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The Maximum Throughput on a Golf Course

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We develop stochastic models to help manage the pace of play on a conventional 18-hole golf course. These models are for group play on each of the standard hole types: par-3, par-4, and par-5. These models include the realistic feature that $k-2$ groups can be playing at the same time on a par-k hole, but with precedence constraints. We also consider par-3 holes with a "wave-up" rule, which allows two groups to be playing simultaneously. We mathematically determine the maximum possible throughput on each hole under natural conditions. To do so, we analyze the associated fully loaded holes, in which new groups are always available to start when the opportunity arises. We characterize the stationary interval between the times successive groups clear the green on a fully loaded hole, showing how it depends on the stage playing times. The structure of that stationary interval evidently can be exploited to help manage the pace of play. The mean of that stationary interval is the reciprocal of the capacity. The bottleneck holes are the holes with the least capacity. The bottleneck capacity is then the capacity of the golf course as a whole.

Key words: pace of play in golf; the capacity of a golf course; queueing models of golf; throughput; production lines; queues in series
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1. Introduction

We develop mathematical models to study the pace of play in golf. It is natural to dismiss the topic as frivolous, because golf is "only" a game. However, golf courses provide important recreational services, with multi-billion-dollar economic impact. Indeed, in 2008 Haydu et al. (2008) published the results of a research study of the economic impact of golf courses in the United States, in which they concluded that "The golf sector is the largest component of the turfgrass industry, accounting for a 44% share. The nearly 16,000 golf courses generated $33.2 billion in (gross) output impacts, contributed $20.6 billion in value added or net income, and generated 483,000 jobs nationwide."

In order for golf courses to be successful and achieve their mission they must be properly designed and well managed. Unfortunately, there is concern that the pace of play has become too slow, that is, that the amount of time spent waiting and the overall time required to play a full round of 18 holes have become excessively long. Indeed, Riccio (2014a) established the Three/45 Golf Association "dedicated to leading, educating, and advocating for a quicker pace of play, including golfers, owners, managers, superintendents and designers." Riccio (2014b) also conducted a study of the pace of play on on a sample of 175 American golf courses using GPS collected data on 40,000 completed 18-hole rounds of golf during June 2013. This study showed that 70% of the rounds lasted more than 4 hours and 10% lasted more than 5 hours. A statistically significant positive relationship was found between the time of play and the number of rounds per course.

It is natural to respond to this challenge by applying the principles of production and operations management (POM), as Riccio (2012, 2013, 2014a) has advocated. POM principles should apply because successive groups of golfers playing on a conventional 18-hole golf course can be viewed, at least roughly, as a production line. The groups can be regarded as "jobs" that flow through a serial network of 18 queues, with unlimited waiting space at each queue and service in order of arrival. However, there are several complicating features. First, to satisfy the high demand and exploit valuable resources, golf courses are typically quite heavily loaded. Second, the system starts empty at the beginning of each day and should terminate with the last group completing play on all holes. Thus, the system is a transient network of queues operating under heavy-traffic conditions. Consequently, conventional steady-state analysis of a stationary queueing model is of doubtful relevance. Nevertheless, POM principles suggest seeking to balance the desire to put more golfers on the course in order to maximize the use of a valuable resource and the desire to put fewer golfers on the course in order provide a good experience by keeping delays low.

Closer examination of group play on golf courses reveals other complications. The one that we primarily address is the fact that more than one group can play at the same time on many of the holes, but under precedence constraints. There are three types of holes on a golf course: par 3, par 4, and par 5. Typically, two groups can be playing on a par-4 hole at the same time, while three groups can be playing on a par-5 hole at the same time. A conventional par-3 hole is more elementary because only one group can play on it at the same time, but there also is the modified par-3 hole "with wave-up," which allows two groups to play at the same time there too, while still maintaining the order determined by their arrival; see Tiger and Salzer (2004), Riccio (2013), and section 5 here. This simultaneous play on most holes has the important consequence that the times between successive groups completing play on a hole will tend to be less than the time required for each group to play the hole.

