SUFFICIENT CONDITIONS FOR FUNCTIONAL-LIMIT-THEOREM VERSIONS OF $L = \lambda W$

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Received 27 May 1986
(Revised 23 June 1986)

Abstract

The familiar queueing principle expressed by the formula $L = \lambda W$ can be interpreted as a relation among strong laws of large numbers. In a previous paper, we showed that this principle can be extended to include relations among other classical limit theorems such as central limit theorems and laws of the iterated logarithm. Here we provide sufficient conditions for these limit theorems using regenerative structure.

Keywords

Queueing theory, conservation laws, Little's Law, limit theorems, central limit theorem, law of the iterated logarithm, invariance principle, regenerative processes.

1. Introduction

This paper is a sequel to [6] in which we established functional-central-limit-theorem (FCLT) and functional-law-of-the-iterated-logarithm (FLIL) versions of the fundamental queueing principle $L = \lambda W$ [15]. Here we present sufficient conditions for both kinds of functional limit theorems (FLTs), exploiting regenerative structure.

*Supported by the National Science Foundation under Grant No. ECS-8404809 and by the U.S. Army under Contract No. DAAG29-80-C-0041.

Sufficient conditions for the FLTs are important because the FLT versions of $L = \lambda W$ state, roughly speaking, that the customer-average wait satisfies an FLT if and only if the time-average queue length satisfies an FLT. Consequently, to obtain these FLTs for specific models, one FLT must be established exploiting available probabilistic structure. The sufficient conditions here, based on regenerative structure, are natural for queueing models; e.g., they contain light-traffic limit theorems for GI/G/s queues in [8] and [18] as special cases. In fact, regenerative structure is more appropriate than might be apparent because, as shown by Athreya and Ney [1] and Nummelin [10,11], regenerative structure exists for almost any irreducible Markov chain. (There is an extra minorization condition; see chapter 4 of [11].) For example, regenerative structure exists for the vector-valued Kiefer-Wolfowitz workload process in a stable GI/G/s queue [3,14] and p. 67 of [11]. Of course, FLTs can be established without the independence associated with regenerative structure, but the regenerative structure covers many applications. Our purpose here is to bring the $L = \lambda W$ FLT relations in [6] one step closer to applications.

This paper is organized as follows. We briefly review the $L = \lambda W$ framework in sect. 2 and give sufficient conditions for the FCLTs and FLILs in sects. 3 and 4.

2. The $L = \lambda W$ framework

The standard $L = \lambda W$ framework is a sequence of ordered pairs of random variables $\{(A_k, D_k): k = 1, 2, \ldots\}$, where $0 \leqslant A_k \leqslant A_{k+1}$, $A_k \leqslant D_k < \infty$ for all k and $A_k \to \infty$ as $k \to \infty$ w.p. 1. We interpret A_k and D_k as the arrival and departure epochs of the kth customer. Other processes of interest are defined in terms of $\{(A_k, D_k)\}$. The associated interarrival times are $U_k = A_k - A_{k-1}$, $k > 1$, $U_1 = A_1$ and the waiting times are $W_k = D_k - A_k$. The arrival counting process is $N(t) = \max\{k \geqslant 0 : A_k \leqslant t\}$ and the output counting processes is $O(t) = \max\{k \geqslant 0 : D_k \leqslant t\}$, $t \geqslant 0$. The queue length at time t, $Q(t)$, is the number of k with $A_k \leqslant t \leqslant D_k$. The principle $L = \lambda W$ relates the cumulative processes

$$\left\{ \sum_{j=1}^{k} W_j : k \geqslant 1 \right\} \quad \text{and} \quad \left\{ \int_0^t Q(s)\,\mathrm{d}s : t \geqslant 0 \right\},$$

using the arrival and departure process $\{A_k\}$, $\{D_k\}$, $\{N(t)\}$ and $\{O(t)\}$; see [6] and [15] for further discussion.

