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TRANSIENT BEHAVIOR OF REGULATED BROWNIAN MOTION, II: NON-ZERO INITIAL CONDITIONS

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Abstract

This paper continues an investigation of the time-dependent behavior of regulated or reflecting Brownian motion (RBM). Part I focused on RBM starting at the origin; Part II focuses on RBM starting at a fixed positive state. The first two moments of RBM as functions of time are analyzed by representing them as the difference of two increasing functions, one of which is the moment function starting at the origin studied in Part I. By appropriate normalization, the two monotone components can be converted into cumulative distribution functions that can be analyzed probabilistically, e.g., their moments can be calculated. Simple approximations are then developed by fitting convenient distributions to these moments. Overall, the analysis yields a better understanding of the way RBM and related stochastic flow systems approach steady state.

TIME-DEPENDENT BEHAVIOR; RELAXATION TIMES; DIFFUSION PROCESSES; FIRST-PASSAGE TIME; INVERSE GAUSSIAN DISTRIBUTION; COUPLING; FITTING DISTRIBUTIONS BY MATCHING MOMENTS

7. Introduction and summary

This paper is a sequel to Abate and Whitt (1987a) in which we described the moments of regulated or reflecting Brownian motion (RBM) as functions of time, under the condition that RBM starts at the origin. Here we focus on RBM starting at a positive state x. We continue the numbering from Part I, so that we can conveniently refer to the theorems and equations of Part I without special mention. As in Part I, we restrict attention (without loss of generality) to canonical RBM (with drift coefficient -1 and diffusion coefficient 1), which we simply refer to as RBM. Section 2 shows how to obtain results for the general case from the canonical version.

Let $R(t, x)$ refer to RBM starting at x, which can be defined in terms of canonical unregulated Brownian motion (BM) starting at the origin (again with

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drift -1 and diffusion coefficient 1), denoted as in Section 2 by $B(t; -1, 1, 0)$, by

$$R(t, x) = \max\left\{x + B(t; -1, 1, 0),\ B(t; -1, 1, 0) - \inf_{0 \leqq s \leqq t} B(s; -1, 1, 0)\right\}, \quad t \geqq 0; \tag{7.1}$$

see (2.5) in Part I or pp. 19, 49 of Harrison (1985). We describe the moment functions $m_k(t, x) = E[R(t, x)^k]$, $t \geqq 0$. We are primarily interested in the case $k = 1$, but we also discuss other k to some extent. For $k = 1$ and 2, explicit formulas for $m_k(t, x)$ are given in Theorem 1.1 (Part I) and the asymptotic behavior as $t \to \infty$ is given in Corollary 1.1.2. Here we develop simple approximations and general theorems that expose the essential structure.

In Part I we saw that a relatively simple analysis is possible when $x = 0$ because $m_k(t, 0)$ is *increasing* for all k (Theorem 1.2) and has a *completely monotone derivative* when $k = 1$ and 2 (Theorem 1.7). However, $m_k(t, x)$ is more complicated when $x > 0$. For $x > 0$, $m_1(t, x)$ is always initially decreasing (for t near 0). In Section 8 we prove that $m_1(t, x)$ is decreasing for all t when $x \geqq 1$, but $m_1(t, x)$ is initially decreasing and then increasing for $0 < x < 1$. The general shape of the first-moment function $m_1(t, x)$ for various initial states x is shown in Figure 1.

7.1. *A moment-function decomposition.* Our main approach here is to decompose the moment functions into two parts by writing $m_k(t, x) = m_k(t, 0) + d_k(t, x)$ where

$$d_k(t, x) = m_k(t, x) - m_k(t, 0). \tag{7.2}$$

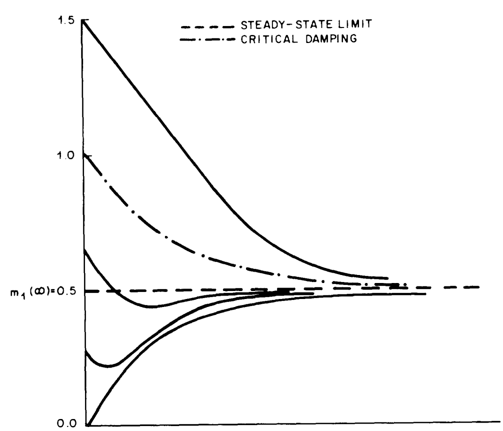

Figure 1. Possible forms of the first-moment function $m_1(t, x) = E(R(t) \mid R(0) = x)$ for different initial states x

For $k = 1$ and 2, we show that $d_k(t, x)$ is decreasing in t for all $x > 0$. Since $m_k(t, 0)$ is increasing in t, $m_k(t, x)$ is the *sum of two monotone functions* for these k.

We can approximate $d_k(t, x)$ for $k = 1$ and 2 the same way we approximated $m_k(t, 0)$; in particular, we can normalize to obtain a c.d.f. (cumulative distribution function) and fit a convenient c.d.f. to the first three moments. Just as with $m_k(t, 0)$, it is significant that we are able to calculate explicitly the first three moments of the c.d.f. related to $d_k(t, x)$. In Part I we worked with the *moment c.d.f.'s* $H_k(t) = m_k(t, 0)/m_k(\infty)$; here we work with the *moment-difference c.d.f.'s* $G_1(t, x) = 1 - d_1(t, x)/x$ and $G_2(t, x) = 1 - d_2(t, x)/x^2$. We primarily focus on the first-moment-difference c.d.f. $G_1(t, x)$, but we also obtain important insights about the second-moment-difference c.d.f. $G_2(t, x)$.

The first-passage times of BM play a fundamental role. Let T_{y0} be the first-passage time of the process $B(t, -1, 1, y)$ from y to 0, which has the inverse Gaussian distribution displayed in (1.5)–(1.7). It turns out that the first-moment c.d.f. and the first-moment-difference c.d.f. can both be simply expressed in terms of the first-passage-time c.d.f. The first-moment c.d.f. $H_1(t)$ is the randomized first-passage time to 0 starting in equilibrium with the exponential stationary distribution, i.e.,

$$H_1(t) = \int_0^\infty 2 \exp(-2y) P(T_{y0} \leqq t)\, dy; \tag{7.3}$$

see Corollary 1.3.1. The difference c.d.f. $G_1(x, t)$ is the average of the first-passage-time c.d.f.'s for initial states in the interval $[0, x]$, i.e.,

$$G_1(t, x) = \frac{1}{x} \int_0^x P(T_{y0} \leqq t)\, dy; \tag{7.4}$$

see Theorem 9.2 and Corollary 11.1.1 here. Our main structural result is the combination of (7.3) and (7.4).

Theorem 7.1. The first moment of RBM can be represented as

$$\begin{aligned} m_1(t, x) &\equiv ER(t, x) = \tfrac{1}{2} H_1(t) + x[1 - G_1(t, x)], \qquad t \geqq 0, \\ &= \int_0^\infty \exp(-2y) P(T_{y0} \leqq t)\, dy + \int_0^x P(T_{y0} > t)\, dy, \qquad t \geqq 0, \end{aligned} \tag{7.5}$$

where T_{y0} is the first-passage time from y to 0 of BM with the inverse Gaussian distribution in (1.5)–(1.7).

It is significant that a similar relationship holds for the queue-length process associated with the $M/M/1$ queue; see Abate and Whitt (1987b), (1988). Then the integrals, the exponential density and the c.d.f. $P(T_{y0} \leqq t)$ in (7.5) are replaced by sums, the geometric distribution and convolutions of the $M/M/1$ busy-period c.d.f., respectively. A generalization of Theorem 7.1 including the $M/M/1$ queue is stated as Theorem 7.3 below.

It is natural to expect that we could easily recognize (7.5) in the explicit formula of Theorem 1.1(a), but the connection seems pretty complicated. However, we can derive Theorem 1.1(a) by substituting (1.6) into (7.5).

We also obtain a remarkably simple characterization of the second-moment function $m_2(t, x)$ in terms of the components of $m_1(t, x)$ in (7.5). The following is established in Corollaries 1.3.2 and 1.5.1 in Part I and Theorems 8.3, 9.3 and 10.1 here. Recall that the stationary-excess (or equilibrium residual-life) c.d.f., say $G_e(t)$, associated with a c.d.f. $G(t)$ on $[0, \infty)$ with mean m_1 is

$$G_e(t) = m_1^{-1}\int_0^t [1 - G(u)]\, du, \qquad t \geqq 0; \tag{7.6}$$

see Whitt (1985). It is well known that the moments m_{ek} of $G_e(t)$ are related to the moments m_k of $G(t)$ by $m_{ek} = m_{k+1}/(k+1)m_1$, so that the moments of $G_e(t)$ are easily obtained from the moments of $G(t)$.

Theorem 7.2. The second moment of RBM can be represented as

$$\begin{aligned} m_2(t, x) \equiv E[R(t, x)^2] &= \tfrac{1}{2}H_2(t) + x^2[1 - G_2(t, x)] \\ &= x^2 + t - 2\int_0^t m_1(u, x)\, du \end{aligned} \tag{7.7}$$

where the c.d.f. $H_2(t)$ is simultaneously the convolution of $H_1(t)$ with itself and the stationary-excess c.d.f. associated with $H_1(t)$, i.e.,

$$H_2(t) = \int_0^t H_1(t-u)\, dH_1(u) = H_{1e}(t) = 2\int_0^t [1 - H_1(u)]\, du \tag{7.8}$$

and $G_2(t, x)$ is the stationary-excess c.d.f. associated with $G_1(t, x)$, i.e., since the mean of $G_1(t, x)$ is $x/2$,

$$G_2(t, x) = G_{1e}(t, x) = (2/x)\int_0^t [1 - G_1(u, x)]\, du. \tag{7.9}$$

A major feature of this work has been the interplay between Laplace transform analysis and probabilistic methods. Discoveries with one approach led to verification and further discoveries with the other approach. As a consequence, the story can be told in two ways, and we present some of both. In particular, we prove Theorems 7.1 and 7.2 both ways.

In Section 9 we apply Laplace transforms to show that the moment-difference c.d.f. $G_1(t, x)$ has the simple representation (7.4). This representation is obtained by exploiting what we believe is a new factorization of the double transform of the density of RBM starting at x. We also establish (7.9) in Section 9. In Section 10 we describe additional properties of the moment-difference c.d.f.'s.

7.2. *A stochastic process decomposition.* In Section 11 we present an alternate derivation of (7.3)–(7.5) using probabilistic methods, in particular, an elementary coupling construction in the spirit of Lindvall (1983), Sonderman (1980) and references there. This coupling construction produces a probabilistic analog of (7.2) for the stochastic processes. Given the stochastic process $\{R(t, x): t \geqq 0\}$ on an underlying probability space, we construct another process $\{R(t, 0): t \geqq 0\}$ on the same probability space having the correct finite-dimensional distributions such that the associated *RBM-difference process*

$$D(t, x) = R(t, x) - R(t, 0), \qquad t \geqq 0, \tag{7.10}$$

has non-increasing sample paths w.p.1 (with probability 1). Theorem 7.1 then is obtained by simply taking expectations. The stochastic process decomposition in (7.10) is of course stronger than the moment-function decomposition in (7.2) because we obtain important conclusions about the entire process. In particular, the process $\{R(t, 0): t \geqq 0\}$ is stochastically increasing in t, while the process $D(t, x)$ has decreasing sample paths w.p.1. Moreover, this decomposition remains valid if the Brownian motion net input process $B(t; -1, 1, 0)$ in (7.1) is replaced by a random walk (partial sums of i.i.d. random variables) or a Lévy process (a process with stationary independent increments; see Prabhu (1980)). It is easy to see that the argument of Section 11 also establishes the following.

