamma}_c(s/2; c^2 + \sqrt{c^4-1})$	$\hat{b}(s)$
$(1+bs)^{-1}$	$(1+bs)^{-1/b}$
$(1+s)^{-2}$	$\hat{\beta}(s) \equiv \exp(-s/(1+s))$
$\mathscr{E}(\hat{\gamma}(s;2)) = \hat{\gamma}(s;2)\hat{h}_1(s;2)$	$\hat{h}_1(s;2)$
$\hat{\gamma}(s;1)$	$\exp(-r_2(s))$
$\hat{h}_1(s;2)$	$\frac{\exp(-r_2(s))}{\hat{h}_1(s;2)}$

Here is where the quantities in Table 5 can be found: γ in (10.5), ρ in (8.9), r_2 in (8.2), γ_c in (10.7), b in (10.19), $\mathscr{E}$ in Table 2, β in Section 11 below, and h_1 in (10.12). To save space, we omit the supporting algebra. The third, fifth and seventh entries are also covered by Examples 6.1.1, 3.2.3, and 9.2.2 of Bondesson (1992). All the entries $\mathscr{I}(\hat{f})$ in Table 5 except entry 7 are GGC because the associated p.d.f.'s f are CM. ($\gamma_c(t; \mu)$ is CM because the Bessel p.d.f. in (10.7) is CM.) Since the Erlang E_2 density is not CM, the seventh transform $\exp(-s/(1+s))$ is not GGC, but it is GCMED.

11. Feller's second Bessel distributions

The second family of Bessel distributions starts with the transform $\hat{\beta}(s) = \exp(-s/(1+s))$ in (3.8) obtained by applying the inverse cumulant-moment-transfer operator $\mathscr{T}^{-1}$ to the exponential LST $(1+s)^{-1}$. Equivalently, we can apply the Poisson-random-sum operator $\mathscr{PS}(\hat{f}) \equiv \exp(-(1-\hat{f}(s))$ to the exponential LST (Feller (1971), p. 438). Either by these means or directly, we see that $\hat{\beta}$ is ID.

More generally, Feller (1971) considers the convolution of β with a gamma p.d.f., i.e., the one-parameter family of LSTs

$$\hat{\beta}_r(s) = (1+s)^{-(r+1)} \exp(-s/(1+s)), \tag{11.1}$$

which has p.d.f.

$$\beta_r(t) = e^{-(t+1)} t^{r/2} I_r(2\sqrt{t}), \qquad t \geqq 0, \tag{11.2}$$

for $r > -1$, where again I_r is the Bessel function of order r. For $r = -1$, β_r reduces to $\hat{\beta}$. For $r = -1$, the c.d.f. has an atom of e^{-1} at 0. The rest of the c.d.f. has a density

$$\beta(t) \equiv \beta_{-1}(t) = e^{-(t+1)} t^{\frac{1}{2}} I_1(2\sqrt{t}), \qquad t \geqq 0. \tag{11.3}$$

To see that formula (11.3) is the analog of (11.2) for $r = -1$, note that $I_k(t) = I_{-k}(t)$ for k integer. These Bessel distributions arise as Poisson sums of gamma distributions (Feller (1971), p. 58). The distribution β arises when a gamma process is directed by the Poisson process (Feller (1971), p. 349). Gaver (1954), p. 147, considers $\hat{\beta}$ as a possible service-time LST in the M/G/1 queue.

We now give the moments and cumulants of β_r. For this purpose, let $L_n^{(r)}(x)$ be the generalized Laguerre polynomials in (22.9.15) on p. 784 of Abramowitz and Stegun (1972). Their role is indicated on p. 194 of Riordan (1968) and (9.39) of Odlyzko (1993).

Proposition 11.1. *The cumulants and moments of* β_r *are*

$$c_k(\beta_r) = (r+1+k)(k-1)! \tag{11.4}$$

and

$$m_k(\beta_r) = k!\, L_k^{(r)}(-1) = \sum_{j=0}^{k} \binom{k+r}{k-j} \frac{k!}{j!}, \tag{11.5}$$

where $L_n^{(r)}(x)$ *are the generalized Laguerre polynomials.*

Proof. The cumulants are easy. From (22.9.15) of Abramowitz and Stegun (1972),

$$\hat{\beta}_r(s) = \sum_{n=0}^{\infty} L_n^{(r)}(-1)(-s)^n.$$

The explicit expression is (22.3.9) on p. 775 of Abramowitz and Stegun (1972).