To explain in greater detail, we describe the steps of group play on a par-4 hole. There are *five steps*, each of which must be completed before the group moves on to the next step. These five steps can be diagrammed as

$$T \rightarrow W_1 \rightarrow F \rightarrow W_2 \rightarrow G. \qquad (1)$$

The first step T is the tee shot (one for each member of the group); the second step W_1 is walking up to the balls on the fairway; the third step F is the fairway shot; the fourth step W_2 is walking up to the balls on or near the green; the fifth and final step G is clearing the green, which may involve one or more approach shots and one or more shots (putts) on the green for each player in the group. The goal in golf is to put the ball into the hole on the green using as few strokes (shots) as possible. A hole is rated par 4 because good play should require four shots: one from the tee, one from the fairway, and two more to clear the green (put it in the hole on the green).

The rules of play allow two groups to play at the same time on a conventional par-4 hole. Two successive groups can be simultaneously playing on the hole, because each group is allowed to hit its initial tee shots after the previous group has hit its fairway shots, and so will be safely out of the way, while each successive group is allowed to hit its fairway shots only after the previous group has cleared the green. Usually about 12 of the 18 holes are par-4 holes. The par-5 holes are longer, allowing three groups to play at the same time, while the par-3 holes are shorter, allowing only one group to play at one time, except under the wave-up rule.

1.1. A Stochastic Model of Group Play

In this paper, we contribute by developing a tractable stochastic model of group play on each hole of the golf course, paying special attention to the inevitable randomness in the times required for each group to complete each stage of play. We develop three models, one for each of the standard hole types: par-3, par-4, and par-5. Putting these models together, we obtain a queueing network model of successive groups of golfers playing on the successive holes of a conventional 18-hole golf course over a single day. In the overall queueing network model, there could be 18 different models for the 18 holes, if the parameters for the holes with the same par value are different.

We have begun using this model to develop useful performance formulas and to simulate the play of successive groups of golfers over the 18-hole golf course during a day; see Fu and Whitt (2014). For example, we are studying alternative schedules for group start (tee) times. We have found that *both* the number of groups to complete play can be increased and the maximum expected time required to play a round per group can be decreased by using a nonconstant tee schedule, making the earlier intervals between tee times shorter than the later ones appropriately. Thus, the present paper is a first step toward applying POM principles to improve the performance of golf courses.

In this paper, we apply the stochastic model to analytically determining the capacity of each hole. The *capacity* is the maximum possible throughput, where the *throughput* is the rate that groups of golfers complete play on the hole. The maximum possible throughput is realized as the limiting throughput in an idealized *fully loaded hole*, where there always are groups ready to start play (tee off) at the first opportunity.

These maximum throughput results for individual holes translate into the capacity of the golf course as a whole. The holes with the least capacity are called the *bottleneck holes*. The capacity of the entire golf course is the capacity of the bottleneck holes. As emphasized by Riccio (2013), it is important to know the capacity of the golf course when setting tee time schedules. No gain in the throughput can be achieved when the starting rate (reciprocal of the interval between tee times) exceeds the capacity. Since par-3 holes tend to be the bottleneck holes, Riccio (2014a) recommends that course managers set the tee interval on the first hole to at least the time it takes to play the longest par 3. Course designers can make that rule easier to follow by putting that longest par-3 hole at the beginning of the course; that makes any queue buildup easier to see. These principles are supported by our analysis; see Corollary 3.

More generally, course designers can use the hole capacity values to help choose arrangements of the holes that are efficient as well as satisfying for golfers and spectators. This follows POM principles as in P2 on p. 481 and section 7.2 of Whitt (1985) and Yamazaki et al. (1992).

1.2. Stage Playing Times

Our stochastic queueing models for group play on golf courses are closely related to previous models in Kimes and Schruben (2002), Tiger and Salzer (2004), Riccio (2012, 2013); for example, see the single-hole bottleneck model on p. 32 of Riccio (2013). However, we innovate by converting the basic *steps of group play* into critical stages, so that our model primitives become the *stage playing times*; see section 2.1. In addition, we provide the first direct mathematical analysis of these stochastic models. Like the previous models, our models can also be analyzed with computer simulation, but mathematical methods facilitate analysis of the pace of play.

