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UNIFORM CONDITIONAL VARIABILITY ORDERING OF PROBABILITY DISTRIBUTIONS

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Abstract

Variability orderings indicate that one probability distribution is more spread out or dispersed than another. Here variability orderings are considered that are preserved under conditioning on a common subset. One density f on the real line is said to be less than or equal to another, g, in *uniform conditional variability order* (UCVO) if the ratio $f(x)/g(x)$ is unimodal with the model yielding a supremum, but f and g are not stochastically ordered. Since the unimodality is preserved under scalar multiplication, the associated conditional densities are ordered either by UCVO or by ordinary stochastic order. If f and g have equal means, then UCVO implies the standard variability ordering determined by the expectation of all convex functions. The UCVO property often can be easily checked by seeing if $f(x)/g(x)$ is log-concave. This is illustrated in a comparison of open and closed queueing network models.

DISPERSION; SPREAD; VARIABILITY; STOCHASTIC COMPARISONS; STOCHASTIC ORDER; CONDITIONAL PROBABILITIES; UNIFORM CONDITIONAL STOCHASTIC ORDER; MONOTONE LIKELIHOOD RATIO; LOG-CONCAVITY; RELATIVE LOG-CONCAVITY; NETWORKS OF QUEUES; STATIONARY-EXCESS DISTRIBUTION

1. Introduction

Variability orderings compare probability distributions according to their spread or dispersion; see Bickel and Lehmann (1976), (1979), Birnbaum (1948), Marshall and Olkin (1979), Oja (1981), Shaked (1980), (1982), (1984), Stoyan (1983), Yanagimoto and Sibuya (1976), (1980) and references in these sources. Our point of departure is the ordering $X \leqq_v Y$ that holds if

$$Eh(X) \leqq Eh(Y) \tag{1}$$

for all convex real-valued functions h, which requires that $EX = EY$. For the case of unequal means, we also consider the related orderings based on all non-decreasing convex functions or all non-increasing convex functions. Our object is to consider stronger variability orderings that are preserved under

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conditioning and so tend to hold locally. Our analysis of *uniform conditional variability order* (UCVO) parallels and extends previous work on *uniform conditional stochastic order* (UCSO) in Keilson and Sumita (1982), Milgrom (1981), Milgrom and Weber (1982), Simons (1980), Whitt (1980), (1982) and references cited there.

We begin in Section 2 by giving background on several standard variability orderings. In Section 3 we introduce our notion UCVO which corresponds to the ratio of the densities being unimodal, without the densities being stochastically ordered. By relating UCVO to sign-change orderings (Theorem 1), we obtain a very simple proof that UCVO implies standard variability orderings involving the expectation of convex functions.

In Section 4 we discuss a sufficient condition for UCVO, which involves one distribution being log-concave relative to another, but not stochastically ordered. A convenient way to check UCVO is available via what we call the *index of log-concavity*. In fact, the index of log-concavity is even useful for identifying ordinary variability ordering. For a probability mass function f on the integers, the index of log-concavity is the function

$$r_f(k) = f(k)^2/(f(k-1)f(k+1)). \tag{2}$$

For a mass function f, its *support* is $\text{supp}(f) = \{k : f(k) > 0\}$. One probability mass function f with support on a connected set of integers is said to be *log-concave relative* to another g if $\text{supp}(f) \subseteq \text{supp}(g)$ and $r_f(k) \geqq r_g(k)$ for all k in $\text{supp}(f)$. Recall that a single mass function f is log-concave if $r_f(k) \geqq 1$ for all k; see Chapter 5 of Keilson (1979) and references there. We exploit the property that $f(k)/g(k)$ is unimodal if f is log-concave relative to g. We illustrate the ideas in the following elementary example.

Example 1. The index of log-concavity for a binomial mass function f with parameters n and p is easily seen to be $r_f(k) = (k+1)(n-k+1)/(k(n-k))$. For a Poisson mass function g with parameter λ, $r_g(k) = (k+1)/k$. Obviously, $\text{supp}(f) \subseteq \text{supp}(g)$ and $r_f(k) \geqq r_g(k)$ for all k, $0 \leqq k \leqq n$. Hence, f is less than g in UCVO provided that f and g are not stochastically ordered, i.e., provided that $g(0) = \exp(-\lambda) > f(0) = (1-p)^n$. If $np = \lambda$, so that f and g have the same mean, ordinary stochastic order is not possible, so that UCVO holds. UCVO with equal means also implies $f \leqq_v g$ as defined in (1). Moreover, if f_n represents the binomial distribution with parameter n and $np = \lambda$ for all n, then $r_{f_n}(k) \geqq r_{f_{n+1}}(k)$, so that $f_n \leqq_v f_{n+1}$ for all n. In other words, we see that $\{f_n\}$ converges monotonically to g in $\leqq_v$ and thus in distribution too (Theorem 7). If $np \neq \lambda$, then orderings involving all non-decreasing convex functions or all non-increasing convex functions hold. Moreover, because of the stronger UCVO property, the ordering has strong implications for comparing associated conditional distributions, as described in Theorem 2 below. Of course, much is already known about binomial and Poisson distributions; e.g., Anderson and Samuels (1965). However, we believe that UCVO and relative log-concavity provide a convenient characterization of the variability orderings.