12. The generalized Pollaczek–Khintchine operator

In Section 4 we noted that there is a one-to-one correspondence between ID LSTs $\hat{f}(s)$ and Lévy stochastic processes $\{X(t): t \geqq 0\}$ for which $X(u+t) - X(u)$ has LST $\hat{f}(s)^t$ for all positive u and t. Any such Lévy process X with non-decreasing sample paths in turn can serve as the input to a stochastic storage system with constant unit release rate; i.e. so that the net input process is $Y(t) = X(t) - t$ and the associated stochastic storage process (starting out empty) is

$$Z(t) = Y(t) - \inf\{Y(u): 0 \leqq u \leqq t\}, \qquad t \geqq 0. \tag{12.1}$$

Assuming that $\boldsymbol{E}X(1) = \rho < 1$, the storage content has a proper steady-state content Z with LST

$$\hat{z}(s) \equiv \boldsymbol{E}\mathrm{e}^{-sZ} = \frac{1-\rho}{1 - s^{-1}\log \hat{f}(s)}; \tag{12.2}$$

see Theorem 4 on p. 78 of Prabhu (1980). We call the steady state LST in (12.2) the *generalized Pollaczek–Khintchine LST.* An interesting special case is when the input is a gamma process, i.e. when $X(t)$ has LST $(1+\mu s)^{-t}$; see p. 72 of Moran (1959) and p. 72 of Prabhu (1980).

The standard $M/G/1$ queue is the special case in which X is a compound Poisson process, so that

$$\log \hat{f}(s) = \lambda(1 - \hat{h}(s))$$

and

$$\frac{\log \hat{f}(s)}{s} = \rho \frac{(1 - \hat{h}(s))}{s m_1(H)} = \rho \mathcal{S}\mathcal{E}(\hat{h})(s). \tag{12.3}$$

Consequently, as noted in Section 1, the standard Pollaczek–Khintchine operator can be represented as $\mathcal{P}\mathcal{K} = \mathcal{G}\mathcal{S} \circ \mathcal{E}$.

In general, using the representation of an ID LST with finite mean in (4.2), we can write

$$\log \hat{f}(s) = \rho \int_0^s \hat{h}(z)\, dz \tag{12.4}$$

for a bona fide LST $\hat{h}$, so that

$$\hat{z}(s) = \frac{1-\rho}{1 - \rho s^{-1} \int_0^s h(z)\, dz}. \tag{12.5}$$

From (12.5), we see that the *generalized Pollaczek–Khintchine operator*, say $\mathscr{GPK}$, can be written as

$$\mathscr{GPK} = \mathscr{GS} \circ \mathscr{U}, \tag{12.6}$$

where $\mathscr{U}$ is the unimodal operator in Section 2.

A simple formula like (12.2) still holds for storage models with more general net input processes $Y(t)$, provided only that Y has *no negative jumps*. Even though $Y(t)$ can assume negative values the transform can be represented as

$$\boldsymbol{E}\mathrm{e}^{-sY(t)} = \mathrm{e}^{-\psi(s)t}, \qquad t \geqq 0, \tag{12.7}$$

and, assuming that $\boldsymbol{E}Y(1) < 0$, the steady-state storage content has LST

$$\boldsymbol{E}\mathrm{e}^{-sZ} = s\psi'(0)/\psi(s); \tag{12.8}$$

see for example Bingham (1975), Theorem 4.2 of Kella and Whitt (1991) and Example 4(a) of Kella and Whitt (1992). As a special case of (12.7), we can have the exponent function

$$\psi(s) = cs + \frac{\sigma^2 s^2}{2} + a \int_0^s h(z)\, dz, \tag{12.9}$$

for an arbitrary LST $\hat{h}$. The first two components in (12.9) correspond to deterministic and Brownian motion contributions to the net input process. The Brownian motion contribution makes (12.8) more general than (12.5).

The exponent ψ in (12.7) has the property that $\psi(0) = 0$, $\psi'(s) < 0$ and $\psi''(s) > 0$ for all real positive s. Hence we can define an inverse exponent $\psi^{-1}(s)$. It is significant that the first-passage time to level $-t$ can be represented as a new Lévy process $\{T(t) : t \geqq 0\}$ with LST

$$\boldsymbol{E}\mathrm{e}^{-sT(t)} = \exp(-\psi^{-1}(s)t), \qquad t \geqq 0. \tag{12.10}$$

In summary, any LST $\hat{h}$ can be a component of the exponent of a Lévy process without negative jumps as in (12.9). Associated with this Lévy process, is the steady-state storage LST in (12.8) and the first-passage-time LSTs in (12.10).