3. The functional central limit theorem

Sufficient conditions for the FCLTs in theorems 3 and 4 of [6] can be obtained from regenerative structure. Our argument closely follows Iglehart's [8] treatment of the GI/G/1 queue and the extension to the GI/G/*s* queue in [18], but the results are not limited to those special models. We extend [8] by not using the special structure of the GI/G/1 queue and by treating joint limits. To make the connection to [8] clear, we use much of the same notation.

We work in the general framework of sect. 2, i.e., with the sequence $\{(A_k, D_k), k = 1, 2, \ldots\}$, where $A_0 = D_0 = 0$ without there being a 0^{th} customer. Let $X_k = (U_k, W_k), k \geq 1$; let $\{\alpha_k, k \geq 1\}$ be a sequence of i.i.d. (independent and identically distributed) nonnegative integer-valued random variables with $E(\alpha_k) = m$, $0 < m < \infty$; and let $\beta_k = \alpha_1 + \ldots + \alpha_k$. We assume that $\{\beta_k\}$ provides a sequence of regenerative points for $\{X_k\}$; i.e. the random vectors $V_k = \{\alpha_k, X_{\beta_{k-1}+1}, \ldots, X_{\beta_k}\}$ are i.i.d. (Extra measurability conditions are unnecessary; see [5].) The construction of V_k for the GI/G/1 queue is described in detail in [8]. There, α_k is the number of customers served in the kth busy cycle.

Let $\{L(k), k \geq 1\}$ be the discrete renewal process associated with $\{\alpha_k\}$, i.e.,

$$L(k) = \max\{j \geq 0 : \beta_j \leq k\}, \quad k \geq 1, \tag{3.1}$$

where $\beta_0 = 0$. Now define two functionals of V_k:

$$\xi_k = U_{\beta_{k-1}+1} + \ldots + U_{\beta_k} \quad \text{and} \quad Y_k^{(1)} = W_{\beta_{k-1}+1} + \ldots + W_{\beta_k}, \tag{3.2}$$

so that the vectors $(\alpha_k, \xi_k, Y_k^{(1)})$ are i.i.d.,

$$\left(A_{\beta_k}, \sum_{j=1}^{\beta_k} W_j\right) = \left(\sum_{j=1}^{k} \xi_j, \sum_{j=1}^{k} Y_j^{(1)}\right)$$

and

$$\left(A_k, \sum_{j=1}^{k} W_j\right) = \left(\sum_{j=1}^{L(k)} \xi_j + r_k, \sum_{j=1}^{L(k)} Y_j^{(1)} + s_k\right), \tag{3.3}$$

where r_k and s_k are remainder terms that will be asymptotically negligible in the FCLT. In the GI/G/1 queue, ξ_k is the length of the kth busy cycle and $Y_k^{(1)}$ is the sum of the waiting times during the kth busy cycle. We assume that first two moments of α_k, ξ_k and $Y_k^{(1)}$ are finite. We *define* λ and w by

$$\lambda^{-1} = m^{-1} E(\xi_k) \quad \text{and} \quad w = m^{-1} E(Y_k^{(1)}). \tag{3.4}$$

Of course, w equals the mean of the stationary distribution in the GI/G/1 queue, but here W_k need not be either stationary or converge in distribution as $k \to \infty$. For example, W_k could be periodic.

Let $\bar{\sigma}_1^2 = \mathrm{Var}(\xi_k)$, $\bar{\sigma}_2^2 = \mathrm{Var}(Y_k^{(1)})$, $\bar{\sigma}_3^2 = \mathrm{Var}(\alpha_k)$, $\bar{\sigma}_{12} = \mathrm{Cov}(\xi_k, Y_k^{(1)})$, $\bar{\sigma}_{13} = \mathrm{Cov}(\xi_k, \alpha_k)$ and $\bar{\sigma}_{23} = \mathrm{Cov}(Y_k^{(1)}, \alpha_k)$. By the second moment assumption, these are all finite. Let $\bar{\Sigma}$ be the covariance matrix of $(\xi_k, Y_k^{(1)}, \alpha_k)$, i.e.