Theorem 7.3. Suppose that $R(t, x)$ is defined by (7.1) except that the net input process $B(t, -1, 1, 0)$ is replaced by any Lévy process $Y(t)$ such that $R(t, x)$ converges in distribution to $R(\infty)$ as $t \to \infty$ with $E[R(\infty)] < \infty$. Then (7.10) still holds with $\{R(t, 0): t \geqq 0\}$ stochastically increasing in t and the sample paths of $\{D(t, x): t \geqq 0\}$ decreasing w.p.1. Moreover, when the state space is $[0, \infty)$,

$$\frac{E[R(t, 0)]}{E[R(\infty)]} = \int_0^\infty \frac{P(R(\infty) > y)}{E[R(\infty)]} P(T_{y0} \leqq t)\, dy \tag{7.11}$$

and

$$E[D(t, x)] = \int_0^x P(T_{y0} > t)\, dy \tag{7.12}$$

where T_{y0} is the first-passage time from y to 0 of the net input process $Y(t)$.

When the state space is the non-negative integers, then

$$\frac{E[R(t,0)]}{E[R(\infty)]}=\sum_{k=0}^{\infty}\frac{P(R(\infty)>k)}{E[R(\infty)]}P(T_{k0}\leqq t) \tag{7.13}$$

and

$$E[D(t,k)]=\sum_{j=0}^{k-1}P(T_{j0}>t). \tag{7.14}$$

Note that the term $P(R(\infty)>y)/ER(\infty)$ in (7.11) is just the density of the stationary-excess c.d.f. associated with $R(\infty)$. (Similarly, $P(R(\infty)>k)/E[R(\infty)]$ in (7.13) is the stationary-excess probability mass function when the c.d.f. $P(R(\infty)\leqq x)$ is absolutely continuous with respect to the integer counting measure.) When the net input process $Y(t)$ is BM, then $R(\infty)$ has the exponential distribution, whose stationary-excess distribution is of course also exponential. The BM case is convenient because much is known about the component c.d.f.'s $P(R(\infty)\leqq y)$ and $P(T_{y0}\leqq t)$ in (7.11) and (7.12).

7.3. *Approximations.* In Section 12 we discuss simple approximations for the first-moment-difference c.d.f. $G_1(t,x)$ associated with RBM, which together with Part I yield simple approximations for the entire moment function $m_1(t,x)$. Our goal is not so much to obtain numbers, which can be obtained directly from Theorem 1.1, but to obtain relatively simple formulas that provide insight. These approximations are supported by extensive numerical comparisons with exact values based on Theorem 1.1.

For $x\leqq 3$, we approximate the difference c.d.f. $G_1(t,x)$ by an H_2 c.d.f. (hyperexponential: mixture of two exponentials). In this case, both components of $m_1(t,x)$ are approximated by H_2 c.d.f.'s, so that $m_1(t,x)$ is approximated by a linear combination of four exponentials. In particular, for $x\leqq 3$, we propose the approximations

$$\begin{aligned} m_1(\infty)-m_1(t,x) &\approx \tfrac{1}{2}[1-\tilde{H}_1(t)]-x[1-\tilde{G}_1(t,x)] \\ &\approx \tfrac{1}{2}[p_1\exp(-t/\tau_1)+(1-p_1)\exp(-t/\tau_2)] \\ &\quad -x[p_2\exp(-t/\tau_3)+(1-p_2)\exp(-t/\tau_4)] \end{aligned} \tag{7.15}$$

where $\tau_1\leqq\tau_2$, $\tau_3\leqq\tau_4$, and p_1 and p_2 are probabilities ($\tilde{H}_1(t)$ comes from (1.13): $p_1=0{\cdot}7236$, $\tau_1=0{\cdot}191$ and $\tau_2=1{\cdot}31$). For t sufficiently large, say $t\geqq 1$, the terms with τ_1 and τ_3 in (7.15) become relatively negligible, so that we can replace (7.15) by

$$m_1(\infty)-m_1(t,x)\approx\tfrac{1}{2}(1-p_1)\exp(-t/\tau_2)-x(1-p_2)\exp(-t/\tau_4). \tag{7.16}$$

Moreover, for x sufficiently small, say $0\leqq x\leqq 0{\cdot}5$, the term with τ_4 in (7.16) is relatively small compared to the term with τ_1, so that we obtain *the simple* *exponential approximation*

(7.17) $m_1(\infty) - m_1(t, x) \approx \frac{1}{2}(1 - p_1) \exp(-t/\tau_2) = 0{\cdot}138 \exp(-0{\cdot}764t)$,

which is the simple exponential approximation in (1.3) for the case $x = 0$. In other words, (1.3) is a reasonable approximation for $0 \leqq x \leqq 0{\cdot}5$ as well as $x = 0$ when $t \geqq 1$. Consequently, (1.3) is also reasonable for random initial conditions, provided that the initial distribution is largely concentrated below the steady-state limit $m_1(\infty) = 0{\cdot}5$.

7.4. *Initializations for rapid approach to steady state.* We conclude in Section 13 by investigating initializations to achieve rapid approach to steady state, which is a problem of interest in simulation; see Gafarian et al. (1978), Kelton (1985) and Kelton and Law (1985) for related work on queues. As in Part I, our analysis reveals a gap between what is suggested by asymptotic analysis as $t \to \infty$ in Corollary 1.1.2 (related to relaxation times) and what seems best from a practical point of view. The asymptotic analysis provides a rough idea, but it seems to give good approximations only for times beyond practical interest.

8. Shape of the first two moment functions

In this section we dicuss the shape of the moment functions $m_1(t, x)$ and $m_2(t, x)$ for $x > 0$, e.g., we prove that Figure 1 is correct. We first obtained our characterizations by directly differentiating the formulas for RBM in Theorem 1.1, but below we obtain our characterizations from the fundamental law of motion (the Chapman–Kolmogorov equations and the generator). (These characterizations could also be deduced from corresponding characterizations for the $M/M/1$ queue in van Doorn (1980) and Abate and Whitt (1988). It is rather remarkable that the derivatives have a more elementary form than the moment functions themselves. Recall that $\Phi(t)$ is the c.d.f. and $\phi(t)$ the density of the standard normal distribution. Let $g(y; t, x)$ be the density at y of RBM at time t starting at x.

8.1. *The first-moment function.* Numerical values of $m_1(t, x)$ based on Theorem 1.1 for $x = 0{\cdot}75$, $2{\cdot}0$, $3{\cdot}0$, $6{\cdot}0$ and $12{\cdot}0$ are in Tables 1–5. (The approximate values there will be discussed in Section 12, where all the tables are to be found.)

Theorem 8.1. The first two derivatives of $m_1(t, x)$ with respect to t are

$$m_1'(t, x) \equiv \frac{dm_1(t, x)}{dt} = t^{-\frac{1}{2}}\phi\left(\frac{x - t}{\sqrt{t}}\right) - \Phi\left(\frac{x - t}{\sqrt{t}}\right) \tag{8.1}$$

$$= -1 + 2^{-1}g(0; t, x) = -1 + h_x(t)$$

for $h_x(t)$ in Section (1.8) and

$$m_1''(t, x) \equiv \frac{d^2 m_1(t, x)}{dt^2} = \frac{1}{2t^{\frac{5}{2}}} \phi\left(\frac{x-t}{\sqrt{t}}\right)(x^2 + (x-1)t). \tag{8.2}$$

Proof. Using the fundamental law of motion (generator), we obtain

$$\begin{aligned} m_1'(t, x) &= \frac{\partial}{\partial t} \int_0^\infty y g(y; t, x)\, dy = \int_0^\infty y \left[\frac{\partial}{\partial t} g(y; t, x)\right] dy \\ &= \int_0^\infty y \left[\left(\frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\right) g(y; t, x)\right] dy = \int_0^\infty y\, d\left[g + \frac{1}{2} g'\right] \\ &= \left(y\left[g + \frac{1}{2}g'\right]\right)\Big|_0^\infty - \int_0^\infty \left[g + \frac{1}{2}g'\right] dy = -1 + 2^{-1} g(0; t, x). \end{aligned}$$

By reversibility, $\exp(-2x) g(0; t, x) = g(x; t, 0)$, so that by (1.15) $2^{-1} g(0; t, x) = 2^{-1} \exp(2x) g(x; t, 0) = h_x(t)$.

Corollary 8.1.1. For all $x > 0$, the derivatives at $t = 0$ satisfy $m_1'(0, x) = -1$ and $m_1''(0, x) = 0$, i.e., in the neighborhood of $t = 0$, $m_1(t, x) = x - t + o(t^2)$.

Unlike Corollary 8.1.1, the moment function decomposition in (7.2) does not greatly help understand the behavior of the moment function $m_1(t, x)$ with $x > 0$ for small t. The two component functions of $m_1(t, x)$ both have derivatives of order $t^{-\frac{1}{2}}$ as $t \to 0$. (In particular, for t near 0, $H(t) \approx (8t/\pi)^{\frac{1}{2}}$ from Section 4.3, so that $G_1(t, x) \approx (2t/\pi x^2)^{\frac{1}{2}}$.) For very small t, the decomposition of $m_1'(t, x)$ thus approaches $\infty - \infty$. However, our goal was to develop good approximations for moderately large t, say $t \geqq 1$, after which the RBM mean is within about 15% of the steady-state limit, and we will show that the decomposition helps there.

Corollary 8.1.2. For $x \geqq 1$, $m_1''(t, x) > 0$ for all t, so that the first derivative $m_1'(t, x)$ is increasing for all t. Consequently, $-1 \leqq m_1'(t, x) < 0$ and $m_1(t, x)$ is strictly decreasing and convex for all $t > 0$.

Proof. If $m_1'(\bar{t}, x) \geqq 0$ for some $\bar{t}$, then $m_1'(t, x) > 0$ for all $t > \bar{t}$, which would imply that $m_1(t, x) \to +\infty$ as $t \to \infty$, which can be ruled out.

Corollary 8.1.3. For $0 < x < 1$, there are times $\hat{t}$ and t^* such that $0 < \hat{t} < t^* = x^2/(1-x)$ and $m_1(t, x)$ is decreasing and convex on $(0, \hat{t})$, achieves a minimum at $\hat{t}$, is increasing and convex on $(\hat{t}, t^*)$ and is increasing and concave on (t^*, ∞). The minimum point $\hat{t}$ is the solution to the equation

$$\phi\left(\frac{x-t}{\sqrt{t}}\right) = t^{\frac{1}{2}} \Phi\left(\frac{x-t}{\sqrt{t}}\right). \tag{8.3}$$

Corollary 8.1.3 justifies Figure 1. We can also derive the asymptotic behavior of the derivatives as $t \to \infty$, just as in Corollary 1.1.2.

