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SPECTRAL THEORY FOR SKIP-FREE MARKOV CHAINS

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The distribution of upward first passage times in skip-free Markov chains can be expressed solely in terms of the eigenvalues in the spectral representation, without performing a separate calculation to determine the eigenvectors. We provide insight into this result and skip-free Markov chains more generally by showing that part of the spectral theory developed for birth-and-death processes extends to skip-free chains. We show that the eigenvalues and eigenvectors of skip-free chains can be characterized in terms of recursively defined polynomials. Moreover, the Laplace transform of the upward first passage time from 0 to n is the reciprocal of the nth polynomial. This simple relationship holds because the Laplace transforms of the first passage times satisfy the same recursion as the polynomials except for a normalization.

1. INTRODUCTION

This paper focuses on skip-free continuous-time Markov chains (CTMCs) on the nonnegative integers. A CTMC with infinitesimal generator matrix $A \equiv (a_{ij})$ is skip-free to the right (left) if $a_{ij} = 0$ for all $j \geq i + 2$ (all $j \leq i - 2$), i.e., if the only upward (downward) jumps allowed are $+1(-1)$. A CTMC that is simultaneously skip-free to the left and right is a birth-and-death process. The

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order of the state space will play no role, so that it suffices for the CTMC to be skip-free after relabeling the states.

The purpose of this paper is to describe the spectral representation (eigenvalues and eigenvectors) of skip-free CTMCs and the distribution of associated first passage times. It turns out that part of the spectral theory for birth-and-death processes developed in the fundamental papers by Ledermann and Reuter [8] and Karlin and McGregor [4,5] extends to skip-free chains. Unfortunately, in general, it is not possible to represent the probability transition function as a mixture of exponentials, but it is still possible to recursively define and apply polynomials that are intimately connected to the characteristic polynomials of the $n \times n$ infinitesimal generator matrices associated with absorbing and reflecting chains restricted to states $0,1,\ldots,n-1$ obtained from A. Indeed, with the approach of Karlin and McGregor [4] (also see van Doorn [9]), the nth recursively defined polynomial coincides with $\phi_n(x)/\phi_n(0)$, where $\phi_n(x)$ is the characteristic polynomials of the $n \times n$ submatrix $A_{(n)}$ (Theorem 2.1(e)). Thus, the eigenvalues of the submatrix $A_{(n)}$ are precisely the roots of the nth recursively defined polynomial. From the algebraic point of view, the skip-free structure enables us to obtain the characteristic polynomials of the matrices $A_{(n)}$ recursively. Moreover, as Ledermann and Reuter [8] observed for birth-and-death processes, *all* eigenvectors of $A_{(n)}$ can be expressed solely in terms of the eigenvalues of the n submatrices $A_{(k)}$, $1 \le k \le n$. These are elementary algebraic consequences of the skip-free structure, which do not depend on the probabilistic interpretation.

This spectral theory for skip-free CTMCs provides additional insight into recent results about upward first passage times by Keilson [6] and Brown and Shao [2]. Keilson [6, p. 59] showed that the distribution of the upward first passage time from state 0 to state n in a birth-and-death process is a convolution of n distinct exponential distributions. Brown and Shao [2] extended this result to CTMCs that are skip-free to the right with distinct real eigenvalues. (From Ledermann and Reuter [8], we know that all eigenvalues for finite birth-and-death processes are real and distinct. From Kendall [7] and Keilson [6, p. 33], we know that this property can be explained by symmetry and it holds for all time reversible CTMCs.) Moreover, Brown and Shao showed that the eigenvectors in the spectral representation for the distribution of the upward first passage times from 0 to n can be expressed solely in terms of the eigenvalues of $A_{(n)}$, so that no separate calculation of eigenvectors is necessary. We provide additional insight by giving an alternate proof: we directly relate the Laplace transform $\hat{f}_{0n}(s)$ of the upward first passage time from 0 to n to the nth recursively defined polynomial $R_n(x)$. In particular, we show that $\hat{f}_{0n}(s) = 1/R_n(s)$ (Theorem 4.1). This connection is easy to understand because, except for the initial conditions ($R_0(x) = 1$ for all x and $\hat{f}_{nn}(s) = 1$ for all s), the Laplace transforms $\hat{f}_{0n}(s)$ satisfy the same recursion as the polynomials $R_n(x)$, which is determined by the law of motion (the Chapman–Kolmogorov equations). For birth-and-death processes, this reciprocal relation was discovered by Karlin and

McGregor [5]. (E. van Doorn pointed this out to us after we completed this work.)