$$\bar{\Sigma} = \begin{pmatrix} \bar{\sigma}_1^2 & \bar{\sigma}_{12} & \bar{\sigma}_{13} \\ \bar{\sigma}_{12} & \bar{\sigma}_2^2 & \bar{\sigma}_{23} \\ \bar{\sigma}_{13} & \bar{\sigma}_{23} & \bar{\sigma}_3^2 \end{pmatrix} . \tag{3.5}$$

As in [2], [6] and [19], let $\Rightarrow$ denote weak convergence and let D^k be the k-fold product of the function space $D[0, \infty)$ of right-continuous real-valued functions with left limits. Define the following random functions in D:

$$\begin{aligned}
A_n(t) &= n^{1/2}[A_{[nt]} - \lambda^{-1}nt], \qquad W_n(t) = n^{1/2}\left[\sum_{j=1}^{[nt]} W_j - wnt\right], \\
B_n(t) &= n^{1/2}\left[\sum_{k=1}^{[nt]} \xi_k - m\lambda^{-1}nt\right], \; Y_n(t) = n^{-1/2}\left[\sum_{k=1}^{[nt]} Y_k^{(1)} - mwnt\right], \\
L_n(t) &= n^{-1/2}[L([nt]) - m^{-1}nt], \; S_n(t) = n^{-1/2}\left[\sum_{k=1}^{[nt]} \alpha_k - mnt\right], \qquad (3.6) \\
(SL)_n(t) &= n^{-1/2}\left[\sum_{k=1}^{L([nt])} \alpha_k - nt\right], \; (BL)_n(t) = n^{-1/2}\left[\sum_{k=1}^{L([nt])} \xi_k - \lambda^{-1}nt\right], \\
(YL)_n(t) &= n^{-1/2}\left[\sum_{k=1}^{L([nt])} Y_k^{(1)} - wnt\right], \quad t \geq 0,
\end{aligned}$$

where $[x]$ is the integer part of x.

The following result shows that the conditions of theorems 3 and 4 in [6] are satisfied in this regenerative setting. Expressions are also given for the normalization constants in the FCLTs, in terms of the variances and covariances of $(\xi_k, Y_k^{(1)}, \alpha_k)$. Let $\circ$ be the composition map, defined by $(x \circ y)(t) = x(y(t))$ as in sect. 3 of [19].

THEOREM 1

Under the above regeneration and moment assumptions,

$$(B_n, Y_n, S_n, L_n, (SL)_n, (BL)_n, (YL)_n, A_n, W_n) \Rightarrow (B, Y, S, L, \theta, A, W, A, W) \text{ in } D^9,$$

where (B, Y, S) is three-dimensional Brownian motion with zero drift vector and covariance matrix $\bar{\bar{\Sigma}}$ in (3.5), $e(t) = t$ and $\theta(t) = 0$, $t \geq 0$, and

$$\begin{aligned} L &= m^{-1}(S \circ (m^{-1}e)) \\ A &= B \circ m^{-1}e + \lambda^{-1}mL = (B - \lambda^{-1}S) \circ m^{-1}e \\ W &= Y \circ m^{-1}e + wmL = (Y - wS) \circ m^{-1}e, \end{aligned} \tag{3.7}$$

so that (A, W) is two-dimensional Brownian motion with zero drift and covariance matrix $\bar{\Sigma}$ with elements

$$\begin{aligned} \sigma_1^2 &= m^{-1}(\bar{\sigma}_1^2 + \lambda^{-2}\bar{\sigma}_3^2 - 2\lambda^{-1}\bar{\sigma}_{13}) = m^{-1}\operatorname{Var}(\xi_k - \lambda^{-1}\alpha_k) \\ \sigma_2^2 &= m^{-1}(\bar{\sigma}_2^2 + w^2\bar{\sigma}_3^2 - 2w\bar{\sigma}_{23}) = m^{-1}\operatorname{Var}(Y_k^{(1)} - w\alpha_k) \\ \sigma_{12} &= m^{-1}(\lambda^{-1}w\bar{\sigma}_3^2 + \bar{\sigma}_{12} - w\bar{\sigma}_{13} - \lambda^{-1}\bar{\sigma}_{23}) \\ &= m^{-1}\operatorname{Cov}(\xi_k - \lambda^{-1}\alpha_k, Y_k^{(1)} - w\alpha_k). \end{aligned} \tag{3.8}$$

