COMPARISON CONJECTURES ABOUT THE $M/G/s$ QUEUE

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Received July 1983
Revised September 1983

We conjecture that the equilibrium waiting-time distribution in an $M/G/s$ queue increases stochastically when the service-time distribution becomes more variable. We discuss evidence in support of this conjecture and others, based partly on light-traffic and heavy-traffic limits. We also establish an insensitivity property for the case of many servers in light traffic.

Queues, multi-server queues, waiting time, stochastic comparisons, light traffic

1. Introduction

In this paper we pose some important queuing problems, make some conjectures about their solution, and present some evidence in support of the conjectures. We consider the standard $GI/G/s$ queueing model with s homogeneous servers in parallel, unlimited waiting room, the first-come first-served discipline and iid (independent and identically distributed) service times (with a general distribution) that are independent of a renewal arrival process. We discuss how greater variability in the service-time distribution affects the equilibrium waiting-time distribution. We primarily focus on the case of Poisson arrivals ($M/G/s$).

To make comparisons, we use several stochastic order relations. We say one random variable X or its cdf F' is less than or equal to another X'' or its cdf F'' in the sense of stochastic order (denoted by $\leqslant_{st}$), increasing convex order (denoted by $\leqslant_{ic}$), and convex order (denoted by $\leqslant_{c}$), respectively, if $Eg(X) \leqslant Eg(X'')$ holds for all nondecreasing, all nondecreasing and convex, and all convex real-valued functions g for which the expectations are well defined; see Chapter 1 of Stoyan [24] or Whitt [27] and references there. Convex order expresses greater variability and implies equal means. With equal means, increasing convex order coincides with convex order. Let $W(G)$ denote the equilibrium waiting time (until beginning service) as a function of the service-time cdf G, assuming throughout that there is a fixed arrival process with nonzero rate and that the system is stable. Our boldest conjecture is

Conjecture 1.1. *If $G' \leqslant_{ic} G''$ in an $M/G/s$ queue, then $W(G) \leqslant_{st} W(G'')$.*

The first such comparisons for queues involved the ordering $\leqslant_{st}$ for both the conditon and the conclusion; see Whitt [28] and references there. The orderings $\leqslant_{ic}$ and $\leqslant_{c}$ were first considered by Stoyan and Stoyan [25]; for example, they established the modification of Conjecture 1.1 for $s = 1$ in which the conclusion as well as the condition involve $\leqslant_{ic}$.

Here interest centers on going from the variability orderings $\leqslant_{ic}$ and $\leqslant_{c}$ in the condition to the monotonicity ordering $\leqslant_{st}$ in the conclusion. Even for $s = 1$, Conjecture 1.1 was an open problem, but in response to this paper, a proof has been provided by Daley and Rolski [2]. For $s = 1$, Conjecture 1.1 under the stronger condition $G' \leqslant_{c} G''$ (equal means) was proved by Rolski and Stoyan [20]. Related results were also obtained by Miyazawa [12]. For $s > 1$, Conjecture 1.1 with this stronger condition is also an open problem. In fact, it is an open problem in natural special cases; e.g., it is known that Erlang distributions with a common mean satisfy $E_{k+1} \leqslant_{c} E_k$ (see Example 3 in [27]), but we do not know if Conjecture 1.1 is

0167-6377/84/$3.00

valid for Erlang service-time distributions. This special case was also conjectured by Miyazawa [12].

In this paper we focus on the waiting time, but we remark that the orderings $\leqslant_{st}$ and $\leqslant_{ic}$ for $W(G)$ in two multiserver systems having a common arrival process imply that the equilibrium queue lengths at arbitrary times are ordered similarly, by virtue of (3.21) of Miyazawa [13] and (2.15) of Miyazawa [14], respectively.

In this paper, we show that Conjecture 1.1 is true in both light and heavy traffic. We also define a two-parameter family of service-time distributions (mixtures of an exponential and a point mass at zero) for which the $M/G/s$ queue can be solved exactly and for which we can verify Conjecture 1.1. The results for this special service-time distribution also yield useful approximations for the distribution of $W(G)$ and the distribution of the entire queue-length process in an $M/G/s$ queue given only the partial information provided by the arrival rate and the first two moments of the service time.

For $s > 1$, our weakest conjecture is the following.