Brown and Shao's explicit expression for the upward first passage time distribution (Eq. (1.2) of [2]), is tantamount to an inversion of the Laplace transform $\hat{f}_{0n}(s) = 1/R_n(s)$ using partial fractions; see Feller [3, p. 275]. $R_n(s)$ is a polynomial of degree n, so that $\hat{f}_{0n}(s)$ is the transform of the convolution of n exponentials when the n roots of $R_n(s) = 0$ are distinct and real (and necessarily negative).

2. RIGHT EIGENVECTORS FOR CHAINS SKIP-FREE TO THE RIGHT

Consider a CTMC on the nonnegative integers with infinitesimal generator matrix $A \equiv (a_{ij})$. Assume that the chain is *skip-free to the right*, i.e., $a_{ij} = 0$ for all $j \geq i + 2$. In this section, we also assume that $0 < a_{i,i+1} \leq -a_{ii} < \infty$ for all $i > 0$.

Because of the skip-free property, we can recursively define a sequence of polynomials $\{R_i(x): i \geq 0\}$ in x by letting

$$R_0(x) = 1, \qquad xR_i(x) = \sum_{j=0}^{i+1} a_{ij} R_j(x) \qquad \text{for } i \geq 1. \tag{2.1}$$

It is easy to see that $R_i(x)$ is a polynomial of degree i for each i. Let $R(x)$ be the column vector $(R_0(x), R_1(x), \ldots)^t$; in matrix notation Eq. (2.1) becomes

$$xR(x) = AR(x) \tag{2.2}$$

with the initial condition $R_0(x) = 1$. For the special case of birth-and-death process, this sequence of polynomials was defined by Karlin and McGregor [4]. Aside from the boundary condition $R_0(x) = 1$, $x \geq 0$, Eq. (2.2) corresponds to the Kolmogorov backward equations for the probability transition matrix $P(t), P'(t) = AP(t)$; Eq. (2.2) is obtained by taking Laplace transforms and letting $R(x)$ correspond to one column of the matrix of Laplace transforms.

It is apparent from Eq. (2.2) that $R(x)$ is a right eigenvector of the infinite matrix A corresponding to the eigenvalue x for all x. We now want to obtain corresponding results for the $n \times n$ submatrices $A_{(n)}$ associated with transitions among the states $\{0, 1, \ldots, n-1\}$. We think of the corresponding Markov chain as being absorbing in state n because $a_{n-1,n} > 0$, so that $-a_{n-1,n-1} > \sum_{k=0}^{n-2} a_{n-1,k}$; i.e., from state $n-1$ there is always a possibility of absorption with $A_{(n)}$. Indeed, there is possibility of absorption from any other state i for which

$$-a_{ii} > a_{i,i+1} + \sum_{j=0}^{i-1} a_{ij}. \tag{2.3}$$

When $-a_{00} > a_{01}$, we think of the chain as having the possibility of absorption in -1. The matrix $A_{(n)}$ thus is the infinitesimal generator matrix of a purely transient chain; there is eventual absorption from any initial state with probability 1. We treat the case of a reflecting upper barrier in Section 5.