Our interest in UCVO and relative log-concavity arose during an investigation of open and closed models for networks of queues. In Section 5 we describe this application; additional results relating open and closed networks of queues are contained in Whitt (1984).

In Section 6 we briefly discuss applications of UCVO to comparisons of renewal-process stationary-excess distributions, providing some extensions to Whitt (1985). In Section 7 we discuss additional properties of UCVO. For example, we show that it is preserved under weak convergence, but not convolution. We conclude in Section 8 by briefly discussing connections to Birnbaum's (1948) early variability ordering and multivariate versions of UCVO.

2. The standard variability orderings

Consider random variables X and Y having absolutely continuous c.d.f.'s F and G with densities or mass functions f and g. (As usual, the mass functions are obtained when there is absolute continuity with respect to the counting measure on the integers.) A common way to compare the variability of F and G is via the sign changes of $g-f$ or $G-F$; for recent treatments and references to earlier work, see Oja (1981), Shaked (1980), (1982), (1984) and Sections 1.3–1.6 of Stoyan (1983). Basic references are Karlin and Novikoff (1963) and Karlin (1968).

Let $S(h)$ be the number of sign changes of the function $h(t)$, which can be properly defined for complicated functions by considering successively finer finite sets of time points. A natural condition on $g-f$ corresponding to g being in some sense more variable than f is

(3) $$S(g-f)=2 \quad \text{with sign sequence } +,\ -,\ +.$$

Condition (3) implies that

(4) $$S(G-F)=1 \quad \text{with sign sequence } +,\ -.$$

If $EX \geqq EY$, then (4) implies that

(5) $$\int_{-\infty}^{t} F(x)dx \leqq \int_{-\infty}^{t} G(x)dx$$

for all t or, equivalently, (1) holds for all non-increasing convex real-valued functions h for which the expectations are defined. Similarly, if $EX \leqq EY$, then (4) implies that

(6) $$\int_t^\infty [1-F(x)]dx \leqq \int_t^\infty [1-G(x)]dx$$

for all t or, equivalently, (1) holds for all non-decreasing convex real-valued functions h for which the expectations are defined.

A case of special interest occurs when $EX = EY$. Then these comparisons involve only greater variability, (5) and (6) are equivalent and (1) holds for all convex functions.

Other variability orderings compare the differences of quantiles. These orderings are expressed via the inverse F^{-1} of the c.d.f. F, defined by

(7) $$F^{-1}(t) = \inf\{x : F(x) > t\}, \qquad 0 < t < 1.$$

The distribution F is said to be *less dispersed* than G, denoted by $F \leqq_{\text{disp}} G$, if

(8) $$F^{-1}(v) - F^{-1}(u) \leqq G^{-1}(v) - G^{-1}(u), \qquad 0 < u < v < 1.$$

It is well known that $F \leqq_{\text{disp}} G$ if and only if $S(F(\cdot - c) - G) \leqq 1$ for every c and, in case of equality, the sign sequence is $-, +$. As a consequence, if X and Y are distributed according to F and G with $F \leqq_{\text{disp}} G$ and $EX + c \leqq EY (EX + c \geqq EY)$, then $Eh(X + c) \leqq Eh(Y)$ for all non-decreasing (non-increasing) convex functions h. In other words, if $X \leqq_{\text{disp}} Y$, then $X \leqq_v Y + c$ for some constant c or, equivalently, $(X - EX) \leqq_v (Y - EY)$. On the other hand, it is easy to construct examples such that $(X - EX) \leqq_v (Y - EY)$ but $S(G - F) > 1$, so that $\leqq_{\text{disp}}$ is strictly stronger. The orderings $X \leqq_{\text{disp}} Y$ and $(X - EX) \leqq_v (Y - EY)$ are interesting in part because they are location-free.

Yet another ordering for positive random variables X and Y is $\log X \leqq_{\text{disp}} \log Y$, which is equivalent to the star-shaped ordering, i.e., $G^{-1}(F(x))/x$ is non-decreasing.

3. Uniform conditional variability ordering

Suppose now that we look at the conditional distributions given that X and Y belong to a set A, where A is chosen so that $P(X \in A) > 0$, $P(Y \in A) > 0$ and the conditional distributions are absolutely continuous with densities or mass functions

(9) $$f_A(t) = f(t)/P(X \in A) \quad \text{and} \quad g_A(t) = g(t)/P(Y \in A), \qquad t \in A.$$

Let X_A and Y_A be the corresponding random variables. It is natural for the set A to be an interval (continuous or discrete), but it is not necessary. For example, A might be the set of even integers when f and g have support on the integers.

It is easy to see that in general properties (3)–(6) do not extend to these conditional distributions. For example, to have (3) extend, we would need to have $S(cg - f) = 2$ with sign sequence $+, -, +$ for arbitrary positive scalar c.