The stage playing times that are the primitives of our models depend on the number of players in the group and their characteristics, and require careful modeling and data analysis, but we do not carry out that step here. We aim to help understand how the stage playing times translate into the time required for the group to play each hole and the entire golf course. The analysis here makes it possible to determine how changes in the stage playing times obtained through course design and management decisions will impact capacity.

We think that stage playing times provide a useful modeling framework for the design and analysis of golf courses. We think that it can be fruitful to separate the overall analysis into three parts. In the first part, we study how course design, course management, and golf group behavior affect stage playing times. In the second part, we study how the distribution of stage playing times of all the groups on all holes affects the pace of play on those holes. In the third part, we study how the results for individual holes can be combined to determine the impact on the pace of play on the entire 18-hole golf course. We are concerned with the second part here. We suggest measuring stage playing times of groups and applying the analysis here to see what that implies about the successive times for groups to play each hole and the successive times between successive groups completing play on each hole. The formulas developed here show how changes in the stage playing times will impact the capacity; for example, for a par-4 hole, we can combine equations (14) and (15).

It is significant that the stage playing times are not only useful to expose the key structure determining performance, but they are also convenient to measure on the golf course. It is far easier to measure group stage playing times than to record the times each individual golfer performs each step.

1.3. The Impact of Variability on Performance

Established POM principles have revealed that variability usually seriously inhibits performance efficiency; for example, see Hopp and Spearman (1996). Counter to naive intuition, variability often does not average out, but degrades the average performance. That is illustrated by the impact of variability in the service-time distribution on the steady-state waiting time in the classical $M/GI/1$ queueing model; the Pollaczek–Khintchine formula for the mean waiting time shows that it is directly proportional to the variability of the service-time distribution, as characterized by its *squared coefficient of variation* (scv, variance divided by the square of the mean). Since variability tends to be hard to understand, this important insight is often missed. A major goal of our stochastic model is to address that problem.

There often is significant variability in group play on golf courses, extending beyond the inevitable randomness required for each golfer to make a shot and walk up to the ball. First, many golf courses allow groups to either walk the course or use carts, and this choice may make a significant difference on stage playing times. Second, many golf courses allow groups to consist of different numbers of golfers, anywhere from one to four, or even more; obviously that too should impact group playing times. Third, there may be unusually slow groups, typically because they contain inexperienced golfers.

Consistent with intuition, Riccio (2012, 2013) has shown that the presence of groups that tend to take longer to play all the holes can have a dramatic detrimental impact on the performance of subsequent groups to play the course. We do not address that phenomenon here, but we intend to use variants of the model here to study the impact of slow groups on the performance over the full golf course in the future.

Nevertheless, our analysis in this paper shows that increased variability in stage playing times consistently reduces the maximum possible throughput on each hole separately. Thus, the capacity of the golf course is necessarily reduced when variability of stage playing times increases. That can be explained succinctly by the conclusion of our analysis: For each of the holes-types in which multiple groups can play at the same time, the random variable representing the interval between successive groups clearing the green on a fully loaded hole is a strictly increasing strictly convex function of the stage playing time variables; see equations (15), (65) and the final line of Theorem 9. The explicit formulas quantify the impact.

1.4. Organization of the Paper

Here is how the rest of this paper is organized: First, in section 2 we develop the model of successive groups playing a par-4 hole. We start in section 2.1 by converting the five steps described above into three stages of group play. Then in section 2.2 we develop a concise recursion to model group play, based on specified stage playing times. Afterward, we discuss the performance measures of interest and carefully define the throughput. In section 3 we examine the par-4 model under the condition that it is fully loaded, and determine the capacity of the hole; the formula is given in equation (14), drawing on equation (15). In section 4, we introduce a specific model of the stage playing times and show how they impact the capacity of a par-4 hole.

We analyze the more elementary par-3 hole, with and without wave-up, in section 5. We show that the fully-loaded par-3 hole with wave-up has essentially the same structure as a fully-loaded par-4 hole, but the stage playing times appear in a different way.