Proof

The limit $(B_n, Y_n, S_n) \Rightarrow (B, Y, S)$ is Donsker's FCLT in D^3 because $(\xi_k, Y_k^{(1)}, \alpha_k)$ are i.i.d. with finite second moments. (Donsker's theorem as discussed in [2] extends easily to the multivariate setting, e.g. apply theorem 8 of [17].) The limit for L_n is added on by applying the inverse mapping in sect. 7 of [19]. The limits for $(SL)_n$, $(BL)_n$ and $(YL)_n$ are added on by applying composition plus translation (random sums) in theorem 5.1(i) of [19]. The appropriate translation to apply [19] is illustrated in the proof of theorem 3 in [6]. Note that the limit for $(SL)_n$ is $(S \circ m^{-1}e) - mm^{-1}(S \circ m^{-1}e) = \theta$. Finally, the limit for (A_n, W_n) is added by applying theorem 4.1 in [2] and showing that $\rho(A_n, (BL)_n) \Rightarrow 0$ and $\rho(W_n, (YL)_n) \Rightarrow 0$, where ρ is a metric on D inducing uniform convergence on compact subsets. The argument is essentially the same as in [8] except for this last step. Here we do not have the special structure of the GI/G/1 queue to exploit. Instead, we use the established weak convergence to knock out the remainder terms. In particular, note that

$$W_n(t) = (YL)_n(t) + n^{-1/2} \sum_{k = T_n(t)+1}^{[nt]} W_k, \tag{3.9}$$

where

$$T_n(t) = \sum_{k=1}^{L([nt])} \alpha_k, \quad n^{-1/2}[T_n(t) - nt] = (SL)_n(t),$$

$$(SL)_n \Rightarrow \theta \quad \text{and} \quad (YL)_n \Rightarrow W.$$

We show that $\rho(W_n, (YL)_n) \Rightarrow 0$ by showing that the remainder term in (3.9) is asymptotically negligible. Since $W_k \geqslant 0$ for all k,

$$n^{-1/2} \sum_{k=T_n(t)+1}^{[nt]} W_k \leqslant n^{-1/2} Y^{(1)}_{L([nt])+1} \leqslant n^{-1/2} \max_{k \leqslant L([nt])+1} \left\{Y_k^{(1)}\right\}. \qquad (3.10)$$

Since $Y_n \Rightarrow Y$ where $P(Y \in C) = 1$ and L_n obeys an SLLN, the continuous mapping theorem with the maximum jump functional, as in the proof of theorem 4(e) in [6], shows that this remainder term converges to 0 in probability. A similar argument applies to show $\rho(A_n\ (BL)_n) \Rightarrow 0$. □

Remark 3.1. By lemma 1 in sect. 5 of [6], the asymptotic efficiency of the estimator $\beta_k^{-1} \Sigma_{j=1}^{\beta_k} W_j$ is the same as for $(mk)^{-1} \Sigma_{j=1}^{[mk]} W_j$, where $m = E\beta_1$, namely $m^{-1/2} \sigma_2^2$. The regeneration cycles affect the asymptotic efficiency only through the expected number of arriving customers. This remains true for other ratio estimators [4,9] and for arbitrary random times (without the regenerative structure) provided they obey an FWLLN.