Since (3) does not control the local behavior of g and f, $S(cg-f)$ could be arbitrarily large with (3) holding.

We now introduce conditions implying that (3)–(6) are inherited by the conditional distributions. We say that f is *uniformly conditionally less variable* than g, and write $X \leqq_{uv} Y$ and $f \leqq_{uv} g$, if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$, the ratio $f(t)/g(t)$ is unimodal where the mode is a supremum, but f and g are not stochastically ordered; i..e., we do not have either $F(x) \leqq G(x)$ for all x or $F(x) \geqq G(x)$ for all x. Just as the conditioning event A need not be an interval, $\mathrm{supp}(g)$ need not be an interval. We understand unimodality of the function $f(t)/g(t)$ to be for t restricted to $\mathrm{supp}(g)$.

It is elementary that the unimodality is preserved under scalar multiplication of the densities and under restriction to a subset, so that it extends to $f_A(t)/g_A(t)$. It is of course possible for the ratio $f_A(t)/g_A(t)$ to be monotone, which corresponds to monotone likelihood ratio order, here denoted by $\leqq_{mr}$; see Keilson and Sumita (1982), Whitt (1980) and references there. The following theorem characterizes UCVO in terms of sign changes.

Theorem 1. Assume that $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. $f \leqq_{uv} g$ if and only if $S(cf-g) \leqq 2$ for all positive constants c with equality for $c=1$ and, in the case of equality, the sign sequence is $-$, $+$, $-$.

Proof. First, suppose that $f \leqq_{uv} g$. Since f/g is unimodal with the mode yielding a supremum, so is cf/g for $c>0$. Hence, $S(cf-g)=S([cf/g]-1) \leqq 2$ and the sign sequence is $-$, $+$, $-$ when $S(cf-g)=2$. It is easy to see that f and g would be stochastically ordered if $S(f-g)<2$ so that $S(f-g)=2$.

Next, suppose that $f \not\leqq_{uv} g$. If $S(f-g)<2$ or if $S(f-g)=2$ but $f-g$ has the wrong sign sequence, then the other condition fails too, so it suffices to assume that $S(f-g)=2$ with sign sequence $-$, $+$, $-$, but that $h=f/g$ is not unimodal. Then there are three points $t_1<t_2<t_3$ such that $h(t_1)>h(t_2)<h(t_3)$. Let c be a constant such that $h(t_2)<c^{-1}<\min\{h(t_1),h(t_3)\}$. Then either $S(h-c^{-1})>2$ or $S(h-c^{-1})=2$ with sign sequence $+$, $-$, $+$. However, $h-c^{-1}$ has the same number of sign changes and the same sign sequence as $cf-g$.

The following theorem summarizes the implications of the $\leqq_{uv}$ order for stochastic comparisons of the conditional distributions. We omit the elementary proof. Let $X \overset{d}{=} Y$ mean that X and Y have the same distribution.

Theorem 2. If $X \leqq_{uv} Y$, then for each subset A in the support of Y one and only one of the following holds:

(i) f/g is constant on A, so that $X_A \overset{d}{=} Y_A$;
(ii) f/g is increasing on A, so that $X_A \geqq_{mr} Y_A$;
(iii) f/g is decreasing on A, so that $X_A \leqq_{mr} Y_A$;

(iv) f/g is unimodal on A but not monotone, and $S(g_A - f_A) = 1$ with sign sequence $+$, $-$, so that $X_A \geqq_{st} Y_A$;

(v) f/g is unimodal on A but not monotone, and $S(g_A - f_A) = 1$ with sign sequence $-$, $+$, so that $X_A \leqq_{st} Y_A$;

(vi) f/g is unimodal on A but not monotone, and $S(g_A - f_A) = 2$ with sign sequence $+$, $-$, $+$, so that $X_A \leqq_{uv} Y_A$.

Of course, only Cases (i)–(iii) are possible in Theorem 2 if A falls entirely on one side of the mode of f/g. Only Case (vi) corresponds to our intuitive idea of how we would like UCVO to behave, i.e., to preserve variability ordering, but of course the other cases cannot be ruled out. It is often easy to check if Case (vi) prevails. Suppose that the subset A has minimal and maximal elements a_1 and a_2. Given that $X \leqq_{uv} Y$, we simply need to have

$$f(a_i)/P(X \in A) < g(a_i)/P(Y \in A)$$

for $i = 1, 2$.

Corollary 1. Suppose that $X \leqq_{uv} Y$.

(a) If $E(X \mid X \in A) \leqq E(Y \mid Y \in A)$, then

$$E(h(X) \mid X \in A) \leqq E(h(Y) \mid Y \in A)$$

for all non-decreasing convex h.

(b) If $E(X \mid X \in A) \geqq E(Y \mid Y \in A)$, then

$$E(h(X) \mid X \in A) \leqq E(h(Y) \mid Y \in A)$$

for all non-increasing convex h.