We develop the corresponding exact model of a par-5 hole and analyze the associated fully loaded model in section 6. As might be anticipated, since three groups can be simultaneously on each par-5 hole, the stochastic analysis is more complicated for a par-5 hole, so that it is more complicated to compute the capacity. However, we provide a remarkably tractable simplification under an additional approximation assumption in section 6.2. In section 6.3, we give a simulation example of group play on a par-5 hole. Finally, in section 7 we draw conclusions.

1.5. Related Queueing Literature

This paper is self-contained, but there is related work in queueing theory. The simultaneous play and the conventions for managing it introduces precedence constraints, as studied in the sophisticated queueing theory based on the max-plus algebra in Baccelli et al. (1992, 1989), Heidergott et al. (2006), Mairesse (1997), but we do not see how to apply that theory. Even if it could be applied, the direct analysis here is appealing because it is more accessible.

The linear flow with constraints makes the overall network model a serial or tandem queueing network with a form of blocking, as in Perros (1994) and the many references therein, but the form of blocking here is evidently not covered by that literature. Our determination of maximum throughput is in the spirit of the throughput analysis for linear loss networks in Momcilovic and Squillante (2011), but that is a different model.

2. Stochastic Model of Groups Playing a Par-4 Hole

In this section, we develop a stochastic model of successive groups of golfers playing a par-4 hole.

2.1. Representation of the Group Play in Three Stages

Recall the five steps of group play on a par-4 hole: T, W_1, F, W_2, and G, depicted in equation (1). Each step must be completed before the group proceeds to the next step. An important part of our modeling approach is to *not* directly model the performance of these individual steps. Instead, we aggregate the five steps into *three stages*, which are important to capture the way successive groups interact while playing the hole. In particular, we represent the three stages as:

$$(T, W_1) \to F \to (W_2, G) \tag{2}$$

Stage 1 is (T, $W1$), stage 2 is F, and stage 3 is (W_2, G). We use this particular aggregation, because it turns out to be the maximum aggregation permitted by the precedence constraints, which we turn to next.

We now describe the precedence constraints, which follow common conventions in golf. Assuming an empty system initially, the first group can do all the stages, one after another without constraint. However, for $n \geq 1$, group $n + 1$ cannot start stage 1 until *both* group $n + 1$ arrives at the tee and group n has completed stage 2, that is, has cleared the fairway. Similarly, for $n \geq 1$, group $n + 1$ cannot start on stage 2 until *both* group $n + 1$ is ready to begin there and group n has completed stage 3, that is, cleared the green. These rules allow two groups to be playing on a par-4 hole simultaneously, but under those specified constraints. We may have groups n and $n + 1$ on the course simultaneously for all n. That is, group n may first be on the course at the same time as group $n - 1$ (who is ahead), but then later be on the course at the same time as group $n + 1$ (who is behind). The groups remain in their original order, but successive groups interact on the hole. The group in front can cause extra delay for the one behind.

We now formalize those rules with a mathematical model. Let A_n be the *arrival time* of the nth group at the tee of this hole on the golf course. Let $S_{j,n}$ be the time required for group n to complete stage j, $1 \leq j \leq 3$; these are called the *stage playing times*. The *mathematical model data* for a par-4 hole consists of a sequence of 4-tuples: $\{(A_n, S_{1,n}, S_{2,n}, S_{3,n}): n \geq 1\}$, where the four components for each n are nonnegative random variables.

We now turn to the performance measures, showing the result of the groups playing on the hole. Let B_n be the time that group n starts playing on this hole, that is, the instant when one of the group goes into the tee box. Let T_n be the time that group n completes stage 1, including the tee and the following walk; let F_n be the time that group n completes stage 2, its shots on the fairway; and let G_n be the time that group n completes stage 3, and clears the green.

2.2. The Fundamental Recursion

We now give a concise mathematical representation of the description above. This representation relates the model primitives to the performance random variables by the following four-part recursion:

$$B_n \equiv A_n \vee F_{n-1}, \qquad T_n \equiv B_n + S_{1,n},$$
$$F_n \equiv (T_n \vee G_{n-1}) + S_{2,n} \quad \text{and} \qquad G_n \equiv F_n + S_{3,n}, \quad (3)$$

where $\equiv$ denotes "equality be definition" and $a \vee b \equiv \max\{a, b\}$. As initial conditions, assuming that the system starts empty, we set $F_0 \equiv G_0 \equiv 0$. The two maxima capture the two precedence constraints.

