MANAGEMENT SCIENCE
Vol. 25 No. 6, June 1979
Printed in U.S.A.

COMPARING PROBABILITY MEASURES ON A SET WITH AN INTRANSITIVE PREFERENCE RELATION*

WARD WHITT†

Recently Fishburn (1978) proposed a definition of stochastic dominance for probability measures defined on a set with an intransitive preference-or-indifference relation. A significant feature of Fishburn's definition is that the stochastic dominance relation inherits the intransitivity. We propose several alternative definitions of stochastic dominance that are transitive even though the preference relation on the underlying sample space is not. The basic idea is to use ordinary first-order stochastic dominance after appropriately transforming the intransitive relation into a transitive relation. As extreme cases or bounds, strong (weak) stochastic dominance is defined to be ordinary stochastic dominance for probability measures defined on the same set after the intransitive relation is replaced by the transitive relation obtained by regarding any two alternatives that were connected by an intransitive cycle as noncomparable (equivalent). Fishburn's definition is shown to be included within these bounds.

(STOCHASTIC DOMINANCE; INTRANSITIVE PREFERENCES)

1. Introduction and Summary

The purpose of this paper is to discuss ways to compare actions with uncertain outcomes when the possible outcomes are ordered by an intransitive preference-or-indifference relation. Intransitive preference relations commonly arise in group decision making. For example, if each member of a group ranks three alternatives x, y, and z, then it is possible for the majority to simultaneously prefer x to y, y to z, and z to x. For instance, this occurs with three people and the rankings (x, y, z), (y, z, x) and (z, x, y). In fact, this phenomenon occurs throughout the theory of social choice; see Fishburn [2]. Intransitive preferences can also arise in the empirical estimation of individual preferences. In order to discover an individual's preferences, a consultant can administer a battery of test questions requiring the individual to make pairwise comparisons. Unfortunately, this can result in intransitivity. Of course, intransitivity is only likely to arise with complex outcomes that cannot easily be represented in monetary terms, e.g., in multiple criteria decision making. When intransitivities do occur, the "rational" approach is to confront the individual with these "inconsistencies" in the expressed preferences so that the individual can make adjustments toward transitivity.

However, for comparing probability distributions on this set of possible outcomes, such adjustments may be either difficult or unnecessary. They may be difficult because the decision maker may not be accessible or the decision maker may not adjust easily. They may be unnecessary because it may be possible to make useful comparisons between probability distributions without actually eliminating the intransitivity. For example, suppose that one probability distribution would be preferred to another for all transitive complete preference-or-indifference relations that could result from "reasonable" adjustments toward transitivity. Then it seems appropriate to say the two probability distributions are ordered without making any adjustment. Similarly, if the two measures are unordered for all transitive complete preference-or-indifference relations that could result from such adjustments, then it seems appropriate to say the two measures are unordered without making any adjustment.

* Accepted by John C. Cox, former Departmental Editor; received March 14, 1978. This paper has been with the author 2 weeks for 1 revision.
† Bell Laboratories, Holmdel, New Jersey.

0025-1909/79/2506/0505$01.25

This idea leads us to define two notions of stochastic dominance for probability measures on a set endowed with an intransitive preference-or-indifference relation. (Definitions will be given in full detail in §3.) First, we say adjusting or changing a comparison between two alternatives in the underlying set is *consistent* if transitivity is inconsistent with the original relation, i.e., if the two alternatives with their original ordering are part of an intransitive cycle. Then, we say *strong stochastic dominance* holds if ordinary stochastic dominance holds for all transitive complete preference-or-indifference relations that could be obtained on the underlying space of possible outcomes by consistent adjustments; we say *weak stochastic dominance* holds if ordinary stochastic dominance holds for at least one transitive complete preference-or-indifference relation that could be obtained on the underlying space of possible outcomes from consistent adjustments. In this paper, we investigate these concepts further. In particular, we provide complete definitions and additional characterizations in §3, where we also introduce additional definitions of stochastic dominance that fall between strong and weak stochastic dominance.

