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A probabilistic generalization of Taylor's theorem

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Abstract: We derive probabilistic generalizations of the fundamental theorem of calculus and Taylor's theorem, obtained by making the argument interval random. The remainder terms are expressed in terms of iterates of the familiar stationary-excess or equilibrium residual-lifetime distribution from the theory of stochastic point processes. The probabilistic generalization of Taylor's theorem can be applied to approximate the mean number of busy servers at any time in an $M_t/G/\infty$ queueing system.

Keywords: Taylor's theorem; fundamental theorem of calculus; stationary-excess distribution; residual lifetime; stochastic point processes; infinite-server queues; nonstationary queues.

1. The result

We present probabilistic generalizations of the fundamental theorem of calculus and Taylor's theorem, obtained by making the argument interval random. For this purpose, let X be a nonnegative random variable with finite mean $E[X]$ and let X_e be a nonnegative random variable with distribution

$$P(X_e \leqslant x) \equiv \frac{\int_0^x P(X \geqslant y)\,dy}{E[X]}, \quad x \geqslant 0, \tag{1}$$

which has kth moment

$$E[X_e^k] = k\int_0^\infty x^{k-1}P(X_e \geqslant x)\,dx = \int_0^\infty \frac{y^k P(X \geqslant y)\,dy}{E[X]} = \frac{E[X^{k+1}]}{(k+1)E[X]}. \tag{2}$$

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The distribution of X_e is called the stationary-excess (stationary forward recurrence-time or equilibrium residual-lifetime) distribution in the context of stochastic point process models; see Daley and Vere-Jones (1988, pp. 53, 71). For example, if the intervals between successive bus arrivals at a bus stop are independent and identically distributed (i.i.d.) according to X, then in the long run the time that a person arriving at the bus stop (independent of the arrival process) must wait for the next bus is distributed according to X_e; see Wolff (1989, pp. 55, 65).

It is evident from (1) that we can regard the stationary-excess distribution as the image of a stationary-excess operator on the space of probability distributions on the interval $[0, \infty]$; if $E[X] = \infty$, then $P(X_e = \infty) = 1$. We will be interested in successive iterates of this operator. For this purpose, let $X_e^{(n)} = (X_e^{(n-1)})_e$ for $n \geqslant 1$ and $X_e^{(0)} = X$ for nonnegative random variables X. For other occurrences of iterates of the stationary-excess operator, see Harkness and Shantaram (1969), Shantaram and Harkness (1972), van Beek and Braat (1973), Whitt (1985), Abate and Whitt (1988) and Eick, Massey and Whitt (1992).

To state our probabilistic generalization of Taylor's theorem, let f be a real-valued function of a real variable. Moreover, let $f^{(n)}$ denote the nth derivative of f with $f^{(0)}=f$.

Theorem. *For any $n \geqslant 1$, suppose that f is n-times differentiable and $f^{(n)}$ is Riemann integrable on the interval $[t, t+x]$ for each $x>0$. If $E[X^n]<\infty$ and $E[\,|f^{(k)}(t+X_e^{(k)})|\,]<\infty$, $0 \leqslant k \leqslant n$, then $E[\,|f(t+X)|\,]<\infty$ and*

$$E[f(t+X)] = \sum_{k=0}^{n-1} E[X^k] \frac{f^{(k)}(t)}{k!} + E\left[f^{(n)}\left(t+X_e^{(n)}\right)\right] \frac{E[X^n]}{n!}.$$

Remarks. (1) Note that for X deterministic the theorem above expresses the fundamental theorem of calculus for $n=1$ and a variant of Taylor's theorem for $n \leqslant 1$; see Rudin (1964, pp. 95, 115). If $P(X=x)=1$, then X_e has a uniform distribution on $[0, x]$, i.e., $P[X_e \leqslant t]=t/x$, $0 \leqslant t \leqslant x$.