The right eigenvector equation for $A_{(n)}$ can be written as

$$xR_{(n)}(x) = A_{(n)}R_{(n)}(x). \tag{2.4}$$

Note that Eq. (2.4) is nearly the same as the first n rows of Eq. (2.2), but not quite. There is a difference in the last row: the last row in Eq. (2.4) is

$$xR_{n-1}(x) = \sum_{k=0}^{n-1} a_{n-1,k}R_k(x), \tag{2.5}$$

whereas the nth row in Eq. (2.2) is

$$xR_{n-1}(x) = \sum_{k=0}^{n-1} a_{n-1,k}R_k(x) + a_{n-1,n}R_n(x). \tag{2.6}$$

This simple difference enables us to relate the eigenvalues and eigenvectors of the finite matrices $A_{(n)}$ to the polynomials $R_i(x)$. The following result is an easy consequence of the construction above, so we give no further proof. The result is purely algebraic; it does not depend on the probabilistic interpretation of A or the sign of any element. To state the result, recall that the characteristic polynomial $\phi_n(x)$ of $A_{(n)}$ is

$$\phi_n(x) = \det(xI - A_{(n)}) = \prod_{j=1}^{n} (x - \alpha_j^n), \tag{2.7}$$

where I is always the identity matrix of appropriate dimension and α_j^n, $1 \le j \le n$, are the n eigenvalues.

THEOREM 2.1:

(a) *The eigenvalues of $A_{(n)}$ coincide with the roots of $R_n(x) = 0$ for $R_n(x)$ defined by Eqs. (2.1) or (2.2).*

(b) *A right eigenvector for $A_{(n)}$ associated with eigenvalue α_j^n is $R_{(n)}(\alpha_j^n) = (R_0(\alpha_j^n), \ldots, R_{n-1}(\alpha_j^n))^t$.*

(c) *Up to a scalar multiple, there is precisely one right eigenvector associated with each eigenvalue, so that $A_{(n)}$ is similar to a diagonal matrix if and only if the n eigenvalues are distinct.*

(d) *If there are n distinct eigenvalues, $\alpha_1^n, \ldots, \alpha_n^n$, then the probability transition matrix $P_{(n)}(t)$ associated with $A_{(n)}$ can be expressed as*

$$P_{(n)}(t) = R_{(n)}e^{D_{(n)}t}R_{(n)}^{-1}, \tag{2.8}$$

where $D_{(n)}$ is the $n \times n$ diagonal matrix with diagonal elements $\alpha_1^n, \ldots, \alpha_n^n$, and $R_{(n)}$ is the right eigenvector matrix $[R_{(n)}(\alpha_1^n), \ldots, R_{(n)}(\alpha_n^n)]$.

(e) *The polynomials $R_n(x)$ defined by Eq. (2.1) can be expressed in terms of the characteristic polynomial $\phi_n(x)$ in Eq. (2.7) and the eigenvalues by*

$$R_n(x) = \frac{\phi_n(x)}{\phi_n(0)} = \frac{\prod_{j=1}^{n} (x - \alpha_j^n)}{\prod_{j=1}^{n} (-\alpha_j^n)}. \tag{2.9}$$

We can combine parts (b) and (e) of Theorem 2.1 to obtain the following property of the eigenvectors.

COROLLARY 2.1: *The right eigenvector of $A_{(n)}$ associated with eigenvalue α_j^n can be expressed solely in terms of the eigenvalues of $A_{(1)}, A_{(2)}, \ldots, A_{(n-1)}$ and α_j^n. All the eigenvectors of $A_{(n)}$ can be expressed solely in terms of the eigenvalues of $A_{(1)}, A_{(2)}, \ldots, A_{(n)}$.*

Remark 2.1: Corollary 2.1 constitutes a transformation of the data from the entries of $A_{(n)}$ to the eigenvalues of $A_{(j)}$ for $j \le n$, but not necessarily a reduction in data because in general for the skip-free chains under consideration $A_{(n)}$ has $\sum_{k=0}^{n-1} (k+2) - 1 = (n^2 + 3n - 2)/2$ nonzero elements, of which $n - 1$ are redundant when the first $n - 1$ row sums are zero. Thus, when the first $n - 1$ row sums are zero, the number of distinct data inputs is $n(n+1)/2$, which coincides exactly with the number of eigenvalues in $A_{(1)}, A_{(2)}, \ldots, A_{(n)}$. When some of the first $n - 1$ rows have nonzero row sums, Theorem 2.2(e) provides a slight reduction in the data determining $P_{(n)}(t)$. Of course, for birth-and-death processes and other skip-free chains with special structure, the number of eigenvalues of $A_{(k)}$ for $k \le n$ may be *greater* than the number of nonredundant entries in $A_{(n)}$.