Our method for comparing probability measures on a set ordered by an intransitive preference relation is motivated by a different definition recently proposed by Fishburn [3]. Fishburn's definition, which we discuss in §2, is a fascinating concept, but its subtlety may make it difficult to understand and apply. Moreover, as we shall show in §2, Fishburn's [3] stochastic dominance relation is not transitive. We believe that in many circumstances it is desirable to have the stochastic dominance relation be transitive even if the preference relation on the underlying space is intransitive. This permits a "rational" comparison between probability measures even if there are "inconsistencies" in the basic preferences.

Moreover, our approach not only produces a transitive stochastic dominance relation, but it also applies to relations on the underlying sample space which are not reflexive or complete, see §2. Finally, even in situations where Fishburn's definition is appropriate, strong and weak stochastic dominance are of interest because they serve as computationally tractable bounds for Fishburn's definition; see Theorem 6.

2. Fishburn's Definition

We use the notation in Fishburn [3]. In particular, let X be the set of all possible outcomes; let R be a reflexive and complete (but not necessarily transitive) preference-or-indifference relation on X, with xRy meaning x is "weakly better" than y; and let I and P be the associated indifference and strict preference relations induced by R, i.e., xIy if xRy and yRx, and xPy if xRy and not yRx. The relation R is a subset of the Cartesian product X^2. Reflexive means xRx for all $x \in X$, and complete means either xRy or yRx or possibly both for all $x, y \in X$. This implies that exactly one of xIy, xPy or yPx holds for all $x, y \in X$. Let $Px = \{y \in X : yPx\}$, $xP = \{y \in X : xPy\}$, $Rx = \{y \in X : yRx\}$, $xR = \{y \in X : xRy\}$, $Ix = \{y \in X : yIx\}$ and $xI = \{y \in X : xIy\}$. Note that $Ix = xI$ for all $x \in X$. Let $\Pi \equiv \Pi(X)$ be the set of all simple probability measures on X, i.e., if $p \in \Pi$, then $p(A) = 1$ for some finite subset A of X. Let the standard notion of (first-order) stochastic dominance based on R be denoted by $SD(R)$ and defined by

$$pSD(R)q \quad \text{if } p(Rx) \geq q(Rx) \quad \text{for all } x \in X. \tag{1}$$

Of course, $SD(R)$ is usually used under the assumption that R is transitive, but it could be used more generally. However, Fishburn defines the following two stochastic dominance relations for probability measures p and q in Π:

$$pD_1(R)q \quad \text{if } p(\{x : q(xP) < \lambda\}) \leq q(\{x : p(xP) < \lambda\})$$

for all λ, and

$$pD_2(R)q \quad \text{if}\, p(\{x : q(Px) > \lambda\}) < q(\{x : p(Px) > \lambda\})$$

for all λ. Fishburn shows that $D_1(R)$ and $D_2(R)$ agree if the relation R is transitive. Moreover, in that case they coincide with the usual definition of stochastic dominance $SD(R)$ in (1). The definition of stochastic dominance actually proposed by Fishburn is denoted by $D(R)$ and is defined by

$$pD(R)q \quad \text{if}\, pD_1(R)q \quad \text{and}\, pD_2(R)q.$$

Fishburn also shows that $D(R)$ satisfies nine basic conditions. However, in some situations these conditions seem difficult to accept. They imply that $D(R)$-dominance for p and q depends only on the relation R restricted to the union of the support of p and the support of q (where the support of a simple probability measure p is the set of all points in X to which p assigns strictly positive probability). This condition seems appropriate if only two measures are to be compared, but it seems questionable if more measures are to be compared. In particular, it rules out transitivity. To see this, let p_x denote the probability measure which assigns probability one to the set $\{x\}$ containing the single point x.

THEOREM 1. *$p_xD(R)p_y$ in Π if and only if xRy in X.*

PROOF. Combine conditions C2 and C3 in Fishburn [3]. Condition C2 says that $pD(R)q$ whenever aRb for all pairs (a,b) with a in the support of p and b in the support of q. Hence, we have established: (i) xRy implies that $p_xD(R)p_y$. Condition C3 says that $qD(R)p$ is ruled out if both aRb for all pairs (a,b) with a in the support of p and b in the support of q and bRa fails to hold for all pairs (b,a) with b in the support of q and a in the support of p. Hence, we have established: (ii) if xPy, then $p_yD(R)p_x$ cannot hold. Combining (i) and (ii), we see that xRy if and only if $p_xD(R)p_y$, as claimed.