4. The functional law of the iterated logarithm

We now obtain sufficient conditions for the FLIL analogs of theorems 3 and 4 of [6] discussed in sect. 6 there. We work in the regenerative setting of sect. 3, which includes the second moment conditions. We also assume that the covariance matrix $\bar{\Sigma}$ in (3.5) is positive definite so that it has an invertible square root Γ, i.e. $\bar{\Sigma} = \Gamma\Gamma^t = \Gamma^t\Gamma$. Let $\|x\|$ be the Euclidean norm in R^k, i.e. $\|x\| = (x_1^2 + \ldots + x_k^2)^{1/2}$ for $x = (x_1, \ldots, x_k)$. We treat vectors as row vectors. Let the compact limit set in D^k associated with Strassen's [16] FLIL for Brownian motion (his theorem 1) or partial sums of i.i.d. random vectors in R^k (Strassen's theorems 2 and 3 for $k = 1$ and corollary 1 to theorem 1 of Philipp [12] for $k > 1$) with covariance matrix I (the identity matrix) be the set K_k of all absolutely continuous functions $x \equiv (x_1, \ldots, x_k)$ in D^k, with $x_i(0) = 0$ for all i and derivatives $\dot{x} \equiv (\dot{x}_1, \ldots, \dot{x}_k)$ satisfying $\int_0^\infty \|\dot{x}(s)\|^2\, ds \leqslant 1$ or, equivalently, $\int_0^t \|\dot{x}(s)\|^2\, ds \leqslant 1$ for all $t > 0$. Note that $x(mt) \in K_k$ if and only if $m^{1/2} x(t) \in K_k$.

Let the random functions be defined as in (3.6) except that the normalization is by $\phi(n) = (2n \log\log n)^{-1/2}$ instead of $n^{-1/2}$, e.g.,

$$W_n(t) = \phi(n)\left[\sum_{j=1}^{[nt]} W_j - wnt\right], \quad t \geq 0. \tag{4.1}$$

As in [6], we write $X_n \sqrt{\to} K$ if w.p. 1 every subsequence of $\{X_n : n \geq 1\}$ has a convergent subsequence and the set of all limit points is K.

THEOREM 2

Under the above regeneration, moment and covariance matrix conditions,

$$(B_n, Y_n, S_n, L_n, (SL)_n, (BL)_n, (YL)_n, A_n, W_n) \sqrt{\to} K_{BYSL\theta AWAW} \text{ in } D^9, \tag{4.2}$$

where

$$K_{BYS} = K_3\Gamma = \{x\Gamma : x \in K_3\} \tag{4.3}$$

and

$$\begin{aligned} K_{BYSL\theta AWAW} = \{(x_1, \ldots, x_9) &: (x_1, x_2, x_3) \in K_{BYS}, \\ x_4 &= -m^{-1}x_3 \circ (m^{-1}e), x_5 = \theta, \\ x_6 &= x_8 = (x_1 - \lambda^{-1}x_3) \circ m^{-1}e, \quad \text{and} \\ x_7 &= x_9 = (x_2 - wx_3) \circ m^{-1}e\}, \end{aligned} \tag{4.4}$$

so that $(A_n, W_n) \sqrt{\to} K_{AW}$, where $K_{AW} = K_{BYS}\Delta$, and

$$\Delta = \begin{pmatrix} m^{-1/2} & 0 \\ 0 & m^{-1/2} \\ -\lambda^{-1}m^{-1/2} & -wm^{-1/2} \end{pmatrix}. \tag{4.5}$$

Proof

Since $\bar{\Sigma}$ has the invertible square root Γ, the random vectors $(\xi_k, Y_k^{(1)}, \alpha_k)\Gamma^{-1}$ are i.i.d. with uncorrelated marginals. We thus can apply theorem 3 of [16] plus corollary 1 to theorem 1 of [12] to conclude that $(B_n, Y_n, S_n) \sqrt{\to} K_{BYS} = K_3\Gamma$, where K_3 is the standard limit set in D^3 associated with uncorrelated marginals defined above. The rest of the proof follows by the continuous-mapping and convergence-together arguments as in the proof of theorem 1. Just as in the proof of theorem 1, we can translate into the framework of [19], but here we use the normalization by $\phi(n)$ instead of $n^{-1/2}$ as in (4.1). □

We now apply theorem 2 to obtain an ordinary LIL. For a direct treatment of ordinary LILs, see [7].