Remark 2.2: Following Ledermann and Reuter [8], we can provide an alternate (more difficult) proof of Theorem 2.1 by directly establishing a recursion for the characteristic polynomials $\phi_n(x)$ in Eq. (2.7). Following Ledermann and Reuter [8], consider the case in which all row sums of A are zero. By induction, it can be shown that

$$\phi_{n+1}(x) = (x - a_{nn})\phi_n(x) - \sum_{k=0}^{n-1} a_{n,k}\phi_k(x) \prod_{j=k}^{n-1} a_{j,j+1} \tag{2.10}$$

with $\phi_1(x) = (x - a_{00})$, $\phi_0(x) = 1$, and $\phi_{-1}(x) = 0$. From Eq. (2.10), it is not difficult to see that $\phi_n(x) \Big/ \prod_{j=0}^{n-1} a_{j,j+1}$ satisfies Eq. (2.1), i.e., $R_n(x) =$

$\phi_n(x)\Big/\prod_{j=0}^{n-1} a_{j,j+1}$. We connect this to Eq. (2.9) by noting (from Eq. (2.10) by induction) that

$$\prod_{j=0}^{n-1} a_{j,j+1} = \prod_{j=1}^{n} \alpha_j^n = \phi_n(0), \qquad n \geq 1. \tag{2.11}$$

Note that in Eq. (2.11), we have used the fact that the row sums are zero.

3. LEFT EIGENVECTORS FOR CHAINS SKIP-FREE TO THE LEFT

The analysis in Section 2 is easily modified to treat CTMCs on the nonnegative integers that are skip-free to the left instead of to the right, i.e., for which $a_{ij} = 0$ for all $j \leq i - 2$. Paralleling Section 2, we assume that $a_{i,i-1} > 0$ for all $i \geq 1$.

Once again we can apply the skip-free property to define a sequence of polynomials, but now we define them by

$$L_0(x) = 1 \qquad \text{and} \qquad xL_i(x) = \sum_{j=0}^{\infty} L_j(x) a_{ji} \qquad \text{for } i = 0. \tag{3.1}$$

If $L(x) = (L_0(x), L_1(x), \ldots)$, then in matrix notation, Eq. (3.1) becomes

$$xL(x) = L(x)A \tag{3.2}$$

with the initial condition $L_0(x) = 1$. Note that $L(x)$ in Eq. (3.2) is a left eigenvector of A whereas $R(x)$ in Eq. (2.2) is a right eigenvector. Analogs of the results of Section 2 describing the absorbing chains associated with $A_{(n)}$ and states $0,1,\ldots,n-1$ then follow easily, now working with left eigenvectors $L_{(n)}(x)$.

The special case of birth-and-death processes studied by Ledermann and Reuter [8] and Karlin and McGregor [4,5] arises when the CTMC is skip-free *both* to the left and right. For the case of birth-and-death processes, we can express the left eigenvectors directly in terms of the polynomials $R_i(x)$ evaluated at $x = \alpha_j^n$ and the potentials $\pi_j = (\lambda_0 \lambda_1 \ldots \lambda_{j-1})/(\mu_1 \ldots \mu_j)$ and $\pi_0 = 0$, where $a_{i,i+1} = \lambda_i$ and $a_{i,i-1} = \mu_i$. In particular, $(1, \pi_1 R_1(\alpha_j^n), \ldots, \pi_{n-1} R_{n-1}(\alpha_j^n))$ is a left eigenvector of $A_{(n)}$ associated with eigenvalue α_j^n. This was discovered by Ledermann and Reuter [8] (see Eqs. (1.38) and (1.39) in [8]), but is perhaps most easily understood via the symmetry of πA (see Keilson [6, p. 33]).