Corollary 8.1.4. As $t\to\infty$, $m_1'(t,x)\to 0$ and $m_1''(t,x)\to 0$; more precisely,

$$
\begin{aligned}
m_1'(t,x) &\sim -\phi\left(\frac{t-x}{\sqrt{t}}\right)((x-1)t^{-\frac{3}{2}}+(x^2-3x+3)t^{-\frac{5}{2}} \\
&\quad +(x^3-6x^2+15x-15)t^{-\frac{7}{2}}+O(t^{-\frac{9}{2}})) \\
&\sim \begin{cases} (2\pi)^{-\frac{1}{2}}e^x e^{-t/2}(1-x)t^{-\frac{3}{2}}, & x\neq 1 \\ -(2\pi)^{-\frac{1}{2}}ee^{-t/2}t^{-\frac{5}{2}}, & x=1. \end{cases}
\end{aligned}
\tag{8.4}
$$

By differentiating (8.1) or (7.5), we can see how the derivative $m_1'(t,x)$ depends on x. We will apply this result in Section 13.

Theorem 8.2. The derivative of $m_1'(t,x)$ with respect to x is

$$\frac{d^2m_1(t,x)}{dx\,dt}=\frac{-x}{t^{\frac{3}{2}}}\phi\left(\frac{x-t}{\sqrt{t}}\right)=-f(t;x,0)<0,$$

where $f(t;x,0)$ is the inverse Gaussian density in (1.5), so that $m_1'(t,x)$ is decreasing in x for each t.

Corollary 8.2.1. For $0<x<1$, the minimum $\hat{t}(x)$, which is the solution to (8.3), is strictly increasing in x.

Proof. By Theorem 8.2, for $x_1<x_2$, $m_1'(\hat{t}(x_1),x_2)<m_1'(\hat{t}(x_1),x_1)=0$, so that $\hat{t}(x_2)>\hat{t}(x_1)$ by Corollary 8.1.4.

8.2. *The second-moment function.* The story for the second-moment function $m_2(t,x)$ is also remarkably simple. As with Theorem 8.1, this can be obtained from Theorem 1.1 or the fundamental law of motion. (Numerical values of $m_2(t,x)$ based on Theorem 1.1 for $x=0{\cdot}25$, $0{\cdot}90$ and $2{\cdot}0$ are in Tables 6–8.)

Theorem 8.3. The first derivative of $m_2(t,x)$ with respect to t is

$$m_2'(t,x)\equiv\frac{dm_2(t,x)}{dt}=1-2m_1(t,x). \tag{8.5}$$

Proof. As in the proof of Theorem 8.1,

$$
\begin{aligned}
m_2'(t,x)&=\frac{\partial}{\partial t}\int_0^t y^2g(y;t,x)\,dy=\int_0^\infty y\left[\frac{\partial}{\partial t}g(y;t,x)\right]dy \\
&=\int_0^\infty y^2\,d\left[g+\frac{1}{2}g'\right]=\left(y^2\left[g+\frac{1}{2}g'\right]\right)\Big|_0^\infty-\int_0^\infty\left[g+\frac{1}{2}g'\right]2y\,dy \\
&=-2m_1(t,x)-\int_0^\infty yg'\,dy
\end{aligned}
$$

where

$$\int_0^\infty yg'\,dy=(yg)\Big|_0^\infty-\int_0^\infty g\,dy=-1.$$

We can apply Theorem 8.3 together with Theorem 7.1 to obtain an easy proof of Theorem 7.2. We now use Theorem 8.3 to establish the shape of $m_2(t, x)$.

Corollary 8.3.1. For $x \geqq 1$, $m_2'''(t, x) < 0$ for all t, so that $m_2''(t, x)$ and $m_2(t, x)$ are decreasing in t and $m_2'(t, x)$ is increasing in t for all t.

Corollary 8.3.2. For $0 < x < 1$, there are times $\hat{t} < t^* = x^2/(1-x)$ such that $m_2'(t, x)$ is increasing and concave on $(0, \hat{t})$, achieves a maximum at $\hat{t}$, is decreasing and concave on $(\hat{t}, t^*)$ and is decreasing and convex on (t^*, ∞). The maximum point $\hat{t}$ is the solution to (8.3).

Corollary 8.3.3. Since $m_2'(0, x) = 1 - 2x$, $m_2'(t, x) \geqq 0$ for all t when $x \leqq 0{\cdot}5$, so that $m_2(t, x)$ is increasing in t for all t if and only if $x \leqq 0{\cdot}5$. ($m_2(0, 0{\cdot}5) = 0{\cdot}25$.)

Corollary 8.3.4. $m_2(t, x) \leqq m_2(\infty) = 1/2$ for all t if and only if $x \leqq \sqrt{2}/2$.

The interesting case for $m_2(t, x)$ is thus $\sqrt{2}/2 < x < 1$; it is depicted in Figure 2. Numerical values for the case $x = 0{\cdot}9$ appear in Table 7.

Corollary 8.3.5. For $\sqrt{2}/2 < x < 1{\cdot}0$, $m_2(0, x) > m_2(\infty) = 1/2$, so that there is a time $\bar{t}$ such that $\bar{t} < \hat{t}$ with $m_2(t, x)$ decreasing on $(0, \bar{t})$, reaching a minimum at $\bar{t}$ where $m_2(\bar{t}, x) < m_2(\infty) = 0{\cdot}5$, and increasing on $(\bar{t}, \infty)$. The time $\bar{t}$ is the unique solution to $m_1(t, x) = m_1(\infty) = 0{\cdot}5$.

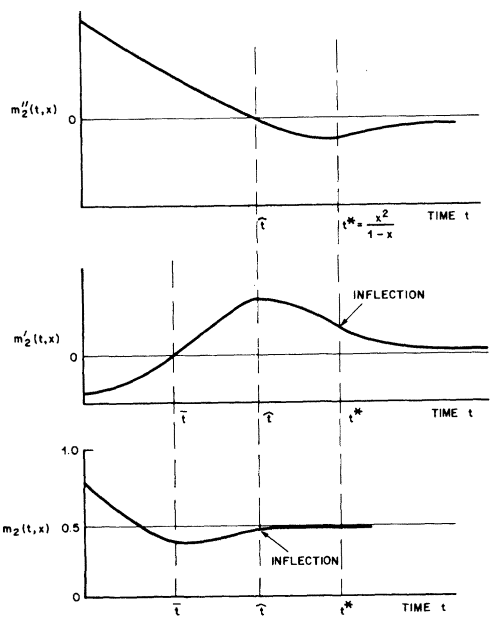

Figure 2. The second-moment function $m_2(t, x)$ and its derivatives for $\sqrt{2}/2 < x < 1{\cdot}0$

Paralleling Theorem 8.2, we now describe the derivatives of $m_1(t, x)$ and $m_2'(t, x)$ with respect to x. The following is an elementary consequence of Theorems 7.1 and 8.3.

Theorem 8.4. The derivative of $m_2'(t, x)$ with respect to x is

$$\frac{dm_2'(t,x)}{dx} \equiv \frac{d^2 m_2(t,x)}{dx\,dt} = \frac{-2\,dm_1(t,x)}{dx} = -2[1 - F(t;x,0)] \tag{8.6}$$

$$= -2\left[\Phi\left(\frac{x-t}{\sqrt{t}}\right) - \exp(2x)\Phi\left(\frac{-t-x}{\sqrt{t}}\right)\right]$$

where $F(t; x, 0)$ is the inverse Gaussian c.d.f. in (1.6).

Paralleling Corollary 8.2.1, we have the following consequence.

Corollary 8.4.1. For $0 \leqq x < 1$, the time $\bar{t} \equiv \bar{t}(x)$ yielding the minimum value of $m_2(t, x)$ is non-decreasing in x with $\bar{t}(0) = \bar{t}(0{\cdot}5) = 0$ and $\bar{t}(x) \to \infty$ as $x \to 1$.

Corollary 8.4.2. For $0 \leqq x < 1$, $\bar{t}(x) < \hat{t}(x)$ where $\bar{t}(x)$ yields the minimum for $m_2(t, x)$ and $\hat{t}(x)$ yields the minimum for $m_1(t, x)$.

9. A transform factorization

Consider (canonical) RBM starting at x and let $g(y; t, x)$ be the density of the state at time t, i.e., the density of (1.1) with initial state x. Let $\hat{g}(y; s, x)$ be the Laplace transform of g with respect to time, defined by

$$\hat{g}(y; s, x) = \int_0^\infty \exp(-st) g(y; t, x)\, dt \tag{9.1}$$

and let $\bar{g}(\sigma; s, x)$ be the double Laplace transform with respect to space and time, defined by

$$\bar{g}(\sigma; s, x) = \int_0^\infty \exp(-\sigma y) \hat{g}(y; s, x)\, dy. \tag{9.2}$$

Gaver (1968) gave an expression for $\bar{g}$ in (2.8) on p. 610 there, namely,

$$\bar{g}(\sigma; s, x) = 2\left(\frac{(\sigma/r_2)\exp(-r_2 x) - \exp(-\sigma x)}{(\sigma + r_1)(\sigma - r_2)}\right), \tag{9.3}$$

where

$$r_1(s) = \Psi(s) - 1, \quad r_2(s) = \Psi(s) + 1 \quad \text{and} \quad \Psi(s) = (1 + 2s)^{\frac{1}{2}}. \tag{9.4}$$

The functions $-r_1(s)$ and $r_2(s)$ are the roots of the equation $r^2 - 2r - 2s = 0$ in the denominator of (2.6) there.

We now show that the double transform $\bar{g}$ in (9.3) can be factored in a useful way.

Theorem 9.1. $\bar{g}(\sigma; s, x) = \bar{g}(\sigma; s, 0)\bar{d}(\sigma, s, x)$, where

$$(9.5)\quad \bar{g}(\sigma; s, 0) = \frac{r_1}{s(\sigma + r_1)} \quad \text{and} \quad \bar{d}(\sigma; s, x) = \frac{\sigma \exp(-xr_2) - r_2 \exp(-x\sigma)}{\sigma - r_2}.$$

Proof. First note that when $x = 0$ Equation (9.3) reduces to

$$(9.6)\qquad \bar{g}(\sigma; s, 0) = 2\left(\frac{(\sigma/r_2) - 1}{(\sigma + r_1)(\sigma - r_2)}\right) = \frac{2}{r_2(\sigma + r_1)}.$$

Since $r_1 r_2 = 2s$, the first terms in (9.5) and (9.6) agree. To complete the proof, factor out $2/r_2(\sigma + r_1) = r_1/s(\sigma + r_1)$ from (9.3).

We can apply Theorem 9.1 to calculate the moments

$$(9.7)\qquad m_k(t, x) = \int_0^\infty y^k g(y; t, x)\, dy.$$

We can obtain the time-transformed moments

$$(9.8)\qquad \hat{m}_k(s, x) = \int_0^\infty \exp(-st) m_k(t, x)\, dt = \int_0^\infty y^k \hat{g}(y; s, x)\, dy$$

by differentiating, i.e.,

$$(9.9)\qquad \begin{aligned} \hat{m}_k(s, x) &= (-1)^k \frac{\partial^k \bar{g}(\sigma; s, x)}{\partial \sigma^k}\bigg|_{\sigma=0} \\ &= (-1)^k \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^j \bar{g}(\sigma; s, 0)}{\partial \sigma^j}\bigg|_{\sigma=0}\right)\left(\frac{\partial^{k-j} \bar{d}(\sigma; s, x)}{\partial \sigma^{k-j}}\bigg|_{\sigma=0}\right). \end{aligned}$$

For the special case $k = 1$,

$$(9.10)\qquad \hat{m}_1(s, x) = \hat{m}_1(s, 0)\bar{d}(0; s, x) + \hat{d}(s, x)\bar{g}(0; s, 0)$$

where

$$(9.11)\qquad \hat{d}(s, x) = \frac{-\partial \bar{d}(\sigma; s, x)}{\partial \sigma}\bigg|_{\sigma=0} = (\exp(-xr_2) + xr_2 - 1)/r_2,$$

$$(9.12)\qquad \bar{d}(0; s, x) = 1 \quad \text{and} \quad \bar{g}(0; s, 0) = s^{-1}.$$

It is significant that we can invert both parts of (9.10). the first part is treated in Part I; now we treat the second part.