The model in equation (3) extends directly to any number of such single-hole models in series. We simply let the completion times G_n from one queue be the arrival times at the next queue.

2.3. Performance Measures

We now define associated performance measures, starting with temporal performance measures for group n. In doing so, we follow the factory physics conventions in Hopp and Spearman (1996) and Riccio (2013) as much as possible. The principal temporal performance measures for group n are: the *waiting time* (before starting play on the hole), $W_n = B_n - A_n$; the *playing time* (the total time group n is actively playing this hole, possibly including some waiting there), $X_n \equiv G_n - B_n$; and the *sojourn time* (the total time spent by group n at the hole, waiting plus playing), $U_n = G_n - A_n = W_n + X_n$. Let X_n^w be the *waiting time while playing the hole* for group n and let X_n^p be the *active playing time* while playing on the hole. Since $X_n^p = S_{1,n} + S_{2,n} + S_{3,n}$ for a par-4 holes, we can easily calculate X_n^w, given the playing time X_n as $X_n^w \equiv X_n - X_n^p$.

We are primarily interested in determining the maximum throughput. For the golf course, the definition of throughput is complicated because the state changes over the course of each day, starting empty, and getting more congested throughout most of the day. However, the rate groups complete play may rapidly approach a limit, even if the system is overloaded. We will be focusing on that limit.

First, we define the *random cycle time* for group n as

$$C_n \equiv G_n - G_{n-1}, \quad n \geq 1, \quad (4)$$

and the *cycle time* for group n is its expected value, $E[C_n]$. Second, the *average random cycle time* for the first n groups is

$$\bar{C}_n \equiv \frac{1}{n}\sum_{k=1}^{n} C_k = \frac{G_n}{n}, \quad n \geq 1. \quad (5)$$

The average cycle time for the first n groups is then just $E[\bar{C}_n]$.

The typical case is to have

$$C_n \Rightarrow C_\infty, \quad E[C_n] \to E[C_\infty] \quad \text{and} \quad \bar{C}_n \Rightarrow E[C_\infty]$$
$$\text{as} \quad n \to \infty, \quad (6)$$

where C_∞ is a random variable and $\Rightarrow$ denotes convergence in distribution, in which case we let $E[C_\infty]$ be the cycle time; That is the standard case, referred to on p. 17 of Riccio (2013).

We define the *random throughput rate* for the first n groups as

$$\Theta_n \equiv 1/\bar{C}_n = \frac{n}{G_n}, \quad n \geq 1. \quad (7)$$

Given that positive finite limits hold in equation (6), we have

$$\Theta_n \Rightarrow \theta \equiv \frac{1}{E[C_\infty]} \quad \text{as} \quad n \to \infty. \quad (8)$$

Thus, the *throughput* is $\theta \equiv 1/E[C_\infty]$.

We define other average performance measures just like equations (5) and (7). For example, the *average sojourn time*, that is, the average time spent at the hole per group (among the first n groups) is

$$\bar{U}_n \equiv \frac{1}{n}\sum_{k=1}^{n} U_k = \frac{1}{n}\sum_{k=1}^{n}(G_k - A_k). \quad (9)$$

We next turn to the performance measures, counting the number of groups at the hole. (Necessarily, any number greater than 2 at a par-4 hole must be waiting in queue, because at most two can be playing at the same time, but there is no limit on the number that can be waiting (unless other assumptions are made). The counting could be done at an arbitrary time, at an arrival epoch (the times A_n) or at a green clearing epoch (the times G_n). At arrival time or departure time n, customer n might or might not be counted. Let N_n^a be the number at the hole, either waiting or playing, as seen by group n upon arrival, but not counting the arrival; then

$$N_n^a \equiv n - 1 - \max\{k \geq 0 : G_k \leq A_n\}, \quad n \geq 1. \quad (10)$$