We thus can express the transition probabilities associated with $A_{(n)}$ as

$$P_{ij}(t) = \pi_j \sum_{k=1}^{n} e^{-\alpha_k^n t} R_i(\alpha_k^n) R_j(\alpha_k^n) \rho_k^n, \tag{3.3}$$

where

$$\pi_j \sum_{k=1}^{n} R_i(\alpha_k^n) R_j(\alpha_k^n) \rho_k^n = \delta_{ij} \tag{3.4}$$

and

$$\rho_k^n = \frac{1}{\sum_{l=0}^{n-1} \pi_l R_l^2(\alpha_k^n)}, \tag{3.5}$$

(see Eq. (1.48) of [8], p. 90 of [9] and p. 34 of [6]). (Recall that left and right eigenvectors associated with different eigenvalues are necessarily orthogonal.)

Ledermann and Reuter also proved for birth-and-death processes that the eigenvalues of $A_{(n)}$ are real, negative, and distinct. In fact, by virtue of symmetry, the eigenvalues are real, negative, and distinct for any reversible CTMC (see Keilson [6, p. 33]), but the reversible skip-free chains are precisely the birth-and-death processes. Ledermann and Reuter also showed that the eigenvalues of $A_{(n)}$ separate those of $A_{(n-1)}$, i.e.,

$$0 > \alpha_0^n > \alpha_0^{n-1} > \alpha_1^n > \alpha_1^{n-1} > \cdots > \alpha_{n-1}^n > \alpha_{n-1}^{n-1} > \alpha_n^n. \tag{3.6}$$

4. FIRST PASSAGE TIMES AND RIGHT EIGENVECTORS

Let $F(t) = (f_{ij}(t))$ be the densities of the first passage times to state j from state i in a CTMC with infinitesimal generator matrix A and let $F(s) \equiv (\hat{f}_{ij}(s))$ be the Laplace transforms. Of course, a proper density is not defined for $i = j$; then the probability distribution has a unit mass at 0, so that we work with Laplace–Stieltjes transforms, and stipulate that $\hat{f}_{ii}(s) = 1$ for all $s \geq 0$. By conditioning on the first transition, we immediately obtain the relation

$$\hat{f}_{in}(s) = \sum_{\substack{k=0 \\ k \neq l}}^{\infty} \frac{a_{ik} \hat{f}_{kn}(s)}{-a_{ii} + s}, \qquad s > 0 \text{ and } i \neq n \tag{4.1}$$

with $f_{ii}(s) = 1$ for all i and $s > 0$ or, equivalently,

$$s\hat{f}_{in}(s) = a_{in}\hat{f}_{nn}(s) + \sum_{k=0}^{\infty} a_{ik} \hat{f}_{kn}(s), \qquad s > 0 \text{ and } i \neq n. \tag{4.2}$$

For the special case of a chain that is skip-free to the right, Eq. (4.2) has the same form as the first n rows of Eq. (2.2) which determines the polynomials $R_i(x)$, except for the normalizations $R_0(x) = 1$ and $f_{nn}(x) = 1$. Since scalar multiples of solutions to Eqs. (2.2) and (4.2) are still solutions, we can relate Eqs. (2.2) and (4.2) by simply renormalizing, i.e., by redefining $R_0(x) = 1/R_n(x)$ or, equivalently, stipulating that $R_n(x) = 1$. Thus, we have determined the Laplace transforms of the first passage time distribution in CTMCs that are skip-free to the right.