Theorem 9.2.

$$\hat{d}(s, x)\bar{g}(0; x, 0) = \int_0^\infty \exp(-st) d_1(t, x)\, dt,$$

where

$$d_1(t, x) = \int_0^x [1 - F(t; y, 0)]\, dy$$

with $F(t; y, 0) = P(T_{y0} \geqq t)$ being the c.d.f. of the inverse Gaussian distribution in (1.6).

Proof. To see that the transform of the integral agrees with (9.11) and (9.12), note that

$$\begin{aligned}\int_0^\infty \exp(-st)\int_0^x [1 - F(t; y, 0)]\, dy\, dt &= \int_0^\infty s^{-1}[1 - \hat{f}(s; y, 0)]\, dy \\ &= s^{-1}\int_0^x [1 - \exp(-yr_2)]\, dy \quad \text{by (1.7)} \\ &= (\exp(-xr_2) + xr_2 - 1)/sr_2,\end{aligned}$$

which is the product of (9.11) and $\bar{g}(0; s, 0) = s^{-1}$ in (9.12).

Applying Theorem 1.3 as well as Theorem 9.2, we have established Theorem 7.1. Since $F(t; y, 0)$ in Theorem 9.2 is a c.d.f. for each y, it is non-decreasing in t.

We now use transforms to establish the corresponding result for the second-moment difference c.d.f. in (7.9). Let $\hat{d}_2(s, x)$, $\hat{G}_1^c(s, x)$ and $\hat{G}_2^c(s, x)$ be the Laplace transforms of $d_2(t, x)$ in (7.2), $G_1^c(t, x) = 1 - G_1(t, x)$ in (7.4) and $G_2^c(t, x) = 1 - G_2(t, x)$ in (7.9), all with respect to time as in (9.1). Let $g_1(t, x)$ and $g_2(t, x)$ be the densities of $G_1(t, x)$ and $G_2(t, x)$ and let $\hat{g}_1(s, x)$ and $\hat{g}_2(s, x)$ be the associated Laplace transforms.

Theorem 9.3. $\hat{d}_2(s, x) = x^2\hat{G}_2^c(s, x) = x^2(1 - \hat{g}_2(s, x))/s$, where

$$\hat{g}_2(s, x) = (2/sx)[1 - \hat{g}_1(s, x)] = 2\hat{d}_1(s, x)/sx^2. \tag{9.13}$$

Proof. From (9.9), with (9.11) and (9.12) we find that

$$\begin{aligned}\hat{m}_2(s, x) - \hat{m}_2(s, 0) &= \frac{2\hat{d}(s, x)}{r_1 s} + \frac{x^2}{s}\left[1 - \frac{2\hat{d}(s, x)}{x^2 r_2}\right] \\ &= \frac{x^2}{s}\left[1 - \frac{2\hat{d}(s, x)}{x^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],\end{aligned}$$

but $r_2^{-1} - r_1^{-1} = s^{-1}$.

10. Properties of the difference c.d.f.'s

Since the first-moment-difference c.d.f. $G_1(t, x)$ in (7.4) is the average of inverse Gaussian c.d.f.'s, many properties are easily deduced. First, since the first three moments of $F(t; y, 0)$ are y, $y + y^2$ and $3y + 3y^2 + y^3$, respectively (see the discussion before Corollary 1.3.4), we can easily calculate the moments of $G_1(x, t)$. These moments can also be calculated from the transform expansion

$$\text{(10.1)}\qquad \frac{1-\exp(-xr_2)}{xr_2} \approx 1 - m_{x1}s + m_{x2}\frac{s^2}{2} - m_{x3}\frac{s^3}{6} + m_{x4}\frac{s^4}{24} + O(s^5).$$

Theorem 10.1. The first four moments of the difference c.d.f. $G_1(t, x)$ in (7.4) are

$$\text{(10.2)}\qquad \begin{aligned} m_{x1} &= \frac{x}{2}, \qquad m_{x2} = \frac{x}{2} + \frac{x^2}{3} = \frac{x}{6}(2x+3) \\ m_{x3} &= \frac{3x}{2} + x^2 + \frac{x^3}{4} = \frac{x}{4}(x^2+4x+6) \\ m_{x4} &= \frac{x}{10}(2x^3 + 15x^2 + 50x + 75). \end{aligned}$$

Corollary 10.1.1. The squared coefficient of variation of $G_1(t, x)$ is

$$\text{(10.3)}\qquad c_x^2 \equiv (m_{x2} - m_{x1}^2)/m_{x1}^2 = (6+x)/3x = \frac{1}{3} + \frac{2}{x}$$

and

$$\text{(10.4)}\qquad \frac{m_{x3}m_{x1}}{m_{x2}^2} = \frac{9(x^2+4x+6)}{2(2x+3)^2} = \frac{3}{2}\left[1 + \frac{9-x^2}{(2x+3)^2}\right]$$

so that an H_2 fit to three moments is possible if and only if $x \leqq 3$. (See Section 3.1 of Whitt (1982).)

The first three moments of $G_1(t, x)$ match an exponential with mean 3/2 exactly for $x = 3$, but when $x = 3$, $m_4 = 124{\cdot}2 < 121{\cdot}5 = 24(3/2)^4$, the fourth moment of an exponential with mean 3/2. This is consistent with numerical evidence (plotting $G_1^c(t, x) \equiv 1 - G_1(t, x)$ on log paper) indicating that $G_1^c(t, x)$ is log-convex for $x \leqq 3$. We formalize our numerical experience in several conjectures.

Conjecture 10.1. For $x \leqq 3$, the first-moment difference c.d.f. $G_1(t, x)$ is DFR; i.e., the complementary c.d.f. $G_1^c(t, x)$ is log-convex; p. 74 of Keilson (1979).

From Theorem 9.3 and Proposition 5.8B of Keilson (1979), we have the following consequence of Conjecture 10.1.

Conjecture 10.2 (corollary to Conjecture 10.1). The second-moment-difference complementary c.d.f. $G_2^c(t, x)$ is log-convex for $x \leqq 3$.

Paralleling Corollary 1.5.1, we also have the following.

Conjecture 10.3 (corollary to Conjecture 10.1). The second-moment-difference c.d.f. $G_2(t, x)$ is stochastically greater than the first-moment-difference c.d.f. $G_1(t, x)$ in the likelihood-ratio ordering for $x \leqq 3$.

We can combine Theorems 9.3 and 10.1 to obtain the first three moments of $G_2(t, x)$. The kth moment of $G_2(t, x)$ is just $m_{x(k+1)}/(k+1)m_{x1}$ for m_{xk} in Theorem 10.1; e.g., the mean of $G_2(t, x)$ is $(2x+3)/6$ and the squared coefficient of variation of $G_2(t, x)$ is $(4/3)(m_{x3}m_{x1}/m_{x2}^2) - 1 = 1 + (18 - 2x^2)/(2x+3)^2$. As with the first-moment-difference c.d.f., $c^2 \geqq 1$ for $x \leqq 3$ and $c^2 \leqq 1$ for $x \geqq 3$. Furthermore, an H_2 fit to three moments of $G_2(t, x)$ is possible if and only if $x \leqq 3$.

Since $F(t; y, 0)$ is stochastically increasing in y, i.e., $F(t; y_1, 0) \geqq F(t; y_2, 0)$ for all $t > 0$ when $y_1 < y_2$, we can draw a corresponding conclusion about $G_1(t, x)$.

Theorem 10.2. The difference c.d.f.'s $G_1(t, x)$ are stochastically increasing in x: $G_1(t, x_1) \geqq G_1(t, x_2)$ for all t when $x_1 < x_2$.

Proof. By the established order,

$$F(t; x_1, 0) \geqq \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} F(t; y, 0)\, dy \geqq F(t; x_2, 0)$$

whenever $x_1 < x_2$. Thus

$$\begin{aligned} G_1(t, x_2) &= \left(\frac{x_1}{x_2}\right) G_1(t, x_1) + \frac{(x_2 - x_1)}{x_2} \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} F(t; y, 0)\, dy \\ &\leqq \left(\frac{x_1}{x_2}\right) G_1(t, x_1) + \left(\frac{x_2 - x_1}{x_2}\right) F(t; x_1, 0) \leqq G_1(t, x_1). \end{aligned}$$

By applying Theorem 1.8 and Corollary 1.8.3, we can also describe the limiting behavior of $G_k(t, x)$ as $x \to \infty$ for $k = 1$ and 2. Of course, by Theorem 10.1, the mean blows up, but if we simply rescale time to keep the mean fixed, then we obtain a non-degenerate limit as $x \to \infty$. (We omit the details of the proof. The idea is that for RBM the first-passage time to 0 from x for large x has mean nearly x with standard deviation of order $x^{\frac{1}{2}}$; Corollary 1.8.3. For $k = 1$ and 2 and large x, $m_k(t, 0)$ is asymptotically negligible compared to $m_k(t, x)$, so that $m_k(t, k) \approx x^k[1 - G_1(t, x)]$.)

Theorem 10.3. For each $t > 0$,

$$\text{(a)} \qquad \lim_{x \to \infty} \frac{m_1(tx, x)}{x} = \lim_{x \to \infty} [1 - G_1(tx, x)] = \begin{cases} 1 - t, & 0 \leqq t \leqq 1 \\ 0, & t \geqq 1. \end{cases}$$

and

$$\text{(b)} \qquad \lim_{x \to \infty} \frac{m_2(tx, x)}{x^2} = \lim_{x \to \infty} [1 - G_2(tx, x)] = \begin{cases} (1 - t)^2, & 0 \leqq t \leqq 1 \\ 0, & t \geqq 1 \end{cases}$$

with the limit in (b) being the stationary-excess distribution associated with the uniform distribution in (a).

11. A coupling construction

11.1. *The difference process.* In this section we derive Theorems 7.1 and 7.3 probabilistically. The idea is to construct convenient versions of RBM starting at x and 0 on the same sample space such that their difference is a stochastic process with decreasing sample paths w.p.1. It is easy to see that the argument extends to yield Theorem 7.3 as well.

This goal is easy to achieve using the usual construction of RBM involving canonical unregulated Brownian motion $B(t; -1, 1, 0)$ as in (7.1). Using the *same* Brownian motion process $B(t; -1, 1, 0)$ for all x, we obtain a family of stochastic processes $\{R(t, x): t \geqq 0\}$ indexed by x on a common probability space. The process $R(t, 0)$ can also be defined directly in terms of the process $R(t, x)$ by

$$(11.1)\quad R(t, 0) = R(t, x) - \inf_{0 \leqq s \leqq t} R(s, x) = B(t; -1, 1, 0) - \inf_{0 \leqq s \leqq t} B(s; -1, 1, 0).$$

It is immediate from (7.1) and (11.1) that the processes $\{R(t, x): t \geqq 0\}$ and $\{R(t, 0): t \geqq 0\}$ individually have the desired finite-dimensional distributions, but of course they are dependent in a complicated way.