(2) The distribution of X_e coincides with the distribution of X when $E[X]<\infty$ if and only if X has an exponential distribution; see Corollary 3.3 and Section 5 of Whitt (1985).

(3) In the first condition of the theorem, it actually suffices for $f^{(k)}$ to be absolutely continuous with respect to Lebesgue measure for $0 \leqslant k \leqslant n-1$, in which case $f^{(n-1)}$ is differentiable almost everywhere and equal to the indefinite integral of $f^{(n)}$; see Royden (1968, pp. 104–107).

(4) Extensions of Dynkin's formula for Markov stochastic processes in Athreya and Kurtz (1973) and references cited there are similar in spirit to our theorem.

2. The proof

Before proving the theorem, we give expression for all moments of the iterated stationary-excess variables. Let $(n)_k = n(n-1)\cdots(n-k+1)$.

Lemma. *For $n \leqslant 1$ and $k \leqslant 1$, if $E[X^n]<\infty$, then*

$$E\left[\left(X_e^{(n)}\right)^k\right] = \frac{n!E[X^{n+k}]}{(n+k)_n E[X^n]}.$$

Proof. Use (2) plus induction on n and k. □

Proof of the theorem. We apply mathematical induction. To treat the case $n=1$, we apply the standard fundamental theorem of the calculus, p.115 of Rudin (1964), and Fubini's theorem (with the moment conditions) to get

$$\begin{aligned} &E[f(t+X)-f(t)] \\ &\quad = E\left[\int_0^X f^{(1)}(t+u)\,\mathrm{d}u\right] \\ &\quad = E\left[\int_0^\infty 1_{\{X \geqslant u\}} f^{(1)}(t+u)\,\mathrm{d}u\right] \\ &\quad = \int_0^\infty P(X \geqslant u) f^{(1)}(t+u)\,\mathrm{d}u \\ &\quad = E\left[f^{(1)}(t+X_e)\right]E[X]. \end{aligned} \tag{3}$$

To carry out the induction, we will apply the result just established for $n=1$ with a new function. For this purpose, for $n \geqslant 1$, define the remainder term

$$R_n f(t, x) \equiv f(t+x) - \sum_{k=0}^{n-1} x^k \frac{f^{(k)}(t)}{k!} \tag{4}$$

and let

$$R_0 f(t, x) \equiv f(t+x).$$

Since

$$\prod_{k=0}^{n-1} E\left[X_e^{(k)}\right] = \frac{E[X^n]}{n!}$$

by the lemma, to prove our theorem, we need to show that

$$E[R_n(t, X)] = E\left[f^{(n)}\left(t+X_e^{(n)}\right)\right] \prod_{k=0}^{n-1} E\left[X_e^{(k)}\right], \tag{5}$$

which we have done for $n=1$. Now note that $R_n f$ has the following properties

$$\frac{\partial}{\partial x} R_n f(t, x) = R_{n-1} f^{(1)}(t, x)$$

and (6)

$$R_n f(t, 0) = 0.$$

Hence, we can apply (6) and the established result for $n=1$ to the new function $f^*(t+x) \equiv R_n f(t, x)$, thinking of t as fixed, to get

$$E[R_n f(t, X)] = E\left[\frac{\partial}{\partial x} R_n f(t, X_e)\right] E[X]$$
$$= E[R_{n-1} f^{(1)}(t, X_e)] E[X]. \tag{7}$$

By induction on n, we obtain (5) from (7). Of course, we must verify the moment conditions in these last two steps, but this is easily done. By (4) and (6),

$$|R_n f(t, x)| \leqslant |f(t+x)| + \sum_{k=0}^{n-1} |x|^k \frac{|f^{(k)}(t)|}{k!}, \tag{8}$$

and

$$\left|\left(\frac{\partial}{\partial x}\right)^k R_n f(t, x)\right|$$
$$\leqslant |f^{(k)}(t+x)| + \sum_{k=0}^{n-k-1} \frac{|x|^k |f^{(k)}(t)|}{k!}, \tag{9}$$

so that the assumed moment conditions together with the Lemma imply that

$$E\left[|R_n^{(k)} f(t, X_e^{(k)})|\right] < \infty \tag{10}$$

for $0 \leqslant k \leqslant n$ and $E[X_e^{(k)}] < \infty$ for $k \leqslant n-1$. □