Let $N(t)$ be the number in the system at time t; then

$$N(t) = N_n^a + 1, \quad A_{n-1} \leq t < A_n. \quad (11)$$

3. Model of a Fully Loaded Par-4 Hole

In order to determine the *capacity* of a par-4 hole, which we understand to be the maximum possible throughput, we now focus on a *fully-loaded hole*, that is, all groups are at the hole at time 0 ready to play, that is, $A_n \equiv 0$ for all n. Given the recursion in equation (3), it actually suffices to have only the weaker condition $A_n \le F_{n-1}$ for all $n \ge 1$, because then the tee box is never idle after the first group starts (at $A_1 = B_1 = 0$). The capacity of a par-4 hole, which we denote by θ^*, is the throughput θ as defined in equation (8) for the fully loaded par-4 hole.

3.1. A Simplified Recursion

It is easy to see that, under this fully loaded condition, the recursion in equation (3) reduces to

$$\begin{aligned} &B_n \equiv F_{n-1}, \quad T_n \equiv B_n + S_{1,n}, \\ &F_n \equiv (T_n \vee G_{n-1}) + S_{2,n} \quad \text{and} \quad G_n \equiv F_n + S_{3,n}, \; n \ge 1, \end{aligned} \tag{12}$$

where $F_0 \equiv G_0 \equiv 0$.

3.2. The Random Cycle Times

Our main result is a *law of large numbers* (LLN) for the cycle times $C_n \equiv G_n - G_{n-1}$ of a fully loaded par-4 hole. The limit is the maximum throughput for a par-4 hole. We make customary independence and identical distribution (i.i.d.) assumptions for the stage playing time vectors. By allowing the three components of that vector to be dependent, we include the phenomenon of an occasional slow group, that tends to play more slowly on all stages. We show that a slow group does not decrease the capacity of a hole beyond its impact on the mean. Riccio (2013), p. 46, shows that groups that are slow on all holes can have a devastating impact on the pace of play for all groups playing after it. That impact is caused by the larger variance for each hole and the dependence over multiple holes.

Theorem 1. (*LLN for the cycle times C_n for the fully loaded model*) *Consider the fully loaded par-4 model in which the sequence of stage playing time random vectors $\{(S_{1,n}, S_{2,n}, S_{3,n}): n \ge 1\}$ is i.i.d. each distributed as the random vector (S_1, S_2, S_3), whose components are strictly positive with finite means. Then*

$$\bar{C}_n \equiv \frac{1}{n}\sum_{k=1}^{n} C_k \equiv \frac{G_n}{n} \to E[Y] \quad \text{as} \quad n \to \infty \quad w.p.1, \tag{13}$$

so that

$$\Theta_n \equiv \frac{1}{\bar{C}_n} \equiv \frac{n}{G_n} \to \frac{1}{E[Y]} \equiv \theta^* \quad \text{as} \quad n \to \infty \quad w.p.1, \tag{14}$$

where

$$Y \equiv (S_1 \vee S_3) + S_2 \quad \text{with} \quad 0 < E[Y] < \infty. \tag{15}$$

and S_3 is independent of (S_1, S_2).

A key step in the proof of Theorem 1 is a representation of the fairway completion times as partial sums of random variables constructed from the stage playing times. Since this is the key structural result, we refer to it as a theorem instead of a lemma. Here, we make *no* stochastic assumptions.

Theorem 2. (*representation for F_n as a partial sum*) *For the fully loaded par-4 hole with recursion in equation* (12),

$$F_n = F_{n-1} + Y_n, \quad n \ge 2, \quad \text{so that} \quad F_n = \sum_{k=1}^{n} Y_k, \quad n \ge 1, \tag{16}$$

where

$$\begin{aligned} Y_n &\equiv (S_{1,n} \vee S_{3,n-1}) + S_{2,n}, \quad n \ge 2, \quad \text{and} \\ Y_1 &= S_1 + S_2. \end{aligned} \tag{17}$$