Conjecture 1.2. *If $G \leqslant_c G''$ and $G' \neq G''$ in an $M/G/s$ queue, then $EW(G') \lneqq EW(G'')$.*

Of course, we obtain several different candidate comparison results as we vary the condition and the conclusion. For example, Conjecture 1.1 becomes valid for $GI/G/s$ systems if we strengthen the condition to $G' \leqslant_{st} G''$. Also Conjecture 1.1 becomes valid for $GI/G/1$ systems if, instead, we weaken the conclusion to $W(G) \leqslant_{ic} W(G'')$, but in that setting the conclusion is not valid with $\leqslant_{st}$; see [29]. The results are summarized in Table 1.

For $GI/G/s$ queues, even the conclusion $EW(G) \leqslant EW(G'')$ is not valid when $G' \leqslant_c G''$; see Wolff [31], Ross [21] and Remark 3.7 in [29]. Since the difficulty for $GI/G/s$ systems in heavy traffic with large s occurs when $c_a^2 > 1$, where c_a^2 is the squared coefficient of variation (variance divided by the square of the mean) of the interarrival time, it is natural to extend the conjecture to cover renewal arrival processes in which the interarrival time is less than an exponential in the convex order. However, even for $D/G/1$ systems, Conjecture 1.1 with the stronger condition $G' \leqslant_c G''$ is not valid; see Remark 3.2 of [29]. We now state an extended version of Conjecture 1.2.

Conjecture 1.3. *Suppose that $F \leqslant_c M$, where M is an exponential cdf. If $G' \leqslant_c G''$ with $G' \neq G''$ in a $GI/G/s$ queue with interarrival-time cdf F, then $EW(G') \lneqq EW(G'')$.*

In support of Conjectures 1.2 and 1.3, we do have the following positive result for the mean waiting times in general $GI/G/s$ systems with the same arrival process. If $G' \leqslant_c G''$, then

$$EW(G') \leqslant EW(G'') + \left(c^2(G'') - c^2(G')\right)/2\mu, \tag{1.1}$$

where μ^{-1} is the common mean and $c^2(G)$ is the squared coefficient of variation of the service-time cdf G; see Theorem 7 of [27]. (It is elementary that $c^2(G') \leqslant c^2(G'')$ is implied by, but does not imply, $G' \leqslant_c G''$.)

Related comparison resuls have been obtained for tandem queues by Niu [16], and for arrival process by Rolski [19] and Lemoine [11] and Daley and Rolski [2]. Delay and Rolski [2] have established for $GI/M/s$ queues the analogue of the Stoyan and Stoyan [25] result: If $-u' \leqslant_{ic} -u''$, where u' and u'' are the interarrival times, and $\mu' \geqslant \mu''$ in two $GI/M/s$ systems, then $W' \leqslant_{ic} W''$. In (5.2) of [12], Miyazawa also obtained $W' \leqslant_{ic} W''$ in some cases. We all conjecture the following analogue of Conjecture 1.1, which would complete the multiserver extension of Rolski and Stoyan [20].

Table 1
Stochastic comparison results for the equilibrium waiting time as a function of the service-time distribution

Condition	Conclusion	Result
$G' \leqslant_{st} G''$	$W(G') \leqslant_{st} W(G'')$	Theorem for $GI/G/s$ Kiefer and Wolfowitz (1955)
$G' \leqslant_c G''$	$W(G) \leqslant_{st} W(G'')$	Theorem for $M/G/1$ Rolski and Stoyan (1976) Counterexamples for $GI/G/1$ Whitt (1983)
	$EW(G) \leqslant EW(G'')$	Counterexamples for $GI/G/s$ Wolff (1977) and Ross (1978)
$G' \leqslant_{ic} G''$	$W(G) \leqslant_{ic} W(G'')$	Theorem for $GI/G/1$ Stoyan and Stoyan (1969)

Conjecture 1.4. *Under the assumptions above,* $W' \leqslant_{st} W'$ *in two GI/M/s systems.*

The rest of this paper is organized as follows. Section 2 discusses the comparison implications of a recent light-traffic theorem for the $M/G/s$ queue by Burman and Smith [1]. Section 3 contains an asymptotic insensitivity result for many servers in light traffic (as the number of servers increases). Section 4 discusses the implications of scheduling results by Pinedo [17] for queues with bursty (highly variable) arrival processes in light traffic. Section 5 discusses the comparison implications of heavy-traffic limits for the $GI/G/s$ queue by Köllerström [8]. Finally, Section 6 introduces the special family of service-time distributions for which the $M/G/s$ queue can be solved exactly. The solution not only makes it easy to verify Conjecture 1.1 in this class, but it also lends support for simple approximations in general $M/G/s$ queues.