THEOREM 4.1: *For a CTMC that is skip-free to the right, the Laplace transform $\hat{f}_{in}(s)$ of the upward first passage time from i to n can be expressed in terms of the polynomials $R_i(x)$ defined in Eq. (2.2) and thus by the eigenvalues by*

$$\hat{f}_{in}(s) = \frac{R_i(s)}{R_n(s)} = \frac{\phi_n(0)\phi_i(s)}{\mu_n(s)\phi_i(0)} = \frac{\prod_{j=1}^{n}(-\alpha_j^n)\prod_{j=1}^{i}(s-\alpha_j^i)}{\prod_{j=1}^{n}(s-\alpha_j^n)\prod_{j=1}^{i}(-\alpha_j^i)}. \tag{4.3}$$

Remark 4.1: Note that $\hat{f}_{0n}(s) = 1/R_n(s)$, so that the distribution of the upward first passage time from 0 to n can be expressed solely in terms of the n eigenvalues of $A_{(n)}$; we do not need the eigenvalues of $A_{(k)}$ for $k < n$.

Remark 4.2: It is easy to go from $\hat{f}_{0n}(s)$ to $\hat{f}_{in}(s)$ because

$$\hat{f}_{0n}(s) = \hat{f}_{0i}(s)\hat{f}_{in}(s), \qquad 1 \le i \le n-1. \tag{4.4}$$

To go from 0 to n for the first time in a chain that is skip-free to the right, you must pass through the intermediate state i for a first time.

Remark 4.3: When the eigenvalues are real, negative, and distinct, as is the case for birth-and-death processes, $\hat{f}_{0n}(s)$ is the Laplace transform of the convolution of n exponential distributions. For birth-and-death processes, this result was established in Keilson [6, p. 59]. (The analysis here provides some additional insight; see Remark 5.1(b) there.) For skip-free chains, this result coincides with Eq. (1.2) of Brown and Shao [2]. However, note that Eq. (4.3) holds without assuming the eigenvalues are real or distinct.

Remark 4.4: The connection between the first passage time transforms and the polynomials $R(x)$ is not limited to skip-free chains. In general, the system of equations (4.2) coincides with the system of equations $xR(x) = AR(x)$ with $R_0(x) = 1$ but without the nth equation $xR_n(x) = \sum_{k=0}^{\infty} a_{nk}R_k(x)$. If there is a unique solution for $R_n(x)$ without this nth equation, then the first passage time transform $\hat{f}_{in}(s)$ can be expressed as

$$\hat{f}_{in}(s) = R_i(s)/R_n(s) \qquad \text{for } s > 0 \text{ and } i \neq n \tag{4.5}$$

with $\hat{f}_{nn}(s) = 1$. Of course, the skip-free property is needed to get the connection to the eigenvalues in Eq. (2.9).

Example 4.1: To see that the nice connection to the eigenvalues for upward first passage time distributions does not extend beyond skip-free chains, consider a simple stochastically monotone CTMC with state space {0,1,2} and infinitesimal transition rates $a_{02} = -a_{00} = a_{10} = a_{12} = -a_{11}/2 = \lambda$ and $a_{ij} = 0$ otherwise. The first passage time distribution from 0 to 2 is exponential with rate λ, i.e., $\hat{f}_{02}(s) = \lambda/(\lambda + s)$, but

$$A_{(2)} = \begin{pmatrix} -\lambda & 0 \\ \lambda & -2\lambda \end{pmatrix}$$

and $\phi_2(x) = (x+\lambda)(x+2\lambda)$, so that the last part of Eq. (4.3) does not hold (although Eq. (4.5) does hold).

By Theorem 4.1, the recursion (2.1) for $R_i(x)$ provides a recursion for the Laplace transforms. As noted in Keilson [6, p. 61] for the special case of birth-and-death processes, recursions for the moments can be found by differentiating the recursion for the Laplace transforms, but the transforms themselves seem somewhat inaccessible via Eq. (2.1). In fact, however, the transforms can easily be determined because they are reciprocals of polynomials of finite degree. We can use Eq. (2.1) to establish a finite recursion for the *coefficients* of the polynomials.