We apply (7.1) and (11.1) to create a *stochastic processes decomposition* for each k by setting

$$(11.2)\quad \begin{aligned} D_k(t, x) &= R(t, x)^k - R(t, 0)^k, \qquad t \geqq 0, \\ &= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} R(t, 0)^{k-j} I(t, x)^j, \qquad t \geqq 0, \end{aligned}$$

where

$$(11.3)\quad I(t, x) = \inf_{0 \leqq s \leqq t} R(s, x), \qquad t \geqq 0.$$

For $k = 1$, (11.2) takes a particularly simple form.

Theorem 11.1. $D_1(t, x) = I(t, x)$, $t \geqq 0$, so that $D_1(t, x)$ is decreasing in t w.p.1.

By taking expections, we obtain an alternative derivation of (7.4) and Theorem 9.2.

Corollary 11.1.1. $d_1(t, x) = E[I(t, x)] = \int_0^x [1 - F(t; y, 0)]\, dy$.

Proof. Note that

$$E[I(t,x)] = \int_0^\infty P(I(t,x) > y)\,dy = \int_0^x P(I(t,x) > y)\,dy$$
$$= \int_0^x P(T_{xy} > t)\,dy = \int_0^x P(T_{x-y,0} > t)\,dy = \int_0^x P(T_{y0} > t)\,dy.$$

By taking expectations in (11.2), we obtain the following representation for the moment function with general k.

Theorem 11.2. The kth moment-difference function in (7.2) can be expressed as

$$d_k(t,x) = \sum_{j=1}^{k} (-1)^{j-1}\binom{k}{j} E[R(t,0)^{k-j} I(t,x)^j] \tag{11.4}$$

for I in (11.3).

The representation (11.4) is not easy to apply for $k > 1$ because $R(t, 0)$ and $I(t, x)$ are dependent. For $k > 1$, (9.9) seems more useful.

It is not difficult to see that the second-moment-difference stochastic process $D_2(t, x)$ does not have monotone sample paths. However, it is not difficult to give a probabilistic proof showing that the second-moment-difference function $d_2(t, x)$ is nevertheless monotone. (Numerical values for $d_2(t, x)$ and $m_2(t, x)$ appear in Tables 6–8.)

Theorem 11.3. The second-moment-difference function $d_2(t, x)$ is decreasing in t.

Proof. As in the proof of Theorem 1.4, we do a detailed analysis for the $M/M/1$ queue-length process, i.e., a birth-and-death process with constant birth and death rates, and obtain the desired diffusion process result in the limit, invoking Iglehart and Whitt (1970) or Stone (1963). We can thus apply the result in our later work on the $M/M/1$ queue too. We use the same construction. Let $Q_n(t)$ denote the number in system in the process starting at n. Let the traffic intensity be less than 1, so that there is a negative drift, and the process starting at n hits 0 w.p.1. Hence, the distance between the processes converges monotonically to 0 w.p.1. By the construction, for each sample path $Q_n(t) - Q_0(t)$ is decreasing in t. We have to do something extra for $k = 2$ because $Q_n^2(t) - Q_0^2(t)$ as constructed above is typically *not* decreasing in t. We use the construction above, however, to see that it suffices to focus on the embedded jump chain of the process starting at n. Moreover, we use the consequence that $Q_n(t)$ is stochastically greater than or equal to $Q_0(t)$ for all t. Let $\tilde{Q}_n(k)$ be the embedded process starting at n at the kth transition epoch.

Let $p = P(\tilde{Q}_n(k+1) = m+1 \mid \tilde{Q}_n(k) = m)$. Since the processes have negative drift, $p < 1/2$. Note that

$$(11.5) \quad E(\tilde{Q}_n^2(k+1) \mid \tilde{Q}_n(k) = m) - m^2 = \begin{cases} 2m(2p-1)+1, & m > 0 \\ 1, & m = 0. \end{cases}$$

By the stochastic order between $\tilde{Q}_n(k)$ and $\tilde{Q}_0(k)$,

$$[E\tilde{Q}_n^2(k+1) - E\tilde{Q}_0^2(k+1)] - [E\tilde{Q}_n^2(k) - E\tilde{Q}_0^2(k)]$$
$$= \sum_{m=1}^{\infty} 2m(2p-1)[P(\tilde{Q}_n(k) = m) - P(\tilde{Q}_0(k) = m)] \leqq 0.$$

We remark that it is easy to see that Theorem 11.3 does *not* extend to the third and other higher moments. to see this, consider simple symmetric random walks with a barrier at 0 starting at n and 0. At 0, let the walks go up or stay put, each with probability 1/2. The change in the expected third moments after one step are $+3n$ and $+1/2$, respectively.

11.2. *Alternative proof of Corollary* 1.3.1. A minor variation of this coupling construction yields an alternative proof of Corollary 1.3.1. In particular, let $R_e(t)$ denote a stationary version of RBM starting with the exponential stationary distribution, which can be defined using (7.1) and randomizing the initial state x according to an exponential distribution with mean 1/2. In this way, $R_e(t)$ and $R(t, 0)$ are defined in terms of the same Brownian motion process $B(t; -1, 1, 0)$. Let

$$(11.6) \qquad D_e(t) = R_e(t) - R(t, 0), \qquad t \geqq 0.$$

By the same reasoning as above,

$$D_e(t) = \inf_{0 \leqq s \leqq t} R_e(s), \qquad t \geqq 0,$$

so that we can do the proof as follows. (We express it in a general form to cover half of Theorem 7.3 as well.)

$$E[R(\infty) - R(t, 0)] = E[D_e(t)] = \int_0^\infty dP(R(\infty) \leqq x) E[I(t, x)]$$
$$= \int_0^\infty dP(R(\infty) \leqq x) \int_0^x (T_{y0} > t)\, dy \qquad \text{by Corollary 11.1.1}$$
$$= \int_0^\infty P(R(\infty) > x) P(T_{x0} > t)\, dx,$$

using integration by parts for Riemann–Stieltjes integrals; e.g., Apostol (1957). First, the integral exists by Theorem 9–26 there. Second, the integration by parts is justified by Theorem 9-6 there. (For the case of RBM, $P(T_{x0} > t)$ is also continuous and bounded in x, so that we can apply p. 150 of Feller (1971).) Hence,

$$1 - H_1(t) = \frac{E[R(\infty) - R(t, 0)]}{E[R(\infty)]} = \int_0^\infty \frac{P(R(\infty) > x)}{E[R(\infty)]} P(T_{x0} > t)\, dx$$

$$= 1 - \int_0^\infty f_{R^*(\infty)}(x) P(T_{x0} \leqq t)\, dx$$

where $f_{R^*(\infty)} = P(R(\infty) > x)/E[R(\infty)]$ is the stationary-excess distribution associated with the c.d.f. $P(R(\infty) \leqq x)$, which for RBM is the exponential distribution. For RBM,

$$1 - H_1(t) = 1 - \int_0^\infty 2 \exp(-2x) F(t; x, 0)\, dx.$$

12. Approximation for the difference c.d.f.'s and the first-moment c.d.f.'s

In this section we develop simple closed-form approximations for the first-moment-difference c.d.f. $G_1(t, x)$ in (7.4). We propose a two-moment H_2 fit when $x \leqq 3$ and a certain two-moment stationary-excess shifted-exponential fit when $x \geqq 3$, both of which reduce to an ordinary exponential when $x = 3$. By Theorem 10.1, the first three moments (but not higher moments) match an exponential distribution exactly when $x = 3$.

12.1. *First case*: $x \leqq 3$. By Corollary 10.1.1, $c_x^2 \geqq 1$ when $x \leqq 3$, so that in this sense the c.d.f. $G_1(t, x)$ is more variable than exponential. In fact, numerical evidence indicates that the complementary c.d.f. $G_1^c(t, x) \equiv 1 - G_1(t, x)$ is log-convex (Conjecture 10.1 above). Hence, it is natural to consider an H_2 approximation. By Corollary 10.1.1, it is possible to make a three-moment H_2 fit to $G_1(t, x)$ for all $x \leqq 3$. The three-moment H_2 fitting procedure is described in Section 5.1. In this case, the parameter γ in (5.7) is

$$\gamma = \frac{3(c^2 - 1)^2}{(d^3 - 9c^2 + 3)} = \frac{6 - 2x}{6 - 3x}. \tag{12.1}$$

The H_2 parameters appear in Table 9. The general form of the approximations for $G_1(t, x)$ and $m_1(x, t)$ is shown in (7.11). Numerical values of the H_2 approximations are displayed along with the exact values based on Theorem 1 for $x = 0{\cdot}75$, $2{\cdot}00$ and $3{\cdot}00$ in Tables 1–3. As for the case in which $x = 0$ in Part I, the quality of the approximations for both $G_1(t, x)$ and $m_1(t, x)$ is excellent in the region of primary interest, $0{\cdot}5 \leqq t \leqq 8{\cdot}0$.

12.2. *The second case*: $x > 3$. By Corollary 10.1.1, $c_x^2 < 1$ when $x > 3$, so

TABLE 1

A comparison of the hyperexponential approximations with the exact first-moment functions starting at $x = 0{\cdot}75$ obtained from Theorem 1.1. (The minimum values are shown in bold.)

time	exact				approximate	
t	$H_1(t)/2$	$xG_1^c(t, x)$	$m_1(t, x)$	$G_1^c(t, x)$	$\tilde{m}_1(t, x)$	$\tilde{G}_1^c(t, x)$
0·00	0·000	0·750	0·750	1·000	0·750	1·000
0·05	0·155	0·545	0·700	0·727	0·696	0·811
0·10	0·206	0·448	0·654	0·597	0·654	0·662
0·15	0·242	0·376	0·618	0·502	0·620	0·545
0·20	0·269	0·322	0·591	0·429	0·594	0·452
0·25	0·290	0·280	0·570	0·373	0·573	0·379
0·50	0·360	0·159	0·519	0·212	0·518	0·184
0·75	0·399	0·103	0·502	0·137	0·501	0·114
1·00	0·425	0·071	0·496	0·095	0·495	0·082
1·25	0·442	0·051	0·494	0·069	0·494	0·064
1·50	0·455	0·038	**0·493**	0·051	**0·494**	0·051
1·75	0·464	0·029	0·493	0·039	0·494	0·041
2·00	0·472	0·022	0·494	0·030	0·495	0·033
2·50	0·481	0·014	0·495	0·018	0·496	0·021
3·00	0·488	0·009	0·496	0·0118	0·497	0·0140
3·50	0·492	0·006	0·497	0·0077	0·497	0·0092
4·00	0·494	0·004	0·4981	0·0052	0·4980	0·0060
4·50	0·496	0·003	0·4986	0·0035	0·4985	0·0039
5·00	0·4972	0·002	0·4990	0·0024	0·4989	0·0026
6·00	0·4986	0·001	0·4995	0·0012	0·4994	0·0011
7·00	0·4993	—	0·4997	0·0006	0·4997	0·0005
8·00	0·4996		0·4998	0·0003	0·4998	0·0002
9·00	0·4998		0·4999	0·0002	0·4999	0·0001
10·00	0·4999		0·5000	0·0001	0·5000	-

that in this sense the difference c.d.f. $G_1(t, x)$ is less variable than an exponential when $x > 3$. Numerical evidence suggest that the complementary c.d.f. $G_1^c(t, x)$ tends to be log-concave or nearly log-concave for $x > 3$ and t away from the origin, e.g., $t > 1$. However, since $G_1^c(t, x) \approx 1 - (2t/\pi x^2)^{\frac{1}{2}}$ for t near 0, $G_1^c(t, x)$ is log-convex in the neighborhood of the origin for all $x > 0$. Numerical evidence indicates that $G_1^c(t, x)$ is initially log-convex and then log-concave.