(c) If $E(X \mid X \in A) = E(Y \mid Y \in A)$, then

$$E(h(X) \mid X \in A) \leqq E(h(Y) \mid Y \in A)$$

for all convex h.

Remark. Theorem 2 and Corollary 1 remain valid if we condition on $X \in A$ and $Y \in B$ where $A \subseteq B$. Then the support of X_A is contained in the support of Y_B.

It is significant that UCVO is not a transitive order. As we show below, it is possible to have $f \leqq_{uv} g$ and $g \leqq_{uv} h$, but not $f \leqq_{uv} h$. (However, transitivity does hold for the relative log-concavity in Section 4.)

Example 2. Let f, g and h be probability mass functions on $\{0, 1, 2, 3, 4\}$ with $f(0) = f(1) = f(2) = 0.19$, $f(3) = 0.3$ and $f(4) = 0.13$; $g(k) = 0.2$ for all k; and $h(0) = h(2) = h(3) = 0.21$, $h(1) = 0.1$ and $h(4) = 0.27$. It is easy to see that $f \leqq_{uv} g$ and $g \leqq_{uv} h$, but not $f \leqq_{uv} h$. Here $S(g - f) = 2$, $S(h - g) = 2$ and $S(h - f) = 4$.

In Section 2 we observed that the ordering $\leqq_{\text{disp}}$ in (8) is stronger than $\leqq_{\text{v}}$ in (1). We now show that the orderings $\leqq_{\text{disp}}$ and $\leqq_{\text{uv}}$ are not comparable.

Example 3. First to show that $\leqq_{\text{uv}}$ does not imply $\leqq_{\text{disp}}$, let f and g be densities with support on $[0,1]$, defined by

$$\begin{aligned} &f(t)=1/3<2/3=g(t), && 0\leqq t\leqq 1/3,\\ &f(t)=2>4/3=g(t), && 1/3<t\leqq 2/3,\\ &f(t)=2/3<1=g(t), && 2/3<t\leqq 1. \end{aligned} \tag{10}$$

Obviously, $f(t)/g(t)$ is unimodal with $S(f-g)=2$, so that $f\leqq_{\text{uv}} g$. The c.d.f.'s F and G associated with f and g have inverses F^{-1} and G^{-1} defined by (7). These inverses F^{-1} and G^{-1} in turn are absolutely continuous with densities, say, f^{-1} and g^{-1}. Clearly, (8) holds if and only if $f^{-1}(t)\leqq g^{-1}(t)$ for all t. However, the ratio $f^{-1}(t)/g^{-1}(t)$ assumes the values 2, 1/3, 2/3, 1/2, and 3/2 over successive subintervals of $[0,1]$. Since we do not have $f^{-1}(t)\leqq g^{-1}(t)$ for all t, we do not have $F\leqq_{\text{disp}} G$. Also note that f^{-1}/g^{-1} is not unimodal.

To show that $\leqq_{\text{disp}}$ does not imply $\leqq_{\text{uv}}$, we can use the same example with a slight modification if we change the roles of (f,g) and (f^{-1},g^{-1}). In particular, instead of (10) above, let

$$\begin{aligned} &f^{-1}(t)=1/3<2/3=g^{-1}(t), && 0\leqq t\leqq 1/3,\\ &f^{-1}(t)=1<4/3=g^{-1}(t), && 1/3<t\leqq 2/3,\\ &f^{-1}(t)=2/3<1=g^{-1}(t), && 2/3<t<1, \end{aligned} \tag{11}$$

so that the pair (f^{-1},g^{-1}) is the same as (f,g) in (10), except $f^{-1}(t)=1$ instead of 2 for $1/3<t\leqq 2/3$. The functions f^{-1} and g^{-1} in (11) determine F^{-1} and G^{-1} by integration and then F and G by inversion. Since $F^{-1}(1)=2/3$, the support of F is now $[0,2/3]$. From (11) it is clear that (8) holds. However, $f(t)/g(t)$ assumes the values 2, 2/3, 4/3, 2 and 0 over successive subintervals of $[0,1]$, so that $S(f-g)=3$. In this case $S(F-G)=0$, i.e., $F(t)\geqq G(t)$ for all t.

4. Relative log-concavity

A convenient sufficient condition for f/g to be unimodal is for f/g to be log-concave; see Chapter 5 of Keilson (1979). For the discussion of log-concavity, we assume that the supports of f and g are subintervals of the real line (or connected sets of integers in the case of mass functions). A density f is log-concave if

$$f(\lambda x+(1-\lambda)y)\geqq f(x)^{\lambda}f(y)^{1-\lambda},\qquad 0\leqq\lambda\leqq 1, \tag{12}$$

for all x and y in the domain or, equivalently, if

$$f(y_2 - x_2)f(y_1 - x_1) \geqq f(y_2 - x_1)f(y_1 - x_2) \tag{13}$$

for $x_1 < x_2$ and $y_1 < y_2$. A probability mass function f is log-concave if $r_f(k) \geqq 1$ for r_f in (2). We say that one density or mass function f is *log-concave relative* to another g, and write $f \leqq_{lc} g$, if f/g is log-concave. This holds for densities if the support of f is a subset of the support of g and $r_f(x_1, x_2, y_1, y_2) \geqq r_g(x_1, x_2, y_1, y_2)$ for all $x_1 < x_2$ and $y_1 < y_2$, where r_f is the *index of log-concavity*, defined by

$$r_f(x_1, x_2, y_1, y_2) = \frac{f(y_2 - x_2)f(y_1 - x_1)}{f(y_2 - x_1)f(y_1 - x_2)}. \tag{14}$$

For probability mass functions, f is log-concave relative to g if the support of f is a subset of the support of g and $r_f(k) \geqq r_g(k)$ for all k, where r_f is defined in (2).