We conclude the introduction by mentioning that there has recently been considerable success in developing algorithms for computing the equilibrium distributions in the $M/G/s$ queue and various $GI/G/s$ generalizations; see [3-6,15,22, 23,26]. The data we have looked at are consistent with the conjectures.

2. The $M/G/s$ queue in light traffic

Burman and Smith [1] have recently described the asymptotic behavior of $W(G)$ as the arrival rate λ approaches zero. Their description involves the stationary-excess cdf G_e associated with the service-time cdf G with mean μ^{-1}, defined by

$$G_e(t) = \mu \int_0^t [1 - G(u)]\, du, \quad t \geqslant 0; \tag{2.1}$$

For their proofs, Burman and Smith require that the service-time distribution be phase-type, but the results hold more generally. In fact, Reiman and Simon [18] have a new proof without this condition. We extract the following from their results.

Theorem 2.1 (Burman and Smith). *In an $M/G/s$ queue,*

$$\lim_{\lambda \to 0} \lambda^{-s} P(W > 0) = 1/\mu^s s! \tag{2.2}$$

and, for each x,

$$\lim_{\lambda \to 0} P(W(G) \leqslant x \mid W(G) > 0) = P(\min\{X_1, \ldots, X_s\} \leqslant x), \tag{2.3}$$

where $X_1, \ldots, X_s$ are iid with stationary-excess cdf G_e.

Corollary 2.1. *Under the conditions of Theorem* 2.1,

$$\lim_{\lambda \to 0} \lambda^{-s} P(W(G) > x) = \frac{1}{s!}\left(\int_x^\infty [1 - G(u)]\, du\right)^s \tag{2.4}$$

and

$$\lim_{\lambda \to 0} \frac{P(W(G'') > x)}{P(W(G') > x)} = \left(\frac{\int_x^\infty [1 - G''(u)]\, du}{\int_x^\infty [1 - G'(u)]\, du}\right)^s. \tag{2.5}$$

If $G' \leqslant_{ic} G''$, then the right side of (2.5) is greater than or equal to 1 for all x. When the right side is strictly greater than 1, we obtain a conclusive inequality in support of Conjecture 1.1 in light traffic, but any x for which the right side equals 1 is inconclusive. (Under the stronger condition $G' \leqslant_c G''$, we have equal means and the right side of (2.5) is 1 for $x = 0$.) Nevertheless, we regard (2.5) as strong evidence in support of Conjecture 1.1. If it turns out that Conjecture 1.1 is false, then it is natural to strengthen the condition in Conjecture 1.1 by requiring that the right side of (2.5) be strictly greater than 1 for all x or, equivalently.

$$\int_x^\infty [G'(u) - G''(u)]\, du > 0 \tag{2.6}$$

for all x.

The light-traffic approach of Reiman and Simon [18] may be viewed as taking successive derivatives of $P(W(G) > x)$ with respect to the Poisson arrival rate λ at $\lambda = 0$. Theorem 2.1 depicts the first nonzero derivative. For stochastic order, it is obviously necessary to have all higher derivatives ordered as well. We conjecture: (i) that all higher derivatives are indeed ordered, (ii) that such comparisons are necessary and sufficient for stochastic order, and (iii) that $P(W(G) > x)$ is analytic as a function of λ, so that it is characterized by all these derivatives.

Formula (2.3) suggests another conjecture concerning the conditional waiting times given that a customer must wait before beginning service.

Conjecture 2.1. *If $G \leqslant_c G''$ in an $M/G/s$ queue, then*

$$(W(G')|W(G')>0) \leqslant_{st} (W(G'')|W(G'')>0).$$

Since $G' \leqslant_c G''$ if and only if $G'_e \leqslant_{st} G''_e$, Conjecture 2.1 is valid in light traffic. It is important to realize that we have used the condition of equal means. Conjecture 2.1 is not valid for the weaker condition $G' \leqslant_{ic} G''$. In fact, it is not valid under the condition $G' \leqslant_{st} G''$.