COROLLARY. *$D(R)$ is transitive if and only if R is transitive.*

PROOF. If R is transitive, $D(R)$ coincides with the ordinary stochastic dominance in (1), which is transitive. If R is not transitive, then there exists x, y and z in X such that (1) xPy, yPz, and zPx or (2) xPy, yPz and zIx or (3) xPy, yIz and zIx. In each case, the same pairwise relations hold for p_x, p_y and p_z, which implies that $D(R)$ is not transitive.

Another issue is that mixtures are not well behaved. By mixtures, we mean the probability measures $\alpha p + (1-\alpha)q$ in Π obtained from each p and q in Π and each real number α with $0 < \alpha < 1$. It is easy to see that none of the three axioms in the expected utility theorem for simple measures, pp. 107–110 of Fishburn [1], are valid for $D(R)$. In particular, suppose p, q and r are three measures in Π. If $pD(R)q$, it does not follow that $[\alpha p + (1-\alpha)r]D(R)[\alpha q + (1-\alpha)r]$, i.e., the independence axiom fails. Moreover, if $pD(R)q$ and $qD(R)r$, it does not follow that either $[\alpha p + (1-\alpha)r]$ $D(R)q$ or $qD(R)$ $[\beta p + (1-\beta)r]$ for any α, $\beta \in (0,1)$, i.e., the Archimedean axiom fails. These properties are easy to check using x, y, and z in X such that xPy, yPz, and zPx; the desired contradictions hold for p_x, p_y and p_z in Π.

3. Transitive Stochastic Dominance Relations

We believe that in many applications it is desirable to have the dominance relation on Π be transitive even when the underlying preference relation R on X is not. Then comparisons which most decision makers would consider rational can be made among the probability measures. A natural way to obtain such transitive relations on Π is to use a standard stochastic dominance relation together with various transitive relations on X generated from the given relation R. The idea is to judiciously assume either that a is related to b when it was not before or that a is not related to b when it was before. However, when we eliminate related pairs from the relation R, then we may violate the completeness assumption, i.e., we may no longer require that every pair of elements be related. This means that we will need to apply notions of stochastic dominance for simple probability measures defined on a space with a partial order relation. Fortunately, a substantial theory of stochastic dominance for probability measures defined on a space with a partial order relation already exists; see Kamae, Krengel and O'Brien [4] and references there. In particular, we use the following partial order relation in Π associated with any relation R on X:

$$pSO(R)q \quad \text{if } \sum_i u(x_i)p(\{x_i\}) > \sum_i u(x_i)q(\{x_i\}) \tag{2}$$

i.e., if $E_p(u) > E_q(u)$, for all nondecreasing real-valued functions u, where the summation is over the supports of p and q respectively. It is well known that $SO(R)$ in (2) is equivalent to $SD(R)$ in (1) when R is transitive and complete, but if R is a partial order, then $SO(R)$ is strictly stronger than $SD(R)$; see pp. 767–771 of Veinott [5]. It is significant that set inclusions for the relations R on X are inherited by the relations $SO(R)$ on Π:

THEOREM 2. *If R_1 and R_2 are two relations on X such that $R_1 \subseteq R_2$, then $SO(R_1) \subseteq SO(R_2)$.*

PROOF. Any real-valued function on X which is nondecreasing with respect to R_2 is nondecreasing with respect to R_1. Thus the class of test functions in the definition of $SO(R)$ is larger for R_1.

It is significant that for our treatment the relation R need not be either reflexive or complete, but for simplicity we shall assume R is both.

We begin by defining a relation C on X by saying xCy if either xIy or there is an inconsistency cycle connecting x and y, i.e., a cycle of binary relations extending from x to y and returning back to x such that transitivity would contradict xIx. More formally, an *inconsistency cycle* connecting x and y is a finite sequence $\{x_1, \ldots, x_n\}$ with $x_1 = x_n = x$ and $x_k = y$ for some k, $1 < k < n$, such that x_iRx_{i+1} for all i and x_iPx_{i+1} for some i. It is easy to verify that C is an equivalence relation (reflexive, symmetric and transitive). As usual, the equivalence relation C induces a partition of equivalence classes in X. An important property is:

THEOREM 3. *Any two distinct C-equivalence classes A_1 and A_2 in X are strictly ordered: with the proper labeling, a_1Pa_2 for all $a_1 \in A_1$ and $a_2 \in A_2$.*

PROOF. First, if $a_1 \in A_1$ and $a_2 \in A_2$, then either a_1Pa_2 or a_2Pa_1 because a_1Ca_2 if a_1Ia_2, which would imply that a_1 and a_2 would belong to the same equivalence class. Suppose a_1Pa_2. We shall show that there is a contradiction if $b_1 \in A_1$, $b_2 \in A_2$ and b_2Rb_1. If so, there would be a cycle from b_2 extending to b_1 and returning back to b_2 contradicting b_2Ib_2: use b_2Rb_1 and a_1Pa_2 plus segments connecting b_1 and a_1 in A_1 and a_2 and b_2 in A_2 so that transitivity would imply b_1Ra_1 and a_2Rb_2. This implies that b_1Cb_2, so that b_1 and b_2 would belong to the same equivalence class, which is a contradiction.

There are actually two kinds of C-equivalence classes. We call an equivalence class A an *indifference class* if a_1Ia_2 for all $a_1, a_2 \in A$; otherwise, we call the equivalence class an *inconsistency class*. It is easy to see that every pair of elements in an inconsistency class is connected by an inconsistency cycle.

We propose to obtain transitive order relations on Π by altering the relation R on X, but in such a way that we change aRb only if a and b belong to a common inconsistency class. The first new relation on X is the *incomplete order* R_e obtained by making any two distinct elements in the same inconsistency class noncomparable, i.e., aR_eb holds if and only If aRb holds and a and b do not belong to a common inconsistency class. (Thus, aR_eb does not hold whenever a and b belong to a common inconsistency class.) The second new relation on X is the *complete order* R_c obtained by making any two elements in the same inconsistency class indifferent, i.e., aR_cb holds if and only if aRb holds or a and b belong to a common inconsistency class. It follows immediately from the definitions that $R_e \subseteq R \subseteq R_c$. It is easy to see that R_c is the transitive completion of R, i.e., R_c is the smallest transitive relation containing R:

THEOREM 4. *Suppose $R \subseteq R' \subseteq X^2$ and R' is transitive, then $R_c \subseteq R'$.*

PROOF. For any two elements a and b in a common C-equivalence class, there exists a finite sequence $\{x_1, \ldots, x_n\}$ with $x_1 = a$, $x_n = b$ and $x_i R x_{i+1}$ for all i, $1 < i < n-1$. Since $R \subseteq R'$, the same is true for R'. Since R' is transitive $aR'b$. Since $aR'b$ holds for all a and b in a common C-equivalence class, $R_c \subseteq R'$.

We regard $SO(R_e)$ and $SO(R_c)$ as two extreme cases or bounds for stochastic dominance in $\Pi(X,R)$. This is justified by

THEOREM 5. (a) $SO(R_e) \subseteq SO(R) = SO(R_c)$.
(b) *If R' is any relation satisfying $R_e \subseteq R' \subseteq R_c$, then $SO(R_e) \subseteq SO(R') \subseteq SO(R_c)$.*

PROOF. (a) Since $R_e \subseteq R \subseteq R_c$, $SO(R_e) \subseteq SO(R) \subseteq SO(R_c)$ by virtue of Theorem 2. To see that $SO(R) = SO(R_c)$, note that to be nondecreasing a real-valued function on (X,R) must be constant on each C-equivalence class. Hence, the set of all nondecreasing real-valued functions on (X,R) coincides with the set of all nondecreasing real-valued functions on (X,R_c).

(b) Apply Theorem 2 again.

Theorem 5(b) provides the connection to the other characterization of stochastic dominance bounds discussed in the introduction. In this other characterization, we consider all possible adjustments that can be made to the original relation R, allowing only consistent changes, i.e., changes for pairs in a common inconsistency class, in order to produce a complete transitive preference-or-indifference relation R_t. We say strong (weak) stochastic dominance holds if $SO(R_t)$-dominance holds for each (at least one) complete transitive relation R_t obtained through such an adjustment. Theorem 5(b) implies the following

COROLLARY. *$SO(R_e)$ is strong stochastic dominance and $SO(R_c)$ is weak stochastic dominance.*

We thus regard $SO(R_e)$ and $SO(R_c)$ as natural bounds for any reasonable definition of stochastic dominance in $\Pi(X,R)$. By this we mean that if $pSO(R_e)q$, then we should have $pS(R)q$ for any other reasonable stochastic dominance relation $S(R)$. Similarly, if $pSO(R_c)q$ fails, then $pS(R)q$ should fail for any other reasonable definition $S(R)$. Hence, we would hope that Fishburn's [3] orderings $D_1(R)$, $D_2(R)$ and $D(R)$ should be contained within our bounds, and this is indeed the case.