The key fact is that $f \leqq_{uv} g$ when $f \leqq_{lc} g$ and $S(f - g) = 2$. It is also significant that, unlike $\leqq_{uv}$, $\leqq_{lc}$ is transitive.

Note that a density on $[0, \infty)$ is log-concave if and only if it is log-concave relative to an exponential density. Similarly, a probability mass function on the non-negative integers is log-concave if and only if it is log-concave relative to a geometric distribution. For intervals $[a, b]$ or connected sets of integers $\{k, k+1, \cdots, k+m\}$, the reference distribution is uniform. Hence, we have the following relationship among classes of distributions:

$$\text{log-concave} \leqq_{lc} \left\{ \begin{array}{l} \text{exponential on } [0, \infty), \\ \text{geometric on } \{k\colon k \geqq 0\}, \\ \text{uniform on } [a, b] \text{ or on} \\ \quad \{k, k+1, \cdots, k+m\} \end{array} \right\} \leqq_{lc} \text{log-convex.}$$

Moreover, log-concavity is preserved under convolution and log-convexity by mixtures; see Keilson (1979).

Example 4. If f is the *normal* density with mean μ and variance σ^2, then its index of log-concavity is

$$r_f(x_1, x_2, y_1, y_2) = \exp(-\{(y_2 - x_2 - \mu)^2 + (y_1 - x_1 - \mu)^2 - (y_2 - x_1 - \mu)^2 - (y_1 - x_2 - \mu)^2\}/2\sigma^2).$$

If g is normally distributed with the same mean and a larger variance, then f/g is log-concave and $S(f - g) = 2$, so that $f \leqq_{uv} g$.

Example 5. We consider three mass functions with finite support. The *hypergeometric* mass function

$$f(k) = \binom{N_1}{k}\binom{N_2}{K-k} \Big/ \binom{N_1 + N_2}{K} \tag{15}$$

has support on the set of integers between $(K - N_2)^+$ and $\min\{N_1, K\}$. It has index of concavity

$$r_f(k) = \frac{(k+1)(N_1 - k + 1)(N_2 - K + k + 1)(K - k + 1)}{k(N_1 - k)(N_2 - K + k)(K - k)} \tag{16}$$

for $(K - N_2)^+ < k < \min\{N_1, K\}$.

The *binomial* mass function with parameters N_3 and p has support on the set of integers $\{0, 1, \cdots, N_3\}$ and has index of log-concavity

$$r_f(k) = \frac{(k+1)(N_3 - k + 1)}{k(N_3 - k)}, \qquad 1 \leqq k \leqq N_3 - 1. \tag{17}$$

The *uniform* mass function on $\{0, 1, \cdots, N_4\}$ has index of log-concavity $r_f(k) = 1$. With $N_1 \leqq N_3 \leqq N_4$, we have

$$\text{hypergeometric} \leqq_{\text{lc}} \text{binomial} \leqq_{\text{lc}} \text{uniform}. \tag{18}$$

Example 6. We now consider several mass functions with support on the non-negative integers. The *Poisson* mass function has index of log-concavity $(k+1)/k$; the *negative binomial* $(k+1)(N+k-1)/k(N+k)$; and the *geometric* 1. Any mixture of geometrics is log-convex, so has index of log-concavity $r_f(k) \leqq 1$. Hence, we have

$$\begin{aligned} &\text{binomial} \leqq_{\text{lc}} \text{Poisson} \leqq_{\text{lc}} \text{negative binomial} \leqq_{\text{lc}} \text{geometric} \\ &\qquad \leqq_{\text{lc}} \text{mixture of geometrics}. \end{aligned} \tag{19}$$

A sufficient condition for f/g to be log-concave is for f/g to be concave. This is illustrated in the following example involving mixtures from exponential families, which is taken from Shaked (1980).