Proof. For $n \ge 2$, the recursion in equation (12) can be expressed as

$$\begin{aligned} F_n &= (T_n \vee G_{n-1}) + S_{2,n} = (F_{n-1} + S_{1,n}) \vee (F_{n-1} + S_{3,n-1}) \\ &\quad + S_{2,n} = F_{n-1} + (S_{1,n} \vee S_{3,n-1}) + S_{2,n} = F_{n-1} + Y_n, \end{aligned} \tag{18}$$

so that equation (16) holds. □

Proof of Theorem 1. Since $G_n = F_n + S_{3,n}$ by equation (12), the difference between G_n and F_n is asymptotically negligible when we divide by n. Given the assumed independence among the random vectors $(S_{1,n}, S_{2,n}, S_{3,n})$, the random variables Y_n are only one-dependent and they are identically distributed for $n \ge 2$. (*One-dependent* means that Y_n and Y_{n+1} may be dependent of each n, but $\{Y_j: j \le n\}$ and $\{Y_{n+j}: j > 1\}$ are independent for each n.) The one-dependence implies that the LLN applies directly to the two subsequences $\{Y_{2k}: k \ge 1\}$ and $\{Y_{2k+1}: k \ge 0\}$. Adding yields the LLN for the full sequence $\{Y_k: k \ge 1\}$ itself: $\bar{F}_n \equiv n^{-1}F_n \to E[Y]$ w.p.1 as $n \to \infty$. Since $S_1 \vee S_3 \le S_1 + S_3$ and the individual means are finite, $E[Y] < \infty$. Since the means are positive, $E[Y] > 0$, so that $1/E[Y] < \infty$ too. Since $G_n = F_n + S_{3,n}$ and $n^{-1}S_{3,n} \to 0$ as $n \to \infty$ w.p.1, we also have $\bar{G}_n \equiv n^{-1}G_n \to E[Y]$ w.p.1 as $n \to \infty$, which directly implies equation (13). The final claimed independence follows from equation (17) and the independence assumed in Theorem 1. □

For designing a par-4 hole and for improving the pace of play, it is important to see how the random variable Y depends on the stage playing times S_i for $i = 1, 2, 3$. For example, formula equation (15) implies that Y decreases directly with a decrease in S_2, but Y fails to change if we decrease only the smaller of S_1 and S_3.

We now formalize the notion that greater variability in the stage playing times will tend to increase the random variable Y in equation (15) and thus decrease the maximum possible throughput, as discussed in section 1.3. To do so, we use the notion of convex (or variability) stochastic ordering; see Chapter 9 of Ross (1996). We say that one random vector Z_1 in $\mathbb{R}^k$ is stochastically less variable than another Z_2, and write $Z_1 \leq_c Z_2$, if

$$E[f(Z_1)] \leq E[f(Z_2)] \quad \text{for all convex real-valued functions} \quad f. \tag{19}$$

COROLLARY 1. (*convex stochastic order*) *If the assumptions of Theorem 1 are satisfied for two random vectors* $(S_1^{(i)}, S_2^{(i)}, S_3^{(i)})$ *with* i=1,2 *and*

$$(S_1^{(1)}, S_2^{(1)}, S_3^{(1)}) \leq_c (S_1^{(2)}, S_2^{(2)}, S_3^{(2)}) \quad \text{in} \quad \mathbb{R}^3, \tag{20}$$

then

$$E[f(Y^{(1)}] \leq E[f(Y^{(2)})] \ \text{ for all nondecreasing convex real-valued functions} \ f. \tag{21}$$

PROOF. From equation (15), we see that Y is a convex function of the vector (S_1, S_2, S_3), but then any nondecreasing convex real-valued function of a convex function is itself convex. □

We conclude this subsection by giving an explicit representation of the random cycle times $C_n \equiv G_n - G_{n-1}$ in terms of the stage playing times, from which we can see that they are one-dependent under the conditions of Theorem 1. However, this structural results seems less revealing than Theorem 2. *Let* $(x)^+ = \max\{x, 0\}$.