THEOREM 6. *$SO(R_e) \subseteq D(R) \subseteq D_j(R) \subseteq SO(R_c)$ for $j = 1,2$.*

PROOF. We first show that $SO(R_e) \subseteq D_1(R)$. For this purpose, suppose $pSO(R_e)q$. Then $p(xP) < q(xP)$ for all $x \in X$ because the function u which is the indicator function of $X - xP$, i.e., which is 0 on xP and 1 on $X - xP$, is nondecreasing on (X,R_e). The definition of $SO(R_e)$ thus implies

$$1 - p(xP) = p(X - xP) = \sum_i u(x_i)p(\{x_i\})$$

$$> \sum_i u(x_i)q(\{x_i\}) = q(X - xP) = 1 - q(xP).$$

As a consequence, $\{x:q(xP) \le \lambda\} \subseteq \{x:p(xP) \le \lambda\}$ for all λ, $0 \le \lambda \le 1$. Hence, $p(\{x:q(xP) \le \lambda\}) \le p(\{x:p(xP) \le \lambda\})$ for all λ, $0 \le \lambda \le 1$. Now note that the function which is the indicator function of the set $\{x:p(xP) > \lambda\}$ is also nondecreasing on (X, R_e). Hence,

$$1 - p(\{x : p(xP) \le \lambda\}) = p(\{x : p(xP) > \lambda\})$$
$$\ge q(\{x : p(xP) > \lambda\}) = 1 - q(\{x : p(xP) \le \lambda\}),$$

so that $p(\{x:q(xP) \le \lambda\}) \le q(\{x:p(xP) \le \lambda\})$ for all λ, $0 \le \lambda \le 1$, which completes this part of the proof. By similar reasoning, $SO(R_e) \subseteq D_2(R)$.

We now show that $D_1(R) \subseteq SO(R_c)$. Suppose $pSO(R_c)q$ fails to hold. This means that $p(A) > q(A)$ where A is the union of all the elements in the k lowest C-equivalence classes for some k (recall Theorem 3). We shall show that $p(\{x:q(xP) \le \lambda\}) \le q(\{x:p(xP) \le \lambda\})$ fails for $\lambda = \alpha$ with $q(A) < \alpha < p(A)$. Since $q(A) < \alpha < p(A)$,

$$\{x : p(xP) \le \alpha\} \subseteq A \subseteq \{x : q(xP) \le \alpha\},$$

so that

$$q(\{x : p(xP) \le \alpha\}) \le q(A) < p(A) \le p(\{x : q(xP) \le \alpha\}).$$

Again, similar reasoning shows that $D_2(R) \subseteq SO(R_c)$.

The bounds $SO(R_e)$ and $SO(R_c)$ provide support for using Fishburn's $D(R)$ instead of ordinary stochastic order $SD(R)$ in (1) because, while $SD(R)$ is clearly transitive and easier to determine, the set inclusion $SD(R) \subseteq SO(R_c)$ can fail, as the following example illustrates.

Example 1. To see that we need not have $SD(R) \subseteq SO(R_c)$, let $X = \{1, 2, \ldots, 6\}$ and $P = \{(1,2), (1,3), (1,6), (2,3), (2,4), (2,6), (3,5), (3,6), (4,1), (4,6), (5,1), (5,6)\}$. Note that there are two C-equivalence classes $\{1, \ldots, 5\}$ and $\{6\}$. Let p and q be two probability measures defined by

$$p(\{2\}) = p(\{4\}) = 1/4, \qquad p(\{6\}) = 1/2,$$
$$q(\{1\}) = q(\{5\}) = 1/8, \qquad q(\{3\}) = 1/3 \quad \text{and} \quad q(\{6\}) = 5/12.$$

Since $p(\{6\}) > q(\{6\})$, we cannot have $pSO(R_c)q$. However, it is easy to verify that $p(Rx) \ge q(Rx)$ for all x, so that $pSD(R)q$.