Example 7. Consider the density $f(x, \theta) = \exp(\psi(\theta)x + \chi(\theta))$ which depends on the parameter θ. Let $g_\mu(x)$ be the mixture of $f(x, \theta)$ with respect to a probability distribution μ on the parameter space. It is elementary that $f(x, \theta)$ is convex in x. Since

$$g_\mu(x)/f(x, \theta) = \int \exp\{[\psi(t) - \psi(\theta)]x + [\chi(t) - \chi(\theta)]\} d\mu(t), \tag{20}$$

$g_\mu(x)/f(x, \theta)$ is also convex in x for any θ. Hence, $f(x, \theta)/g_\mu(x)$ is concave and thus log-concave. Shaked (1980) notes that $g_\mu(x)$ is often approximated by $f(x, \bar{\theta})$ where $\bar{\theta}$ is the expected value of μ, and indicates that $f(x, \bar{\theta})$ is less variable than g_μ when they have the same mean. We have seen that $f(\cdot, \theta) \leqq_{\text{uv}} g_\mu$ or there is stochastic order for all μ and θ. See Shaked (1980) for a variety of applications.

5. Comparing open and closed networks of queues

The concept of relative log-concavity introduced in Section 4 is very useful to compare open and closed models for networks of queues. In this section we briefly describe the main results; more discussion appears in Whitt (1984).

We consider the standard Markovian Jackson open network of first-come–first-served service centers with one customer class, as described in Jackson (1963), Kelly (1979) and Sauer and Chandy (1981). There are n service centers and a single external Poisson arrival process with rate λ. Each arrival goes initially to service center i with probability r_i, independent of the history of the system. Each customer completing service at service center i goes immediately to service center j with probability q_{ij}, independent of the history of the system. The customer departs from the network with probability $1-\Sigma_{j=1}^{n} q_{ij}$. The service rate at service center i is $\mu_i(k)$ when there are k customers at service center i. We assume that $\mu_i(k)$ is non-decreasing in k. The special case in which $\mu_i(k)=\min\{k\mu_i, s\mu_i\}$ corresponds to the standard service center with s servers working in parallel, each with exponential service times having mean μ_i^{-1}.

We assume that the system is stable; i.e., there exists a proper equilibrium distribution for the number of customers at each service center. Let N_i^o denote the number of customers at node i in equilibrium. An important property of this open Jackson network is that the distribution of the vector $(N_1^o, \cdots, N_n^o)$ has independent marginals (is of product form), i.e.,

$$P((N_1^o, \cdots, N_n^o) = (k_1, \cdots, k_n)) = \prod_{i=1}^{n} P(N_i^o = k_i). \tag{21}$$

Moreover, the distribution of N_i^o is the distribution of a birth-and-death process on the non-negative integers with constant birth rate λ_i and death rates $\mu_i(k)$, where λ_i is the solution of the basic system of traffic rate equations

$$\lambda_j = \lambda r_j + \sum_{i=1}^{n} q_{ij}\lambda_i. \tag{22}$$

The related Jackson closed model is obtained by eliminating the external arrival process and the possibility of departures ($\Sigma_{j=1}^{n} q_{ij} = 1$ for all i) and having a fixed number K of customers in the network. It is significant that the equilibrium distribution for this closed model can be conveniently expressed in terms of an associated open model. Any service center in the closed model is selected and its arrivals are made to leave the network and are replaced by an external Poisson arrival process with rate λ sufficiently small to ensure stability. (Otherwise, the rate λ is unspecified.) Even though the original arrival process to the designated service center in the closed network is typically not a Poisson process, it turns out that the equilibrium distribution of the number of customers at each service center in the closed model is just the conditional distribution associated with the equilibrium distribution for the open model in (21) given that the total number of customers in the open network is K. (This is well known.) Let N_i^c be the equilibrium number of customers at service center i in the closed model. Then the basic relation between the equilibrium distributions in the open and closed models is

(23) $P(N_1^c = k_1, \cdots, N_n^c = k_n) = P(N_1^o = k_1, \cdots, N_n^o = k_n)/P(N_1^o + \cdots + N_n^o = K)$
where $k_1 + \cdots + k_n = K$.

The equilibrium distribution for the closed model in (23) tends to be much harder to compute than (21) because of the normalization by the conditioning. In Whitt (1984) we propose approximating the distribution of $(N_1^c, \cdots, N_n^c)$ by the distribution of $(N_1^o, \cdots, N_n^o)$ for the associated open model, under the condition that $E(N_1^o + \cdots + N_n^o) = K$. To understand how this fixed-population-mean (FPM) method works, it is useful to make stochastic comparisons between the open and closed models. The key result establishes that N_i^c is log-concave relative to N_i^o for each i for any stable external arrival rate in the open model.

Theorem 3. The distributions of N_i^o and N_i^c are both log-concave and $N_i^c \leqq_{\text{lc}} N_i^o$ for each i.

Proof. The same argument applies to all service centers, so let $i = 1$. Apply (23) to obtain

$$(24) \qquad \frac{P(N_1^c = k+1)}{P(N_1^c = k)} = \frac{P(N_1^o = k+1)P(N_2^o + \cdots + N_n^o = K - k - 1)}{P(N_1^o = k)P(N_2^o + \cdots + N_n^o = K - k)}.$$

Since N_i^o has the distribution of a birth-and-death process with non-increasing birth rates and non-decreasing death rates, the mass function $P(N_i^o = k)$ is log-concave; Example 5.7F in Keilson (1979). Since log-concavity is preserved under convolution, the mass function $P(N_2^o + \cdots + N_n^o = k)$ is also log-concave. Since the right side of (24) is the product of two ratios, both decreasing in k, $P(N_i^c = k)$ is log-concave. Moreover,

$$P(N_1^c = k+1)P(N_1^o = k)/P(N_1^c = k)P(N_1^o = k+1)$$

is decreasing in k, so that $N_1^c \leqq_{\text{lc}} N_1^o$.