THEOREM 4.2: *For CTMCs that are skip-free to the right, if $R_i(x)$ is the ith polynomial determined by Eq. (2.1), then $R_i(x) = c_{i0} + c_{i1}x + \cdots + c_{ii}x^i$ for constants c_{ij} for each i, where the coefficients c_{ij} are defined recursively by $c_{00} = 1$, $c_{i+1,0} = -\sum_{j=0}^{i} a_{ij}c_{j0}/a_{i,i+1}$ and*

$$-a_{i,i+1}c_{i+1,m} = \sum_{j=m}^{i} a_{ij}c_{jm} + c_{i,m-1}, \qquad 1 \le m \le i+1.$$

If $a_{00} = -a_{01}$, then $c_{i0} = 1$ for all i.

PROOF: By Eq. (2.1),

$$-x\sum_{m=0}^{i} c_{im}x^m = \sum_{j=0}^{i+1} a_{ij}\sum_{m=0}^{j} c_{jm}x^m = \sum_{m=0}^{i+1} x^m \sum_{j=m}^{i+1} a_{ij}c_{jm},$$

so that we can solve for the coefficients by identifying the coefficients of x^m for each m, i.e.,

$$0 = \sum_{j=0}^{i+1} a_{ij}c_{j0} \qquad \text{and} \qquad -c_{i,m-1} = \sum_{j=m}^{i+1} a_{ij}c_{jm}, \qquad 1 \le m \le i+1. \qquad \blacksquare$$

5. FINITE SKIP-FREE CHAINS WITH A REFLECTING UPPER BARRIER

The finite CTMCs with infinitesimal generator matrices $A_{(n)}$ treated in Sections 2–4 are absorbing from state $n-1$. We now apply the results for these absorbing chains to treat the corresponding chains with a reflecting upper barrier, just as Ledermann and Reuter [8] did for birth-and-death processes.

We only discuss the case of chains that are skip-free to the right; chains that are skip-free to the left can be treated similarly. Let $B_{(n)}$ be obtained from

$A_{(n)}$ by replacing element $a_{n-1,n-1}$ by $a_{n-1,n-1} + a_{n-1,n}$. The interesting case is when

$$a_{n-1,n-1} + a_{n-1,n} = -\sum_{k=0}^{n-2} a_{n-1,k} < 0, \tag{5.1}$$

so that $B_{(n)}$ has a reflecting barrier at $n-1$.

Just as in Section 2, the right eigenvector equation for $B_{(n)}$,

$$xR_{(n)}(x) = B_{(n)}R_{(n)}(x) \tag{5.2}$$

coincides with the first n rows of Eq. (2.2) except in the last row, where the difference is $a_{n-1,n}(R_n(x) - R_{n-1}(x))$. Consequently, we obtain an analog of Theorem 2.1. We only state part of it, and again we give no additional proof. Let β_j^n, $1 \le j \le n$, be the eigenvalues of $B_{(n)}$ and $\psi_n(x)$ its characteristic polynomial.

THEOREM 5.1:

(a) *The eigenvalues of $B_{(n)}$ coincide with the roots of $R_n(x) - R_{n-1}(x) = 0$ for $R_{(n)}(x)$ defined by Eq. (2.1) or (2.2).*

(b) *A right eigenvector associated with eigenvalue β_j^n is $R_{(n)}(\beta_j^n) = (R_0(\beta_j^n), \ldots, R_{n-1}(\beta_j^n))^t$.*

(c) *The characteristic polynomials of $A_{(n)}$ and $B_{(n)}$ are related by $\psi_n(x) = \phi_n(x) - a_{n-1,n}\phi_{n-1}(x)$.*

(d) *If the row sums of $B_{(n)}$ are all zero, then 0 is an eigenvalue of multiplicity one, say β_1^n. If also $\bar{\psi}_{(n}(x) = \psi_n(x)/x$, then*

$$\frac{R_n(x) - R_{n-1}(x)}{x} = \frac{\bar{\psi}_n(x)}{\bar{\psi}_{n(0)}} = \frac{\prod_{j=2}^{n}(x - \beta_j^n)}{\prod_{j=2}^{n}(-\beta_j^n)}. \tag{5.3}$$