Stochastic order $pSD(R)q$ has been defined in (1), but it also could have been defined as

$$p(xR) \le q(xR), \text{ for all } x \in X. \tag{3}$$

As illustrated by the previous example, (1) neither implies nor is implied by (3). It is also easy to see, reasoning as in Theorem 6, that $pSO(R_e)q$ implies both (1) and (3). However, we have not yet determined if (1) and (3) together imply $pSO(R_c)q$.

Finally, we mention some additional transitive order relations on X that fall between R_e and R. In particular, we say aR_ub if $Ra \subseteq Rb$, aR_lb if $aR \supseteq bR$, and $a\hat{R}b$ if aR_ub and aR_lb. Clearly R_u, R_l and $\hat{R}$ are all transitive.

Theorem 7. *$R_e \subseteq \hat{R} \subseteq R_u \subseteq R$ and $\hat{R} \subseteq R_l \subseteq R$.*

Proof. Since $a \in Ra$, $a \in Rb$ if $Ra \subseteq Rb$, which implies that aRb if aR_ub. Similarly, aRb if aR_lb. If aR_eb, then aPb and a and b belong to different C-equivalence classes. Hence, $a\hat{R}b$, by Theorem 3.

Corollary. $SO(R_e) \subseteq SO(\hat{R}) \subseteq SO(R_u) \subseteq SO(R)$.

It is easy to see by example that the inclusions $R_e \subseteq \hat{R} \subseteq R_u \subseteq R$ can all be strict, while R_u and R_l need not be related by set inclusion.

Another ordering R^* intermediate between R_e and $\hat{R}$ is obtained by saying aR^*b for a and b in different C-equivalence classes if and only if aRB and by saying aI^*b for a and b in a common C-equivalence class A if aIb and, for every other element c in A, we have aIc and bIc. Otherwise, we say neither aR^*b nor bR^*a holds. In this way, we proceed as in the construction of R_e from R except that we do not remove all relations in an inconsistency class; we only remove those which are part of an inconsistency cycle of length three.

THEOREM 8. *I^* is transitive and $R_e \subseteq R^* \subseteq \hat{R}$.*

PROOF. We consider only the second inclusion. Suppose aR^*b. If $(a,b) \notin R_u$, then there exists c such that cRa but not cRb. Since aRb, this implies that a and b are part of an inconsistency cycle of length three, so that aR^*b fails, which is a contradiction. Hence, $R^* \subseteq R_u$. Similarly, $R^* \subseteq R_l$, so $R^* \subseteq \hat{R}$.

The following example shows that all the set inclusions in Theorem 8 can be strict.

EXAMPLE 2. Let $X = \{a,b,c,d\}$ and suppose aPb, aIc, aId, bIc, bId, and cId. It is easy to see that

$$R_e = \{(a,a), (b,b), (c,c), (d,d)\},$$

$$R^* = \{(a,a), (b,b), (c,c), (d,d), (c,d), (d,c)\}$$

and

$$\hat{R} = R^* \cup \{(a,b), (a,c), (a,d), (c,b), (d,b)\}.[1]$$

[1] I am grateful to my colleague Shlomo Halfin for suggesting the promising relation $\hat{R}$ and making other helpful comments.

References

1. FISHBURN, P. C., *Utility Theory for Decision Making*, Wiley, New York, 1970.
2. ———, *The Theory of Social Choice*, Princeton Univ. Press, Princeton, N.J., 1973.
3. ———, "Stochastic Dominance Without Transitive Preferences," *Management Sci.*, Vol. 24 (1978), pp. 1268–1277.
4. KAMAE, T., KRENGEL, U. AND O'BRIEN, G. L., "Stochastic Inequalities on Partially Ordered Spaces," *Ann. Probability*, Vol. 5 (1977), pp. 899–912.
5. VEINOTT, A. F., JR., "Optimal Policy in a Dynamic Single Product, Nonstationary Inventory Model with Several Demand Classes," *Operations Res.*, Vol. 13 (1965), pp. 761–778.