Example 2.1. *Even for $s = 1$, the condition $G' \leqslant_{st} G''$ does not imply that*

$$E(W(G')|W(G')>0) \leqslant E(W(G'')|W(G'')>0). \tag{2.7}$$

Given (2.3), it suffices to show that we need not have $m_1(G'_e) \leqslant m_1(G''_e)$ where $m_k(G)$ is the kth moment of G and $m_1(G_e) = m_2(G)/2m_1(G)$. For a concrete example, let G' be obtained from a pmf (probability mass function) with mass $\frac{1}{3}$ on 1, 2 and 10; let G'' be obtained from a pmf with mass $\frac{1}{3}$ on 10 and $\frac{2}{3}$ on 1; then $G' \leqslant_{st} G''$ but $m(G'_e) = 102/24 > 105/26 = m(G''_e)$, as in Example 2.1 of [30].

We close this section by noting that if $G' \leqslant G''$ in one of the stronger stochastic orderings $\leqslant_f$ (failure rate order) or $\leqslant_r$ (monotone likelihood ratio order) defined in [30], then $G'_e \leqslant G''_e$ in the same ordering by Corollary 3.4 of [30], so that (2.7) and stronger stochastic orderings for the conditional waiting times are valid in light traffic. (For this application we use the continuous stationary-excess operator in (1.1) of [30], for which Corollary 3.4 is still valid.)

3. Insensitivity in light traffic with many servers

The $M/G/s/0$ loss system and the $M/G/\infty$ system are insensitive: the equilibrium distribution of the number of busy servers depends on the service-time distribution only through its mean. However, as demonstrated by (2.3), the $M/G/s$ delay distribution depends on the service-time distribution beyond its mean. We now show that there is asymptotic insensitivity of the delay distribution in light traffic as $s \to \infty$. If we just let $s \to \infty$, then such insensitivity is trivial; the $M/G/s$ system approaches the $M/G/\infty$ system and there is no delay. We consider the conditional delay distribution and normalize so that there is a nontrivial limit as first $\lambda \to 0$ and then $s \to \infty$. (Note that the normalized probability of delay is insensitive in light traffic for all s by (2.2).) We apply a simple extreme-value limit theorem, as on p. 56 of Lamperti [9].

Theorem 3.1. *For an $M/G/s$ queue in which $G(0) = 0$,*

$$\lim_{s\to\infty} \lim_{\lambda\to 0} P(sW(G) > x | W(G) > 0) = e^{-\mu x}$$

for all x.

Proof. From (2.3) and (2.1),

$$\begin{aligned}\lim_{s\to\infty} \lim_{\lambda\to 0} & P(sW(G) > x | W(G) > 0) \\ &= \lim_{s\to\infty} [1 - G_e(x/s)]^2 \\ &= \lim_{s\to\infty} [1 - (\mu x/s) + o(1/s)]^s = e^{-\mu x}.\end{aligned}$$

4. Batch arrivals in light traffic

In this section we illustrate the negative results that can arise with bursty arrival processes. We let arrivals occur in batches and show that greater variability in the service-time distribution systematically causes the equilibrium waiting-time distribution to *decrease* in an appropriate sense in light traffic.

Consider the $GI^D/G/s$ queue with a batch renewal arrival process in which each batch has exactly n customers. The arrival process of batches is assumed to be a renewal process in which the renewal interval has no atom at the origin. Of course, the arrival process of customers is then not a renewal process. Unlike Section 2, if $n > s$, then as the rate of the renewal process converges to 0 (say, by scaling the renewal interval), the equilibrium waiting-time $W(G)$ converges to a nontrivial limit, say $W_L(G)$. The distribution of $W_L(G)$ is a mixture of the waiting-time distributions of customers $1, 2, \ldots, n$ in the stochastic scheduling problem treated by Pinedo [17], in which there are exactly n jobs, all of which are in the system at time 0. As a consequence of Theorem 2 of [17], we have

Theorem 4.1. *If $G' \leqslant_c G''$ in an $GI^D/G/s$ system in* *light traffic, then* $-W_L(G') \leqslant_{ic} -W_L(G'')$, *so that* $EW_L(G'') \leqslant EW_L(G')$.