13. Theta distributions

Continuing our focus on first-passage times in Sections 8, 10 and 12, we start here with the first-passage time for standard RBM (zero drift, unit variance) from a reflecting barrier to an absorbing barrier (Feller (1971), p. 343); we call its distribution our *first theta distribution.* If the distance between the barriers in $\sqrt{2}/2$, then the first-passage time LST is simply $\hat{\theta}_1(s) = 1/\cosh\sqrt{s}$; see (132) on p. 233 of Cox and Miller (1965).

We shall consider four related distributions associated with the classical theta functions; see Bellman (1961), Oberhettinger and Badii (1973) and Chapter 16 of Abramowitz and Stegun (1972). The distributions we consider can all be represented as infinite series involving exponential p.d.f.'s. In particular, the p.d.f.'s have the form

$$\theta_i(t) = \sum_{n=1}^{\infty} a_n \mu_n \exp(-\mu_n t), \qquad t \geqq 0. \tag{13.1}$$

The LSTs $\hat{\theta}_i(s)$ and the parameters $\{(a_n, \mu_n)\}$ for these four theta distributions appear in Table 6. Of course, for our purpose, the significant point is that these theta LSTs have relatively simple expressions in terms of the hyperbolic functions (Abramowitz and Stegun (1972), p. 83).

The second and third theta functions are obviously bona fide mixtures of exponential p.d.f.'s and are thus CM. We will establish operator relations showing that all four p.d.f.'s are ID, with θ_2 and θ_3 being unimodal with a mode at 0. Moreover, the operator relations provide probabilistic interpretations of the distributions. The following relations can be established. We omit the proof, which

TABLE 6
The LSTs and parameters in (13.1) of the four theta distributions

LST	a_n	μ_n
$\hat{\theta}_1(s) = \dfrac{1}{\cosh\sqrt{s}}$	$\dfrac{(-1)^{n+1}4}{\pi(2n-1)}$	$\dfrac{\pi^2(2n-1)^2}{4}$
$\hat{\theta}_2(s) = \dfrac{\tanh\sqrt{s}}{\sqrt{s}}$	$\dfrac{8}{\pi^2(2n-1)^2}$	$\dfrac{\pi^2(2n-1)^2}{4}$
$\theta_3(s) = \sqrt{3/s}\coth\sqrt{3s} - s^{-1}$	$\dfrac{6}{\pi^2 n^2}$	$\dfrac{\pi^2 n^2}{3}$
$\theta_4(s) = \dfrac{\sqrt{s}}{\sinh\sqrt{s}}$	$2(-1)^{n+1}$	$\pi^2 n^2$

uses basic properties of hyperbolic functions (Abramowitz and Stegun (1972), p. 83), and the operators here.

Proposition 13.1. *The following relations hold for the four theta LSTs*:

(a) $\hat{\theta}_1(s) = \mathcal{I}(\hat{\theta}_2(s))$,

(b) $\hat{\theta}_4(3s) = \mathcal{I}(\hat{\theta}_3(s))$,

(c) $\hat{\theta}_1(s) = \hat{\theta}_2(s)\hat{\theta}_4(s)$,

(d) $\hat{\theta}_3(s) = \frac{1}{s}\left(\frac{1}{\hat{\theta}_2(3s)} - 1\right) \equiv \mathscr{E}\mathcal{M}^{-1}(\hat{\theta}_2(3s))$,

(e) $\hat{\theta}_2(s) = \mathcal{I} \circ \mathscr{E}(\hat{\theta}_1(s)\hat{\theta}_4(s))$.

(f) $\mathcal{U}^{-1}(\hat{\theta}_2(s)) = (\hat{\theta}_2(s) + \hat{\theta}_1(s)^2)/2$,

(g) $\mathcal{U}^{-1}(\hat{\theta}_3(s)) = (\hat{\theta}_3(s) + \mathscr{E}(\hat{\theta}_4(3s)^2))/2$,

(h) $\mathcal{U}(\hat{\theta}_2(s)) = \mathcal{T}(\hat{\theta}_1(s)) = 2\log(\cosh\sqrt{s})/s$,

(i) $\mathcal{U}(\hat{\theta}_3(s)) = \mathcal{T}(\hat{\theta}_4(3s)) = -2\log(\sqrt{3s}/\sinh\sqrt{3s})/s$,

(j) $\hat{\theta}_3(s) = \mathcal{I} \circ \mathscr{E}(\mathscr{E}(\hat{\theta}_4(3s)^2/\hat{\theta}_3(s))$,

(k) $\mathcal{T}^{-1}(\hat{\theta}_2(s)) = \exp(-\sqrt{s}\tanh\sqrt{s})$,

(l) $\mathcal{T}^{-1}(\hat{\theta}_3(s)) = \exp(-\sqrt{3s}\coth\sqrt{3s} - 1)$.