Conjecture 12.1. For each $x > 3$, there exists t_x such that $G_1^c(t, x)$ is log-convex on the interval $(0, t_x)$ and log-concave on the interval (t_x, ∞).

The dominant shape for times t of primary interest, say $1 \leqq t \leqq 2x$, is the log-concavity and this is reflected in our approximations. For $x > 3$, we tried three approximations based on matching two moments: gamma, Weibull and stationary-excess shifted-exponential (SESE, to be defined below) and found the

TABLE 2
A comparison of the hyperexponential approximations with the exact first-moment functions starting at $x = 2{\cdot}0$ obtained from Theorem 1.1

time	exact				approximate	
t	$H_1(t)/2$	$xG_1^c(t, x)$	$m_1(t, x)$	$G_1^c(t, x)$	$\tilde{m}_1(t, x)$	$\tilde{G}_1^c(t, x)$
0·00	0·000	2·00	2·00	1·000	2·00	1·000
0·05	0·155	1·80	1·95	0·897	1·97	0·942
0·10	0·206	1·69	1·190	0·847	1·93	0·888
0·15	0·242	1·61	1·185	0·804	1·89	0·838
0·20	0·269	1·53	1·180	0·766	1·84	0·790
0·25	0·290	1·46	1·75	0·730	1·78	0·746
0·50	0·360	1·15	1·51	0·574	1·51	0·565
0·75	0·399	0·90	1·30	0·448	1·28	0·434
1·00	0·425	0·70	1·13	0·351	1·11	0·338
1·25	0·442	0·551	0·99	0·276	0·98	0·266
1·50	0·455	0·436	0·89	0·218	0·88	0·212
1·75	0·464	0·348	0·81	0·174	0·80	0·170
2·00	0·472	0·279	0·75	0·139	0·75	0·138
2·50	0·481	0·182	0·66	0·091	0·66	0·092
3·00	0·488	0·121	0·609	0·061	0·611	0·063
3·50	0·492	0·082	0·574	0·041	0·576	0·043
4·00	0·494	0·056	0·550	0·028	0·553	0·030
4·50	0·496	0·039	0·535	0·019	0·537	0·021
5·00	0·4972	0·027	0·524	0·0136	0·526	0·0145
6·00	0·4986	0·0136	0·512	0·0068	0·513	0·0071
7·00	0·4993	0·0069	0·506	0·0035	0·506	0·0035
8·00	0·4996	0·0036	0·5032	0·0018	0·5031	0·0017
9·00	0·4998	0·0019	0·5017	0·0009	0·5015	0·0008
10·00	0·4999	0·0010	0·5009	0·0005	0·5008	0·0004

SESE to perform best. Roughly, speaking, the distributions with given first two moments become successively more log-concave moving from gamma to Weibull, SESE and the exact difference c.d.f. Numerical comparisons of all these candidate approximations for the case $x = 12$ appear in Table 5.

The ordinary shifted-exponential distribution is an exponential distribution on the interval $[a, \infty)$ for $a > 0$, i.e., the complementary c.d.f. is $\exp(-(t-a)/b)$, $t \geqq a$, which has mean $a + b$, with the two parameters a and b. The associated stationary-excess distribution (SESE) has the density

$$g(t) = \begin{cases} 1/(a+b), & 0 \leqq t \leqq a \\ [1/(a+b)] \exp(-(t-a)/b), & t > a, \end{cases} \tag{12.2}$$

and complementary c.d.f.

$$G^c(t) = \begin{cases} 1 - t/(a+b), & 0 \leqq t \leqq a \\ [b/(a+b)] \exp(-(t-a)/b), & t > a. \end{cases} \tag{12.3}$$

TABLE 3

A comparison of the exponential approximation for the first-moment difference c.d.f. and the resulting approximation for the first-moment function starting at $x = 3{\cdot}0$ with exact values obtained from Theorem 1.1

time	exact				approximate	
t	$H_1(t)/2$	$xG_1^c(t, x)$	$m_1(t, x)$	$G_1^c(t, x)$	$\tilde{m}_1(t, x)$	$\tilde{G}_1^c(t, x)$
0·00	0·000	3·00	3·000	1·000	3·00	1·000
0·05	0·155	2·80	2·950	0·932	2·99	0·967
0·10	0·206	2·69	2·900	0·898	2·96	0·936
0·15	0·242	2·61	2·850	0·869	2·93	0·905
0·20	0·269	2·53	2·800	0·844	2·88	0·875
0·25	0·290	2·46	2·750	0·820	2·83	0·846
0·50	0·360	2·14	2·500	0·713	2·53	0·716
0·75	0·399	1·85	2·252	0·618	2·23	0·607
1·00	0·425	1·59	2·01	0·530	1·97	0·513
1·25	0·442	1·35	1·79	0·450	1·75	0·435
1·50	0·455	1·14	1·60	0·380	1·56	0·368
1·75	0·464	0·96	1·42	0·320	1·40	0·311
2·00	0·472	0·81	1·28	0·269	1·26	0·264
2·50	0·481	0·57	1·05	0·189	1·05	0·189
3·00	0·488	0·40	0·89	0·133	0·89	0·135
3·50	0·492	0·28	0·77	0·094	0·78	0·097
4·00	0·494	0·20	0·69	0·067	0·70	0·069
4·50	0·496	0·143	0·64	0·048	0·65	0·050
5·00	0·4972	0·102	0·60	0·034	0·60	0·036
6·00	0·4986	0·053	0·551	0·0176	0·554	0·0183
7·00	0·4993	0·028	0·527	0·0092	0·528	0·0094
8·00	0·4996	0·015	0·514	0·0049	0·514	0·0048
9·00	0·49980	0·008	0·508	0·0026	0·507	0·0024
10·00	0·49989	0·004	0·504	0·0014	0·504	0·0013
12·00	0·49997	0·001	0·5013	0·0004		
14·00	0·499990	—	0·5004	0·00013		
16·00	0·499997		0·5001	0·00004		

We were motivated to consider SESE because $G^c(t)$ has approximately the right shape, being linear initially and then exponential.

The linearity is nearly exact. In particular, when $t \ll x$, the barrier at 0 has negligible influence and RBM behaves just like BM without a barrier, so that

$$m_1(t, x) \approx x - t \quad \text{for} \quad t \ll x. \tag{12.4}$$

Similarly,

$$m_2(t, x) \approx t + (x - t)^2 \quad \text{for} \quad t \ll x. \tag{12.5}$$

From the tables, it is easy to see that (12.4) and (12.5) are consistent with the numerical results.

TABLE 4

A comparison of the two-moment stationary-excess shifted-exponential (SESE) approximation for the first-moment-difference c.d.f. and the resulting approximation for the first-moment function starting at $x = 6{\cdot}0$ with exact values obtained from Theorem 1.1

time	exact				approximate	
t	$H_1(t)/2$	$xG_1^c(t, x)$	$m_1(t, x)$	$G_1^c(t, x)$	$\tilde{m}_1(t, x)$	$\tilde{G}_1^c(t, x)$
0·00	0·00	6·00	6·00	1·000	6·00	1·000
0·05	0·15	5·80	5·95	0·966	6·02	0·990
0·10	0·21	5·69	5·90	0·949	6·04	0·980
0·25	0·29	5·46	5·75	0·910	5·98	0·949
0·50	0·36	5·14	5·50	0·857	5·77	0·898
0·75	0·40	4·85	5·25	0·808	5·50	0·848
1·00	0·42	4·58	5·00	0·763	5·22	0·797
1·25	0·442	4·31	4·75	0·718	4·93	0·747
1·50	0·455	4·05	4·50	0·674	4·63	0·696
1·75	0·464	3·79	4·25	0·631	4·34	0·645
2·00	0·472	3·53	4·00	0·588	4·04	0·595
2·50	0·481	3·03	3·51	0·505	3·44	0·494
3·00	0·488	2·55	3·04	0·426	2·87	0·398
3·50	0·492	2·11	2·61	0·353	2·41	0·320
4·00	0·494	1·73	2·22	0·288	2·03	0·257
4·50	0·496	1·39	1·89	0·232	1·73	0·207
5·00	0·4972	1·11	1·61	0·185	1·49	0·166
6·00	0·4986	0·69	1·19	0·115	1·14	0·107
7·00	0·4993	0·418	0.92	0·070	0·92	0·069
8·00	0·4996	0·250	0·75	0·042	0·77	0·045
9·00	0·4998	0·147	0·65	0·0246	0·67	0·0290
10·00	0·4999	0·087	0·59	0·0144	0·612	0·0187
12·00	0·5000	0·029	0·529	0·0049	0·547	0·0078
14·00	—	0·010	0·510	0·0017	0·520	0·0033
16·00		0·0034	0·5034	0·0006	0·508	0·0014
18·00		0·0011	0·5011	0·0002	0·5034	0·0006
20·00		0·0004	0·5004	0·0001	0·5014	0·0002

In fact, since RBM starting at x is stochastically greater than BM starting at x (in the strong sample path sense of Kamae et al. (1977); (7.1) provides a proof), we have the following result.

Theorem 12.1. For $0 \leqq t \leqq x$, $m_1(t, x) \geqq x - t$ and $m_2(t, x) \geqq t + (x - t)^2$.

Moreover, for relatively small t, we recognize that (12.4) and (12.5) are the preferred approximations. Approximation (12.4) is relevant to $G_1(t, x)$ because $xG_1^c(t, x)$ will tend to be the dominant part of $m_1(t, x)$ for large x, especially for $t \geqq 1$ because $m_1(1, 0) = 0{\cdot}85\, m_1(\infty)$. Hence, we would expect $G_1^c(t, x)$ to be approximately linear initially (with the exception of a more rapid decrease in the neighborhood of the origin) and the SESE distribution has this property.

TABLE 5

A comparison of several two-moment approximations for the first-moment-difference c.d.f. starting at $x = 12{\cdot}0$ and the resulting stationary-excess shifted-exponential (SESE) approximation for the first-moment function with exact values obtained from Theorem 1.1

time	exact				approximate			
					$\tilde{m}_1(t, x)$	$\tilde{G}_1^c(t, x)$		
t	$H_1(t)/2$	$xG_1^c(t, x)$	$m_1(t, x)$	$G_1^c(t, x)$	SESE	SESE	Weibull	gamma
0·00	0·00	12·00	12·00	1·000	12·00	1·000	1·000	1·000
0·25	0·29	11·53	11·75	0·955	12·01	0·977	0·991	0·997
0·50	0·36	11·14	11·50	0·928	11·83	0·954	—	—
0·75	0·40	10·85	11·25	0·904	11·59	0·931	—	—
1·00	0·42	10·58	11·00	0·881	11·33	0·908	0·935	0·955
1·50	0·455	10·04	10·50	0·837	10·81	0·863	—	—
2·00	0·472	9·53	10·00	0·794	10·27	0·817	0·834	0·856
2·50	0·481	9·02	9·50	0·752	9·73	0·771	0·779	0·797
3·00	0·488	8·51	9·00	0·709	9·19	0·725	0·723	0·736
4·00	0·494	7·51	8·00	0·625	8·10	0·634	0·613	0·615
6·00	0·499	5·51	6·01	0·459	5·91	0·451	0·418	0·406
8·00	0·500	3·63	4·13	0·302	3·75	0·271	0·268	0·255
10·00	—	2·11	2·61	0·176	2·31	0·151	0·163	0·155
12·00		1·10	1·60	0·092	1·51	0·084	0·095	0·092
14·00		0·53	1·03	0·044	1·07	0·047	0·053	0·053
16·00		0·24	0·737	0·020	0·82	0·026	0·029	0·031
18·00		0·101	0·601	0·0084	0·676	0·015	0·015	0·017
20·00		0·042	0·542	0·0035	0·598	0·0082	—	—
22·00		0·017	0·517	0·0014	0·555	0·0046	0·0037	0·0054

In fact, the SESE fit is *asymptotically correct* in the sense of Theorem 10.3 as $x \to \infty$. (We omit the proof, which is not difficult.)