3. An application: Nonstationary queues

Our interest in this problem arose from considering the $M_t/G/\infty$ queueing model, which has infinitely many servers, a nonhomogeneous Poisson arrival process with deterministic time-dependent arrival-rate function λ, and i.i.d. service times that are independent of the arrival process; see Eick et al. (1992). For appropriate initial conditions, the number of busy servers at time t has a Poisson distribution with mean

$$m(t) = E\left[\int_{t-X}^{t} \lambda(u)\, du\right] \tag{11}$$

when X is a random variable with the service-time distribution. This becomes a special case of our situation here by setting

$$f(t+x) = \int_{t-x}^{t} \lambda(u)\, du, \quad x \geqslant 0. \tag{12}$$

From (11) and (12), our theorem here with $n=1$ yields

$$m(t) = E[\lambda(t - X_e)] E[X], \tag{13}$$

as in Theorem 1.1 of Eick et al. (1992).

Our theorem also yields information about *uniform acceleration approximations* for the $M_t/G/\infty$ queue. These approximations are obtained from a given queueing system by constructing a family of queueing systems indexed by ε with $\varepsilon \downarrow 0$. The system indexed by ε has the same arrival-rate function as the original system except that it is divided by ε, which increases or accelerates the rate as $\varepsilon \downarrow 0$. Similarly, the service *time* is scaled by ε, which accelerates the service *rate*. This technique was applied to the analysis of the $M_t/M_t/1$ queue by Massey (1981, 1985), where an asymptotic expansion for its transition probabilities, mean queue length and variance of queue length was obtained. For the $M_t/G/\infty$ queue, the accelerated mean queue length can be written in closed form as

$$m^\varepsilon(t) = \frac{1}{\varepsilon} E\left[\int_{t-\varepsilon X}^{t} \lambda(s)\, ds\right]. \tag{14}$$

Consequently, we can apply our probabilistic generalization of Taylor's theorem to get an expansion in ε, with an exact expression for the remainder term, as we did in Eick et al. (1992). If λ is an $(n+1)$-times differentiable function, then $m^\varepsilon(t) = m_n^\varepsilon(t) + r_n^\varepsilon(t)$, where

$$m_n^\varepsilon(t) = \sum_{j=0}^{n} (-\varepsilon)^j \frac{\lambda^{(j)}(t)}{(j+1)!} E[X^{j+1}] \tag{15}$$

and

$$r_n^\varepsilon(t) = (-\varepsilon)^{n+1} \frac{E\left[\lambda^{(n+1)}(t - \varepsilon X_e^{(n+2)})\right]}{(n+2)!} \times E[X^{n+2}]. \tag{16}$$

As a special case, the zero-th order term in the uniform acceleration expansion, is usually referred to as the *pointwise stationary approximation;* e.g., see Green and Kolesar (1991) and Whitt (1991). Moreover, we have an exact expression for the error induced by this approximation (for the original, unaccelerated system, or $\varepsilon = 1$),

$$|m(t) - \lambda(t)E[X]| = \tfrac{1}{2}E\left[\lambda^{(1)}(t - X_e)\right]E[X^2]. \tag{17}$$

Moreover, all of these results extend to nonstationary networks of infinite server queues, see Theorem 5.4 of Massey and Whitt (1992).

These various approximations for the time-dependent mean $m(t)$ are not so important as direct approximations, because $m(t)$ is quite readily calculated exactly using (13). We are interested in the approximations primarily to gain insight into corresponding approximations when there are only finitely many servers; then no explicit closed-formulas are available.

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