Corollary 2. If $EN_i^o \leqq EN_i^c$, then $Eh(N_i^o) \leqq Eh(N_i^c)$ for all non-decreasing concave real-valued functions h.

Proof. Apply Corollary 1.

Now consider the special case in which each service center has several exponential servers, so that $\mu_i(k) = \min\{k\mu_i, s_i\mu_i\}$. Then the utilization at service center i, denoted by u_i^o and u_i^c for the open and closed models, is the expected number of busy servers there in equilibrium, i.e.,

$$(25) \qquad u_i^o = E(\min\{N_i^o, s_i\}) \quad \text{and} \quad u_i^c = E(\min\{N_i^c, s_i\}).$$

The following result shows that the FPM method always produces lower bounds on the utilizations in the closed model.

Theorem 4. If $E(N_1^o + \cdots + N_n^o) \leqq N_1^c + \cdots + N_n^c = K$, then $u_i^o \leqq u_i^c$ for all i.

Proof. The condition on the total population implies that $EN_i^o \leqq EN_i^c$ for some i. Since u_i^o and u_i^c are realized as the same non-decreasing concave function h of N_i^o and N_i^c, respectively, as given in (25), $u_i^o \leqq u_i^c$ by Corollary 2. Finally, it is well known that $u_i^o/u_j^o = u_i^c/u_j^c$ for all i and j because $n-1$ of the two systems of n traffic rate equations are identical.

Remarks. (1) For the case of single-server and infinite-server service centers only, Theorem 4 was first proved directly by Zahorjan (1983).

(2) As indicated in Section 4 of Whitt (1984), Theorems 3 and 4 extend to multiple customer classes.

6. Renewal-process stationary-excess distributions

For any density f with c.d.f. F and mean μ, the associated renewal-process stationary-excess distribution has density $s(f) = (1 - F(t))/\mu$, $t \geqq 0$. Stochastic comparisons involving f and $s(f)$ are discussed in Whitt (1985). An elementary comparison result (Theorem 3.3 there) is

$$F \leqq_{\mathrm{v}} G \quad \text{if and only if } s(F) \leqq_{\mathrm{st}} s(G). \tag{26}$$

The concept of UCVO provides some elementary extensions.

Theorem 5. (a) If $f \leqq_{\mathrm{uv}} g$, then $s(f)/s(g)$ is unimodal, and $s(f) - s(g)$ has the sign sequence $-, +, -$ if $S(s(f) - s(g)) = 2$.

(b) $f/(\mu s(f))$ is the failure rate function, so that $f/s(f)$ is unimodal, monotone, etc. if and only if the failure rate is.

Proof. (a) By Theorem 1, $S(af - bg) \leqq 2$ for all positive a and b. Let $\bar{F}(t) = 1 - F(t)$. After integrating, $S(a\bar{F} - b\bar{G}) \leqq 2$, with the same sign sequence in case of equality. Since $s(f)(t) = \bar{F}(t)/\mu(f)$, $S(cs(f) - s(g)) \leqq 2$ for all positive c, which implies that $s(f)/s(g)$ is unimodal, by Theorem 1 again. (b) The failure rate is defined as $f(t)/\bar{F}(t)$.

Corollary 3. If the failure rate, say ρ, associated with a density f on $[0, \infty)$ is unimodal and $S(\mu\rho - 1) = 2$ with sign sequence $-, +, -$ $(+, -, +)$, then $f \leqq_{\mathrm{uv}} s(f)$ $(f \geqq_{\mathrm{uv}} s(f))$.

Proof. Apply Theorem 5(b).

7. Additional properties

It is easy to see that $X \leqq_{\mathrm{disp}} aX$ for $a > 1$. This ordering holds in UCVO for certain densities, but not all densities.

Theorem 6. Suppose $a > 1$. If X has a density f such that $f(e^t)$ is log-concave for $t > 0$ and $f(-e^t)$ is log-concave for $t < 0$, then the ratio $f_{aX}(t)/f_X(t)$ is non-decreasing away from the origin, so that $X \leqq_{\mathrm{uv}} aX$.

Proof. Since $f_{aX}(t) = f(t/a)/a$, it suffices to prove that $f(t/a)/f(t)$ is non-decreasing away from the origin or, equivalently,

$$\frac{f(t_1/a)}{f(t_1)} < \frac{f(t_2/a)}{f(t_2)}, \qquad 0 < t_1 < t_2,$$

which is equivalent to

$$\frac{g(\log t_2 - \log a)g(\log t_1)}{g(\log t_2)g(\log t_1 - \log a)} \geqq 1$$

for $g(t) = f(e^t)$. This corresponds to (11) with $y_2 = \log t_2$, $x_2 = \log a$, $y_1 = \log t_1$ and $x_1 = 0$.