Proof. Apply Theorem 2 of Pinedo [17]. The expeted value of the increasing concave function with respect to the mixture is the weighted sum of the corresponding expected values. (Recall that $-f(-x)$ is increasing convex when $f(x)$ is increasing concave.)

5. The $GI/G/s$ queue in heavy traffic

In this section we consider the $GI/G/s$ queue. As before, let λ be the arrival rate, μ the service rate, and c_a^2 and $c^2 \equiv c^2(G)$ the squared coefficients of variation of an interarrival time and a service time, respectively. As $\lambda \to s\mu$ ($\rho \equiv \lambda/s\mu \to 1$) from below (with associated sequence of interarrival-time and service-time distributions appropriately controlled), $(1-\rho)W(G)$ converges to an exponential distribution with mean $(c_a^2 + c^2)/2\mu$; see Köllerstöm [8]. Hence, Conjecture 1.1 with the stronger condition $G' \leqslant_c G''$ is valid for the limit of $(1-\rho)W(G)$ as $\rho \to 1$ because $c^2(G') < c^2(G'')$ if $G' \leqslant_c G''$ and $G' \neq G''$. Conjecture 1.1 with the weaker condition $G' \leqslant_{ic} G''$ is also valid for the limit of $(1-\rho'')W(G)$ because if the means are not equal, then $(1-\rho'')W(G')$ converges to 0. For $W(G)$ itself, the heavy-traffic limit implies that

$$P(W(G') > x) \leqslant P(W(G'') > x) \tag{5.1}$$

for sufficiently large x and ρ. However, the heavy-traffic limit does not imply that (5.1) is every valid for fixed x and $\rho \to 1$. Similarly, the gap in (1.1) trivially disappears when you multiply both sides by $(1-\rho)$ and let $\rho \to 1$. If Conjecture 1.1 is not true, then (1.2) is likely to fail for small x, e.g., $x = 0$.

Since $P(W(G) > 0) \to 1$ as $\rho \to 1$ with s fixed, Conjecture 2.1 is verified in heavy traffic too.

6. A special family of service-time distributions

In this section we introduce a family of service-time distributions for which the orderings $\leqslant_c$ and $\leqslant_{ic}$ can be considered and for which the $M/G/s$ queue can be solved exactly. Let the cdf G_p be the mixture of an exponential cdf with probability p and the cdf of a point mass on 0 with probability $1-p$. The orderings $\leqslant_{ic}$ and $\leqslant_c$ are easily characterized in this class. For this purpose, let $m(G_p)$ be the mean of G_p.

Lemma 6.1. *If* $p_1 \geqslant p_2$ *and* $m(G_{p_1}) \leqslant m(G_{p_2})$, *then* $G_{p_1} \leqslant_{ic} G_{p_2}$.

Proof. If $p_1 = p_2$, then $G_{p_1} \leqslant_{st} G_{p_2}$. Suppose $p_1 > p_2$. As in the examples in Section 2 of [27], apply the characterization in Theorem 2 there. Let X be distributed as G_{p_1}. If $X = 0$, let $Y = 0$; if $X = x$, let $Y = -x$ with probability $1 - (p_2/p_1)$ and let $Y = yx$ with probability p_2/p_1, with y chosen so that $X + Y$ is distributed as G_{p_2}; i.e., $y = [m(G_{p_2}) - m(G_{p_1})]/m(G_{p_1})$. Since $E(Y|X) \geqslant 0$, we have verified that $G_{p_1} \leqslant_{ic} G_{p_2}$.

It is also not difficult to completely describe the behavior of an $M/G/s$ queue with service-time cdf G_p. Let λ be the arrival rate and μ the individual service rate. Let $N(t)$ be the stochastic process representing the number of customers in the system, including any in service. It is not difficult to see that $N(t)$ is a Markov process. For $N(t) \leqslant s-1$, the process evolves as an $M/M/s$ queue with arrival rate λp and individual service rate μp. (Arrivals with zero service time can be ignored.) When the process $N(t)$ hits s, it evolves as an $M/G/1$ queue with arrival rate λ and service-time distribution G_p', modified by having the rate of the exponential component multiplied by s. (Customers with zero service time must wait, but there are batch departures.) Until $N(t)$ hits $s-1$ again from above, the process evolves as this $M/G/1$ queue conditioned on a nonzero busy period having begun at the instant $N(t)$ hit s from below. The successive first passage time from $s-1$ to s and back again from s to $s-1$ form an alternating renewal process. The time to go from $s-1$ to s, say $T(s-1, s)$, has mean