Theorem 12.2. If we change the time scale so that the difference c.d.f. $G_1(t, x)$ and the SESE approximation $G(t)$ specified by (12.3) both have the mean 1/2 for all x, then both converge to the uniform distribution on [0, 1] as $x \to \infty$; i.e.,

$$\lim_{x\to\infty} G(tx/2) = \lim_{x\to\infty} G_1(tx/2, x) = \begin{cases} t, & 0 \leqq t \leqq 1 \\ 1, & t \geqq 1, \end{cases}$$

where, for each x, the SESE c.d.f. $G(t)$ is constructed to have the first two moments as $G_1(t, x)$.

From Theorem 12.2 plus a uniform integrability argument, we can also obtain the following corollary about the moments.

Corollary 12.2.1. For all k, the scaled kth moments m_k/m_1^k of SESE and the difference c.d.f. asymptotically agree as $x \to \infty$.

TABLE 6
The exact second-moment function and its components starting at $x = 0{\cdot}25$ (from Theorem 1.1(b))

time	exact			
t	$H_2(t)/2$	$x^2G_2^c(t, x)$	$m_2(t, x)$	$G_2^c(t, x)$
0·00	0·000	0·063	0·063	1·000
0·05	0·039	0·049	0·089	0·789
0·10	0·071	0·042	0·113	0·678
0·15	0·099	0·037	0·136	0·599
0·20	0·123	0·034	0·157	0·539
0·25	0·145	0·031	0·176	0·489
0·50	0·230	0·020	0·251	0·328
0·75	0·290	0·015	0·305	0·237
1·00	0·333	0·011	0·344	0·177
1·25	0·366	0·0085	0·375	0·136
1·50	0·392	0·0066	0·399	0·106
1·75	0·412	0·0052	0·417	0·084
2·00	0·428	0·0042	0·432	0·067
2·50	0·451	0·0027	0·454	0·044
3·00	0·466	0·0018	0·468	0·029
3·50	0·476	0·00124	0·478	0·0197
4·00	0·483	0·00085	0·484	0·0136
5·00	0·492	0·00041	0·4919	0·0066
6·00	0·496	0·00021	0·4958	0·0033
7·00	0·4977	0·00011	0·4978	0·0017
8·00	0·4988	0·00006	0·4988	0·0009
9·00	0·4993	0·00003	0·49937	0·0005
10·00	0·4996		0·49966	0·0003

For example the ratio m_3/m_1^3 for the difference c.d.f. (SESE) when $x = 3$, 6, 12, 30 and 300 is, respectively, 6·00, (6·00), 3·67 (3·91), 2·75, (2·91), 2·28 (2·34) and 2·03 (2·03).

We now describe moment fitting with SESE. In terms of the parameters a and b, the first three moments are

$$m_1 = \frac{a+b}{2} + \frac{b^2}{2(a+b)}$$

(12.6)
$$m_2 = \frac{(a+b)^2}{3} + b^2 + \frac{2b^3}{3(a+b)}$$

$$m_3 = \frac{(a+b)^3}{4} + \frac{3b^2(a+b)}{2} + 2b^3 + \frac{9b^4}{4(a+b)}.$$

We can obtain a and b in terms of m_1 and m_2 from (12.6). We find that a can

TABLE 7
The exact second-moment function and its components starting at $x = 0{\cdot}9$ (from Theorem 1.1(b)). The minimum value of the second moment among these times is shown in bold type

time	exact			
t	$H_2(t)/2$	$x^2G_2^c(t, x)$	$m_2(t, x)$	$G_c^2(t, x)$
0·00	0·000	0·810	0·810	1·000
0·05	0·039	0·733	0·772	0·905
0·10	0·071	0·669	0·740	0·826
0·15	0·099	0·614	0·712	0·758
0·20	0·123	0·565	0·688	0·698
0·25	0·145	0·523	0·668	0·646
0·50	0·230	0·370	0·600	0·456
0·75	0·290	0·273	0·563	0·337
1·00	0·333	0·208	0·541	0·257
1·25	0·366	0·161	0·528	0·199
1·50	0·392	0·127	0·519	0·157
1·75	0·412	0·101	0·5128	0·124
2·00	0·428	0·081	0·5088	0·100
2·50	0·451	0·053	0·5041	0·066
3·00	0·466	0·036	0·5018	0·044
3·50	0·476	0·024	0·5007	0·030
4·00	0·483	0·0168	0·5001	0·021
5·00	0·492	0·0082	0·49990	0·0102
6·00	0·496	0·0042	**0·49979**	0·0051
7·00	0·4977	0·0021	0·49985	0·0027
8·00	0·4988	0·0011	0·499903	0·0014
9·00	0·4993	0·0006	0·499944	0·00074
10·00	0·4996	0·0003	0·499971	0·00040

be eliminated from (12.6) yielding a single equation for $z = b/m_1$, namely,

$$-(2/3c^2)z^4 + z^2 + [1-(1/3c^2)]z - (3/4)(c^2+1)[1-(1/3c^2)] = 0. \tag{12.7}$$

Setting $z = \sin\theta$, we can express the solution parametrically in terms of θ. In particular

$$\begin{aligned} a &= m_1(1-\sin\theta+\cos\theta), \qquad b = m_1\sin\theta, \\ c^2 &= 1-(2/3)[\cos^2\theta + (1-\cos\theta)(1-\sin\theta)] \\ &= (1/3) + (2/3)(1-\cos\theta)(\sin\theta+\cos\theta). \end{aligned} \tag{12.8}$$

Equation (12.7) can easily be solved numerically. Alternatively, a good approximate solution for some values of c^2 can be obtained from

$$z \approx \hat{z} = \begin{cases} \dfrac{1}{2}\left[1-\dfrac{1}{3c^2}\right]\left(\left[1+\dfrac{3(c^2+1)}{(1-[1/3c^2])}\right]^{\frac{1}{2}}-1\right), & c^2 \leqq 0{\cdot}4 \\ 1-\dfrac{(1-c^2)}{2}\left(1+\dfrac{[(1-c^4)/2]^{\frac{1}{2}}+(1-c^2)}{3c^2}\right), & c^2 > 0{\cdot}425. \end{cases} \tag{12.9}$$

TABLE 8
The exact second-moment function and its components starting at $x = 2{\cdot}0$ (from Theorem 1.1(b))

time	exact			
t	$H_2(t)/2$	$x^2 G_2^c(t, x)$	$m_2(t, x)$	$G_2^c(t, x)$
0·00	0·000	4·00	4·00	1·000
0·05	0·039	3·81	3·85	0·953
0·10	0·071	3·64	3·71	0·910
0·15	0·099	3·47	3·57	0·868
0·20	0·123	3·32	3·44	0·829
0·25	0·145	3·17	3·31	0·792
0·50	0·230	2·52	2·75	0·630
0·75	0·290	2·01	2·30	0·503
1·00	0·333	1·61	1·95	0·403
1·25	0·366	1·301	1·67	0·325
1·50	0·392	1·056	1·45	0·264
1·75	0·412	0·861	1·27	0·215
2·00	0·428	0·705	1·13	0·176
2·50	0·451	0·478	0·929	0·120
3·00	0·466	0·329	0·795	0·082
3·50	0·476	0·228	0·705	0·057
4·00	0·483	0·160	0·643	0·040
5·00	0·492	0·080	0·572	0·020
6·00	0·496	0·041	0·537	0·103
7·00	0·4977	0·0216	0·519	0·0054
8·00	0·4988	0·0115	0·5103	0·0029
9·00	0·4993	0·0062	0·5055	0·00154
10·00	0·4996	0·0033	0·5030	0·00084

The first term in (12.9) is based on ignoring the z^4 term in (12.7), which is only appropriate for relatively small c^2. The second term is obtained by matching coefficients in the power series expansion of $(1 - c^2)$. The second term in (12.9) is a good approximation for a large range of c^2 values. Table 10 contains typical values of θ, c^2, a/m_1, $b/m_1 = z$ and $\hat{z}$. The ranges of possible normalized second and third moments for SESE are $1/3 \leqq c^2 \leqq 1$ and $2 \leqq m_3/m_1^3 \leqq 6$, *coinciding exactly* with the corresponding ranges for the exact first-moment difference c.d.f. $G_1(t, x)$ by Theorem 10.1.

Numerical comparisons of the approximations of $G_1(t, x)$ and $m_1(t, x)$ using SESE are made for the cases $x = 6$ and 12 in Tables 4 and 5. The approximations for $x > 3$ are not as good as for $x \leqq 3$, but they are pretty good in the regions of primary interest. The relevant second regime here (see Section 1.6) might be the time required for RBM to get within $0{\cdot}01x - 0{\cdot}20x$ of the steady-state limit $m_1(\infty) = 0{\cdot}5$. For $x = 12$, then, we would be primarily interested in the region $9{\cdot}5 \leqq t \leqq 17{\cdot}5$ for which $2{\cdot}9 \geqq m_1(t, x) \geqq 0{\cdot}62$.