Remark. For X with support in $(0, \infty)$, the condition in Theorem 6 corresponds to $\log X$ having a log-concave density.

The following relates convergence in distribution to UCVO. We omit the elementary proof.

Theorem 7. Let f, g and $f_n, g_n, n \geqq 1$, be probability mass functions with support on the non-negative integers. Then

(a) $f_n(k) \to f(k)$ as $n \to \infty$ for all k (weak convergence) if and only if $f_n(0) \to f(0)$, $f_n(1) \to f(1)$ and $r_{f_n}(k) \to r_f(k)$, $k \geqq 1$.

(b) If $f_n \leqq_{uv} g_n$ for all n, $f_n(k) \to f(k)$ and $g_n(k) \to g(k)$ as $n \to \infty$ for all k, then $f \leqq_{uv} g$.

We have the following result about mixtures.

Theorem 8. If $f \leqq_{uv} g$, then $f \leqq_{uv} pf + (1-p)g \leqq_{uv} g$, $0 < p < 1$.

Proof. Consider the first inequality. Since

$$[pf + (1-p)g]/f = p + (1-p)(g/f),$$

unimodality is preserved. Moreover,

$$S([pf + (1-p)g] - f) = S((1-p)g - (1-p)f) = S(g - f).$$

If f and g are mass functions with $\text{supp}(f) = \{k, k+1\} \subseteq \text{supp}(g)$, then $f \leqq_{uv} g$ provided that f and g are not stochastically ordered. A sufficient condition for f and g not to be stochastically ordered is for $\text{supp}(g)$ to contain at least one point less than k and one point greater than $k+1$. This case is convenient for constructing counterexamples. For example, $f \leqq_{uv} g$, $f \leqq_{uv} h$ and $g \leqq_{uv} h$ together do not imply that $pf + (1-p)g \leqq_{uv} h$. They also do not imply that $f * g \leqq_{uv} h$, $f * h \leqq_{uv} h * h$ or $f * h \leqq_{uv} g * h$, where $*$ denotes convolution.

The following example shows that the $\leqq_{uv}$ and $\leqq_{lc}$ orderings are not preserved under convolutions, even for distributions with common support.

Example 8. We need not have $f * h \leqq_{uv} g * h$ ($*$ denoting convolution) when $f \leqq_{uv} g$, even when f, g and h are symmetric and log-concave with common support (so that $f * h$ and $g * h$ are log-concave) and $f \leqq_{lc} g \leqq_{lc} h$ (so that $f \leqq_{v} g \leqq_{v} h$). To see this, let

$$f(0) = f(2) = 0.1, \quad f(1) = 0.8;$$

$$g(0) = g(2) = 0.2, \quad g(1) = 0.6;$$

$$h(0) = h(2) = 0.3, \quad h(1) = 0.4.$$

Then $(f * h)$ has masses 0.03, 0.28, 0.38, 0.28, 0.03 on $\{0, 1, 2, 3, 4\}$ while $(g * h)$ has masses 0.06, 0.26, 0.36, 0.26, 0.06. Hence $r_{f*h}(1) = 6.871 > 3.12 = r_{g*h}(1)$ as required, but $r_{f*h}(2) = 1.84 < 1.92 = r_{g*h}(2)$. Moreover, the ratio $(f * h)/(g * h)$ is not unimodal.

8. Concluding remarks

Birnbaum (1948) suggested a variability ordering in which X is said to be less variable (or more peaked) about x than Y about y if

$$|X - x| \leqq_{st} |Y - y| \tag{27}$$

where $\leqq_{st}$ is the usual stochastic order. Paralleling UCSO, it is natural to define stronger variability orderings by replacing $\leqq_{st}$ in (27) by the monotone likelihood ratio ordering $\leqq_{mr}$. If X and Y have distributions that are symmetric about a common point, then $X \leqq_{uv} Y$ corresponds exactly to (27) with $\leqq_{st}$ replaced by $\leqq_{mr}$.

A natural multivariate generalization of UCVO for random vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ is to require that there exists $x = (x_1, \cdots, x_n)$ such that

$$[(X_1 - x_1)^+, \cdots, (X_n - x_n)^+ \mid X \geqq x] \leqq_{mr} [(Y_1 - x)^+, \cdots, (Y_n - x_n)^+ \mid Y \geqq x] \tag{28}$$

after making any of the 2^n possible coordinate sign changes about the new origin x, where now $\leqq_{mr}$ is the *multivariate monotone likelihood ratio ordering* discussed in Karlin and Rinott (1980) and Whitt (1982) and references there. Of course, for $n = 1$, (28) is equivalent to UCVO.

However, a major difficulty in the multivariate case is that there seems to be no natural extension of Section 2, so that we do not automatically get (1) for a large class of functions h on R^n from (28).

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