$$ET(s-1, s) = (1/p\mu)(s-1)! \sum_{k-0}^{s-1} (s\rho)^{k-s}/k!. \tag{6.1}$$

A simple way to obtain (6.1) is to observe that in an $M/M/s/0$ loss system, with arrival rate λp and service rate μp, the well-known equilibrium probability of s busy servers can be expressed as $(1/p\mu s)/[ET(s-1, s) + (1/p\mu s)]$. Similarly, the time to go from s to $s-1$, $T(s, s-1)$, has mean

$$ET(s, s-1) = 1/ps\mu(1-\rho). \tag{6.2}$$

Formula (6.2) is the appropriate mean $M/G/1$ busy period conditioned that it is greater than 0 (which means dividing by p).

Let N be the equilibrium distribution of $N(t)$. From (6.1) and (6.2), we see that $P(N \geqslant s)$ coincides with the $M/M/s$ value. Hence, $P(N = k)$ coincides with the $M/M/s$ value for $k \leqslant s-1$. On the other hand, $P(n = s + k | N \geqslant S)$ coincides with the $M/G/1$ value with $s = 1$ and faster service rate. Since Poisson arrivals see time averages, see Wolff [32], $(W(G_p)|W(G_p) > 0)$ is distributed as the $M/G/1$ value using G_p with service rate multiplied by s. As a consequence, $P(W(G_p) > 0)$ is independent of p and

$$E(W(G_p)|W(G_p) > 0) = \frac{1 + c^2(G_p)}{2s\mu(1-\rho)} = \frac{1}{s\mu p(1-\rho)}, \tag{6.3}$$

where

$$c^2(G_p) = (2-p)/p. \tag{6.4}$$

This convenient characterization of $W(G_p)$ in terms of the $M/M/s$ queue and $c^2(G_p)$ is a natural candidate for approximating more general $M/G/s$ queues. Note that (6.3) is consistent with the established light-traffic and heavy-traffic behavior. (The stationary-excess distribution associated with G_p is exponential with rate μp.) For this special service-time distribution, $s\mu(1-\rho)E(W(G_p)| W(G_p) > 0)$ is independent of ρ, just as in the case $s = 1$. For the special case involving G_p, we obtain the familiar Lee and Longton [10] approximation formula for EW as an exact result, i.e.,

$$EW(G_p)/EW(G_1) = \frac{1 + c^2(G_p)}{2}; \tag{6.5}$$

$EW(G_p)$ is exactly the $M/M/s$ value multiplied by $(c^2(G_p) + 1)/2$.

The established structure allows us to apply the results for $s = 1$ [2,20] to verify Conjectures 1.1 and 2.1

Theorem 6.1. *Within the class of G_p service-time distributions, Conjectures* 1.1 *and* 2.1 *are valid.*

We also obtain useful insights about the time-dependent behavior. Compared to a standard $M/M/s$ system with $p = 1$, the process $N(t)$ with $p < 1$ is slowed down when $N(t) \leqslant s-1$; when $N(t) \geqslant s$, the overall arrival and service rates are the same, but the process is altered by having batch service completions.

We close this section by remarking that we can do a similar detailed analysis of the $GI/G/s$ system in which both the interarrival-time and service-time distributions are from the G_p class. The formulas confirm negative comparison results in [29].

7. Conclusions

We have presented several conjectures about the qualitative behavior of multi-server queues and some supporting evidence based on light-traffic and heavy-traffic limits and a special family of service-time distributions. In the continued search for results, it may be useful to restrict the class of service-time distributions (e.g., with increasing failure rate) and consider other stochastic order relations (as in [30]). It is also interesting to know how the established properties change as we change the model, e.g., as we change the rule for assigning customers to servers.

Acknowledgments

I am grateful to Mike Pinedo for getting me started, to Daryl Daley and Tomasz Rolski for proving Conjecture 1.1 in the case $s = 1$, to Masakiyo Miyazawa for indicating connections to his work, and to Marty Reiman and Burt Simon for discussing their approach to light traffic.

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