We now point out how our four theta distributions are related to the theta functions on p. 421 of Oberhettinger and Badii (1973). The theta functions are functions of two variables, denoted by $\theta_i(z \mid t)$. In terms of these theta functions, our theta p.d.f.'s are:

(13.2)
$$\begin{aligned}
\theta_1(t) &= \frac{1}{2}\frac{\partial}{\partial z}\theta_1(z \mid t)\big|_{z=0} \\
\theta_2(t) &= \theta_2(0 \mid t) \\
\theta_3(t) &= \theta_3(0 \mid t/3) - 1 \\
\theta_4(t) &= \frac{\partial}{\partial t}\theta_4(0 \mid t), \qquad t > 0.
\end{aligned}$$

We now give additional properties of our theta distributions. First, $\theta_2(0) = \theta_3(0) = \infty$, while $\theta_1(0) = \theta_4(0) = 0$.

We now consider moments and cumulants. First,

(13.3)
$$m_k(\theta_1) = \frac{k!}{(2k)!}|E_{2k}|,$$

where E_m is the mth Euler number; see (4.5.66) on p. 85 and p. 804 of Abramowitz and Stegun (1972). For example, $|E_{2n}| = 1$, 5, 61 and 1385, while $m_n = \frac{1}{2}, \frac{5}{12}, \frac{61}{120}$ and $\frac{277}{366}$ ($c^2 = \frac{2}{3}$) for $n = 1, 2, 3, 4$. From the Parseval relation (5.8), we see that the negative moments of the first theta distribution are

$$m_{-n}(\theta_1) = \frac{1}{(n-1)!} \int_0^\infty z^{n-1} \hat{\theta}_1(z)\, dz, \tag{13.4}$$

which is finite for all n.

By Proposition 13.1(a) and (4.10), we can calculate the moments of θ_2, they are

$$m_k(\theta_2) = 2c_{k+1}(\theta_1). \tag{13.5}$$

Directly, from (4.5.64) on p. 85 and p. 804 of Abramowitz and Stegun (1972), we have

$$m_k(\theta_2) = \frac{4^{k+1}(4^{k+1} - 1)k!}{(2(k+1))!} |B_{2(k+1)}|, \tag{13.6}$$

where B_{2k} are the Bernoulli numbers; for example the first four moments are $\frac{1}{3}, \frac{4}{15}, \frac{34}{105}$ and $\frac{496}{945}$ ($c^2 = \frac{7}{5}$).

Using Proposition 13.1(c), we see that

$$c_k(\theta_1) = c_k(\theta_2) + c_k(\theta_4) \tag{13.7}$$

From (4.5.67) on p. 85 of Abramowitz and Stegun (1972) we get

$$m_k(\theta_3) = \frac{12^{k+1} |B_{2(k+1)}|\, k!}{(2(k+1))!}, \tag{13.8}$$

where again B_{2k} are the Bernoulli numbers; the first three moments are $\frac{1}{5}, \frac{4}{35}$ and $\frac{18}{175}$ ($c^2 = \frac{13}{7}$).

By (4.5.65) on p. 85 of Abramowitz and Stegun (1972),

$$m_k(\theta_4) = \frac{(4^k - 2)k!}{(2k)!} |B_{2k}| \quad \text{and} \quad c_k(\theta_4) = \frac{4^k\, |B_{2k}|\, k!}{2k(2k)!} \tag{13.9}$$

where B_k are the Bernoulli numbers; the first three moments are $\frac{1}{6}, \frac{7}{180}$ and $\frac{31}{2520}$ ($c^2 = \frac{2}{5}$).

14. Conclusions

In this paper we have introduced several classes of LSTs and operators mapping LSTs into other LSTs. Most of the p.d.f.'s considered in this paper are monotone or unimodal, which is usually what we want in applications. However, non-monotone p.d.f.'s can be constructed from monotone ones by convolution and non-unimodal ones can be constructed from unimodal ones by simple mixtures. New distributions can be created by choosing location and scale parameters in addition to manipulating given parameters.

Most of the LSTs and operators considered here have been considered before in the literature, but not from this perspective. Considering each operator alone seems to involve little more than a restatement of known relations, but considering several different operators together reveals new relations among them and a surprising overall unity. In retrospect, we are left in awe at the realization that much of this story was known to Feller (1971).

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