TABLE 9

Parameters for the H_2 approximation of the first-moment-difference c.d.f. $G_1(t, x)$ for $0 < x \leqq 3$ based on the H_2-fitting scheme in (5.7)

starting state	moment parameters						moment-fitting parameters			basic H_2 parameters			
										probs.		means	
x	m_1	m_2	m_3	c^2	$\frac{m_3 m_1}{m_2^2}$	$d^3 = \frac{m_3}{m_1^3}$	γ	r	α	p_1	p_2	τ_1	τ_2
0·25	0·125	0·146	0·441	8·33	2·59	225.8	1·048	0·479	0·877	0·939	0·061	0·064	1·061
0·50	0·250	0·333	1·031	4·33	2·32	66·0	1·111	0·458	0·758	0·879	0·121	0·130	1·120
0·75	0·375	0·563	1·793	3·00	2·12	34·0	1·200	0·436	0·640	0·820	0·180	0·199	1·176
1·00	0·500	0·833	2·750	2·33	1·98	22·0	1·333	0·413	0·522	0·761	0·239	0·271	1·229
1·25	0·625	1·146	3·926	1·93	1·87	16·1	1·555	0·388	0·402	0·701	0·299	0·346	1·279
2·00	1·000	2·333	9·00	1·33	1·65	9·0	∞	0·296	0·000	0·500	0·500	0·592	1·408
2·50	1·250	3·333	13·91	1·13	1·56	7·12	−0·666	0·200	−0·361	0·320	0·680	0·781	1·471
3·00	1·500	4·500	20·25	1·00	1·50	6·00	0·000	0·000	1·000	0	1·000	1·500	1·500

TABLE 10

Typical parameter values in moment matching with the stationary-excess shifted-exponential (SESE) distribution

exact				approximation
c^2	θ	a/m_1	$z = b/m_1 = \sin\theta$	$\hat{z}$
0·3333	0.0°	2·000	0·000	0·000
0·3368	5·6°	1·895	0·100	0·097
0·3492	11·5°	1·780	0·200	0·193
0·3731	17·5°	1·653	0·300	0·283
0·3876	20·5°	1·587	0·350	0·318
0·4071	23·6°	1·516	0·400	0·356
0·4288	26·7°	1·443	0·450	0·445
0·4553	30·0°	1·366	0·500	0·494
0·5000	34·9°	1·248	0·572	0·565
0·6000	44·2°	1·020	0·697	0·693
0·6095	45·0°	1·000	0·707	0·703
0·7000	52·6°	0·813	0·794	0·793
0·7067	53·13°	0·800	0·800	0·799
0·7887	60·0°	0·634	0·866	0·866
0·8000	60·9°	0·611	0·874	0·874
0·9000	70·4°	0·392	0·942	0·942
1·0000	90·0°	0·000	1·000	1·000

TABLE 11

Times $t_p(x)$ to each and remain within a fraction p of the steady-state limit $m_1(\infty) = 0{\cdot}50$. (The minimum initial state x for these candidates is shown in bold for each p.)

initial state x	±0·025 $t_{0\cdot05}(x)$	±0·010 $t_{0\cdot02}(x)$	±0·005 $t_{0\cdot01}(x)$	±0·001 $t_{0\cdot002}(x)$
0·00	2·2	3·2	4·2	6·5
0·25	2·1	3·2	4·2	6·5
0·50	1·7	2·8	3·8	6·2
0·60	1·3	2·5	3·5	6·0
0·65	0·9	2·3	3·3	5·7
0·66	**0.2**	2·2	3·2	5·6
0·70	0·3	1·9	3·0	5·5
0·72	0·4	**0·4**	2·8	5·2
0·75	0·4	0·6	2·4	5·0
0·77	0·5	0·7	**0·8**	4·9
0·80	0·6	0·9	1·1	4·1
0·82	0·7	1·0	1·2	**1·7**
0·85	0·9	1·2	1·6	2·1
0·90	1·3	1·5	2·2	3·2
0.95	1·4	2·2	2·7	4·7
1·00	1·7	2·5	3·2	4·8
1·25	2·8	3·8	4·7	6·8
2·00	4·9	6·2	7·2	9·8
2·50	6·2	7·4	8·5	10·5
3·00	7·1	8·5	9·7	12·5
4·00	8·8	10·2	11·7	14·3
6·00	12·2	14·0	15·6	18·2
12·00	21·5			

12.3. *The second-moment-difference c.d.f.* In this paper we do not study approximations for the second-moment-difference c.d.f. $G_2(t, x)$ and the second-moment function $m_2(t, x)$, but approximations can be easily constructed. For $x \leqq 3$ we can use H_2 distributions for $G_2(t, x)$; for $x > 3$ it is natural to use the stationary-excess distribution of the SESE distribution in (12.2), fitting the two parameters a and b to the first two moments of $G_2(t, x)$. This is asymptotically correct as $x \to \infty$ in the sense of Theorem 12.2 (then with the parameters a and b determined by the two-moment fit to $G_1(t, x)$).

13. Initialization for rapid approach to steady state

In this section we see which initial states x cause the first moment $m_1(t, x)$ to approach the steady-state limit $m_1(\infty) = 0{\cdot}50$ most quickly. See Kelton and Law (1985) and Kelton (1985) for related work for the $M/M/1$ queue. (They consider the expected waiting times of successive customers instead of the queue-length process, but the behavior of these processes is similar.)

For any initial state x and probability p, let $t_p(x)$ be the time required to reach and remain within $(100p)\%$ of the steady-state limit, i.e.,

$$t_p(x) = \inf\left\{t \geqq 0 : \sup_{t' \geqq t} \left|\frac{m_1(t', x)}{m_1(\infty)} - 1\right| \leqq p\right\}. \tag{13.1}$$

Let t_p^* be the minimum value of $t_p(x)$ over all x and let x_p^* be the value(s) of x such that $t_p(x) = t_p^*$.

The asymptotic theory in Corollary 1.1.2 suggests that the critical damping level $x = 1$ causes steady state to be approached most quickly. This is so because there is a $t^{-\frac{5}{2}}$ term when $x = 1$ instead of a $t^{-\frac{3}{2}}$ term. However, as in Part I, we find that the asymptotic theory is not adequate for practical purposes. If we want to find x_p^* for $p = 0{\cdot}01$ or $0{\cdot}02$, then the best value of x is actually much less than 1. Corollary 1.1.2 evidently only becomes relevant for t beyond practical interest.

13.1. *Empirical observations.* Table 11 displays the values of $t_p(x)$ for $p = 0{\cdot}05$, $0{\cdot}02$, $0{\cdot}01$ and $0{\cdot}002$ and various initial states x. Included in the list are the values x_p^* that are optimal for these p, which are $0{\cdot}66$, $0{\cdot}72$, $0{\cdot}77$ and $0{\cdot}82$, respectively. This suggest that x_p^* is decreasing in p, which we *prove* in Section 13.2 below. For these values of p which appear to be of practical interest, the initial state $x = 0{\cdot}75$ is nearly optimal, clearly yielding much smaller values of $t_p(x)$ than either the steady-state limit $x = 0{\cdot}50$ or the critical damping level $x = 1{\cdot}00$. More generally, in agreement with Kelton and Law (1985), it appears that one-and-a-half times the steady-state limit might be approximately optimal with one of these criteria.

Furthermore, in agreement with Kelton and Law (1985), we find that the functions $t_p(x)$ increase very sharply in the neighborhood of x_p^*. However, this seems to be primarily due to the criterion (13.1) rather than the nature of RBM. As we show in Theorem 13.2 below, the optimal starting state x_p^* is such that the moment function $m_1(t, x)$ reaches, but does not cross, the level $(1-p)m_1(\infty)$. Then t_p^* is the time t that $m_1(t, x)$ first hits $(1+p)m_1(\infty)$ from above. For $x < x^*$, $m_1(t, x)$ crosses below $(1-p)m_1(\infty)$, so that $t_p(x)$ is the later time t when $m_1(t, x)$ hits $(1-p)m_1(\infty)$ from below. Thus $tp(x)$ has a jump discontinuity at x_p^*.

It is of some interest to compare $x = 1$ and $x = 0$ as initial states because $m_1(t, 1)$ and $m_1(t, 0)$ are both monotone and initially equally distant from the steady-state limit $m_1(\infty) = 0{\cdot}5$. It is easy to see that $m_1'(t, 0) > |m_1'(t, 1)|$ for $t \leqq 0{\cdot}5$ so that initially $m_1(t, 0)$ increases more rapidly than $m_1(t, 1)$ decreases. However, $0{\cdot}5 - m_1(t, 0) > m_1(t, 1) - 0{\cdot}5$ for t about $0{\cdot}75$, as is substantiated by numerical values. One gets within 1% or 0·1% of the steady-state limit quicker by starting with $x = 1$ than with $x = 0$; see Table 11. (This contradicts the conclusion at the bottom of p. 391 of Kelton and Law (1985).)

13.2. *Theoretical conclusions.* We conclude by deriving a few theoretical properties about $t_p(x)$, t_p^* and x_p^*. Note that the moment functions are ordered, i.e., $m_1(t, x_1) \leqq m_1(t, x_2)$ for all $t \geqq 0$ when $x_1 < x_2$. Since $m_1(t, x) \leqq m_1(\infty)$ for all t if and only if $x \leqq 0{\cdot}5$, and $m_1(t, x) \geqq m_1(\infty)$ for all t if and only if $x \geqq 1{\cdot}0$, we have the following result.

Theorem 13.1. For all p, $t_p(x)$ is decreasing in x for $0 \leqq x \leqq 1/2$ and increasing in x for $x \geqq 1$.

Corollary 13.1.1. For all p, $0{\cdot}5 < x_p^* < 1{\cdot}0$.

Since $m_1(\hat{t}(0{\cdot}5), 0{\cdot}5) \approx 0{\cdot}40$ from numerical values, $x_p^* = 0{\cdot}5$ and $t_p^* = 0{\cdot}0$ for $p \geqq 0{\cdot}20$. (For $p > 0{\cdot}20$, there are ties for x_p^*.) Hence, the interesting cases are restricted to $p < 0{\cdot}20$ and $0{\cdot}5 < x < 1{\cdot}0$.

Since $m_1(t, x)$ is unimodal in t with a minimum for $0{\cdot}5 < x < 1{\cdot}0$, we have the following result.

Theorem 13.2. For $p < 0{\cdot}20$, x_p^* is the unique x such that

$$m_1(\hat{t}(x), x) = (1-p)m_1(\infty) \tag{13.2}$$

where $\hat{t}(x)$ is the solution to (8.3) and $t_p^* = t_p(x_p^*)$.

As an easy consequence of Theorem 8.2, we have the following monotonicity result.

Theorem 13.3. The optimal values t_p^* and x_p^* are increasing in $1-p$, with $x_{0{\cdot}2}^* = 0{\cdot}50$ and $t_{0{\cdot}2}^* = 0{\cdot}0$, $x_p^* \to 1$ and $t_p^* \to \infty$ as $p \to 0$.

13.3. *The second-moment function.* A similar story can be told for the second-moment function. Let $t_{2p}(x)$, x_{2p}^* and t_{2p}^* be the analogs of $t_p(x)$, x_p^* and t_p^* for $m_2(t, x)$.

From Section 8.2, we can deduce the following.

Theorem 13.4. For all p, $t_p(x)$ is decreasing in x for $0 \leqq x \leqq \sqrt{2}/2$ and increasing in x for $x \geqq 1$.

Corollary 13.4.1. For all p, $\sqrt{2}/2 < x_{2p}^* < 1{\cdot}0$.

Theorem 13.5. x_{2p}^* is the unique x, if one exists, such that

$$m_2(\bar{t}(x), x) = (1-p)m_2(\infty) = (1-p)/2$$

where

$$m_1(\bar{t}(x), x) = m_1(\infty) = 1/2;$$

then $t_{2p}^* = t_{2p}(x_p^*)$. Otherwise, $x_{2p}^* = \sqrt{2}/2$ and $t_p^* = 0$.

Theorem 13.6. The optimal values t_{2p}^* and x_{2p}^* are increasing in $(1-p)$ with $x_p^* \to 1$ and $t_p^* \to \infty$ as $p \to 0$.

Numerical evidence indicates that $x_{2p}^* \geqq x_p^*$ for all p. Moreover, $m_2(t, x) \geqq m_1(t, x)$ for all t when $x \geqq 1$ and $m_2(t, x) \leqq m_1(t, x)$ for all t when $x \leqq \sqrt{2}/2$, but the functions cross when $\sqrt{2}/2 < x < 1{\cdot}0$. Meaningful comparisons between $m_1(t, x)$ and $m_2(t, x)$ are difficult, however, because the second moment involves the squaring operation.

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