COROLLARY

Under the conditions of theorem 2, $[A_n(1), W_n(1)] \sqrt{\to} K_{AW}$ in R^2, where

$$K_{AW} = \pi_1(K_{AW}) = \{x(1) : x \in K_{AW}\} = \{y\Gamma\Delta : y \in R^3 \text{ and } \|y\| \leqslant 1\}$$

with $\pi_1 : D^2 \to R^2$ defined by $\pi_1(x) = x(1)$.

Proof

The projection map π_1 is continuous. □

We conclude by remarking that the SLLN for $\{(A_n, W_n)\}$ associated with $L = \lambda W$ in [15] is also a corollary to theorem 2; i.e. $n^{-1} A_n \to \lambda^{-1}$ and $n^{-1} \Sigma_{k=1}^n W_k \to w$ w.p. 1 as $n \to \infty$.

References

[1] K.B. Athreya and P. Ney, A new approach to the limit theory of recurrent Markov chains, Trans. Amer. Math. Soc. 245(1978)493.

[2] P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).

[3] F. Charlot, M.A. Ghidouche and M. Hamimi, Irreducibility and recurrence in the sense of Harris for the waiting times in the GI/G/q queue, Z. Wahrscheinlichkeitsth. 43(1978)187, (in French).

[4] M.A. Crane and D.L. Iglehart, Simulating stable stochastic systems, I: General multi-server queues, J. Assoc. Comp. Mach. 21(1974)103.

[5] P.W. Glynn, Some new results in regenerative process theory, Technical Report No. 60, Department of Operations Research, Stanford University (1982).

[6] P.W. Glynn and W. Whitt, A central-limit-theorem version of $L = \lambda W$, Queueing Systems: Theory and Applications 1(1986)191.

[7] P.W. Glynn and W. Whitt, An LIL version of $L = \lambda W$, submitted for publication (1985).

[8] D.L. Iglehart, Functional limit theorems for the queue GI/G/1 in light traffic, Adv. Appl. Prob. 3(1971)269.

[9] D.L. Iglehart, Simulating stable stochastic systems, V: Comparison of ratio estimators, Nav. Res. Log. Qtrly. 22(1975)553.

[10] E. Nummelin, A splitting technique for Harris recurrent Markov chains, Z. Wahrscheinlichkeitsth. 43(1978)309.

[11] E. Nummelin, *General Irreducible Markov Chains and Nonnegative Operators* (Cambridge University Press, Cambridge, England, 1984).

[12] W. Philipp, Almost sure invariance principles for sums of B-valued random variables, in: *Probability in Banach Spaces II*, Lecture Notes in Mathematics 709 (Springer-Verlag, New York, 1979) p. 171.

[13] R.F. Serfozo, Functional limit theorems for stochastic processes based on embedded processes, Adv. Appl. Prob. 7(1975)123.

[14] K. Sigman, A new look at the FIFO GI/G/c queue by use of the random assignment discipline, Department of Operations Research, University of California at Berkeley (1986).

[15] S. Stidham, Jr., A last word on $L = \lambda W$, Oper. Res. 22(1974)417.

[16] V. Strassen, An invariance principle for the law of the iterated logarithm, Z.Wahrscheinlichkeitsth. 3(1964)211.

[17] W. Whitt, Weak convergence of probability measures on the function space $C[0, \infty)$, Ann. Math. Statist. 41(1970)939.

[18] W. Whitt, Embedded renewal processes in the GI/G/s queue, J. Appl. Prob. 9(1972)650.

[19] W. Whitt, Some useful functions for functional limit theorems, Math. Oper. Res. 5(1980)67.