To summarize, the roots of $R_n(x) = 0$ provide the eigenvalues of $A_{(n)}$; the roots of $R_n(x) - R_{n-1}(x)$ provide the eigenvalues of $B_{(n)}$; the eigenvalues $A_{(1)}, \ldots, A_{(n)}$ provide the eigenvectors of $A_{(n)}$; and the eigenvalues of $A_{(1)}, \ldots, A_{(n-1)}$ and $B_{(n)}$ provide the eigenvector of $B_{(n)}$. Thus, to obtain the complete spectral representation for $B_{(n)}$, it suffices to know the roots of $R_i(x) = 0$, $1 \le i \le n-1$, and $R_n(x) - R_{n-1}(x) = 0$. Of course, for computation we would exploit Theorem 5.1(b) and only find the roots of one matrix.

6. THE FULL SPECTRAL REPRESENTATION

For the finite-state skip-free cases, the absorbing chains determined by $A_{(n)}$ and the reflecting chains determined by $B_{(n)}$, we have full spectral representations for the probability transition function as in Eq. (2.8) if and only if the eigenvalues are distinct. (Recall Theorem 2.1(c).) This corresponds to a simple mixture of exponentials when the eigenvalues are also real, as with birth-and-death processes. As discussed by Keilson [6], the extra symmetry associated with reversibility yields Eq. (3.3) for birth-and-death processes. For skip-free chains, we know of no useful conditions for the eigenvalues of $A_{(n)}$ and $B_{(n)}$ to be real or distinct.

For the infinite-state skip-free case, we would expect to have a spectral representation of the form

$$P_{ij}(t) = \int_0^\infty e^{-xt} Y_i(x) Z_j(x) d\Psi(x), \tag{6.1}$$

where $Y_0(x) = Z_0(x) = 1$ if all the eigenvalues of $A_{(n)}$ are real and distinct for all n. We can partially characterize this representation as follows.

THEOREM 6.1: *For CTMCs skip-free to the right (left), Eq. (6.1) holds for all i and j if Eq. (6.1) holds for $i = 0$ ($j = 0$), in which case $Y_i(x) = R_i(x)$ defined in Eq. (2.1) ($Z_j(x) = L_j(x)$ defined in Section 3).*

PROOF: Consider the case of skip-free to the right. The skip-free property immediately yields the relation

$$\hat{P}_{0j}(s) = \hat{f}_{0i}(s)\hat{P}_{ij}(s), \qquad 1 \le i \le j. \tag{6.2}$$

for the Laplace transforms. By Theorem 4.1,

$$\hat{P}_{ij}(s) = R_i(s)\hat{P}_{0j}(s), \qquad 1 \le i \le j, \tag{6.3}$$

where $R_i(s)$ is the polynomial of degree i defined recursively by Eq. (2.1). In the time-domain, Eq. (6.3) translates into

$$P_{ij}(t) = R_i(-D)P_{0j}(t), \tag{6.4}$$

where $D \equiv \dfrac{d}{dt}$ is the differential operator. If Eq. (6.1) holds for $i = 0$, then

$$P_{ij}(t) = R_i(-D)\int_0^\infty e^{-xt} Z_j(x) d\Psi(x), \tag{6.5}$$

$$= \int_0^\infty e^{-xt} R_i(x) Z_j(x) d\Psi(x). \qquad \blacksquare$$

Note that Theorem 6.1 provides the form of the full spectral representation given by Eq. (6.1) for birth-and-death processes and its validity from the validity for $P_{00}(t)$ alone. From Theorems 4 and 7 of Kendall [7], we know that a sufficient condition for Eq. (6.1) is reversibility, but the reversible skip-free CTMCs are just the birth-and-death processes. It remains to determine when Eq. (6.1) holds more generally. For further discussions about Eq. (6.1) in a special case, see [1].

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