THEOREM 3. (*random cycle times*) *For a fully loaded par-4 hole,*

$$C_n \equiv G_n - G_{n-1} = (S_{1,n} - S_{3,n-1})^+ + S_{2,n} + S_{3,n}, \quad n \geq 2. \tag{22}$$

PROOF. From equations (12) and (17), for $n \geq 2$,

$$\begin{aligned} C_n &\equiv G_n - G_{n-1} = F_n + S_{3,n} - F_{n-1} - S_{3,n-1} \\ &= Y_n + S_{3,n} - S_{3,n-1} = (S_{1,n} \vee S_{3,n-1}) + S_{2,n} \\ &+ S_{3,n} - S_{3,n-1} = (S_{1,n} - S_{3,n-1})^+ + S_{2,n} + S_{3,n}. \end{aligned}$$

□

3.3. Playing Times in a Fully-Loaded Model

We now expose the efficiency of having two groups allowed to play on a par-4 hole simultaneously. In the previous subsection, we saw that the long-run average time between successive groups completing play of a fully loaded par-4 hole is $E[Y]$, while the time each group spends on the par-4 hole is the playing time $X_n \equiv G_n - B_n$. We now show that the playing times on a fully loaded par-4 hole are in steady state for all $n \geq 2$, with a mean that is greater than $E[Y]$. Moreover, we quantify the difference. We also identify the waiting time while playing, X_n^w, and its mean.

THEOREM 4. (*the playing times for the fully loaded model*) *In the fully loaded par-4 model, for* $n \geq 2$, *the playing times,* X_n, *and waiting times while playing,* X_n^w, *simplify to*

$$\begin{aligned} (X_n, X_n^w) &\equiv (G_n - B_n, X_n - X_n^p) \\ &= (Y_n + S_{3,n}, (S_{1,n} \vee S_{3,n-1}) - S_{1,n}). \end{aligned} \tag{23}$$

Under the assumptions of Theorem 1, the distribution of (X_n, X_n^w) *is independent of* n *for all* $n \geq 2$, *with*

$$E[X_n] = E[Y] + E[S_3] \quad \text{and} \quad E[X_n^w] = E[(S_1 \vee S_3)] - E[S_1] \quad \text{for all} \quad n \geq 2. \tag{24}$$

PROOF. From the definition of a playing time, the recursion equation (12) and Theorem 2, it follows immediately that

$$X_n \equiv G_n - B_n = F_n + S_{3,n} - F_{n-1} = Y_n + S_{3,n}, \quad n \geq 2.$$

Recall that Y_n itself involves $S_{3,n-1}$, not $S_{3,n}$. Thus, under the assumptions of Theorem 1, the distribution of X_n is independent of n for all $n \geq 2$. For the waiting time while playing,

$$\begin{aligned} X_n^w &\equiv X_n - X_n^p = X_n - (S_{1,n} + S_{2,n} + S_{3,n}) \\ &= (S_{1,n} \vee S_{3,n-1}) - S_{1,n}, \end{aligned}$$

as claimed. □

3.4. The Moments of the Random Variable Y

We now derive the moments of the random variable Y, which plays a key role for the fully loaded par-4 hole. Let H and H_i be the cdf's of Y and S_i, respectively, for example, $H_i(x) \equiv P(S_i \leq x)$ and let $\bar{H}_i(x) \equiv 1 - H_i(x)$ be the complementary cdf (ccdf).

THEOREM 5. (*moments of* Y) *Consider the fully loaded par-4 model.* (*a*) *If the assumptions of Theorem 1 hold, then*

$$E[Y] = \int_0^\infty H^*(x)dx, \quad \text{where} \quad H^*(x) \equiv \bar{H}_1(x) + \bar{H}_2(x) + \bar{H}_3(x) - \bar{H}_1(x)\bar{H}_3(x). \tag{25}$$

(*b*) *If, in addition, the random variables* S_1, S_2, *and* S_3 *are mutually independent and the cdf's* H_1 *and* H_3 *have densities* h_1 *and* h_3, *then*

$$\mathrm{Var}(Y) = \mathrm{Var}(S_2) + \mathrm{Var}(S_1 \vee S_3), \tag{26}$$

where,

$$E[(S_1 \vee S_3)^2] = \int_0^\infty E[S_1^2 | S_1 > x] P(S_1 > x) h_3(x) dx + \int_0^\infty E[S_3^2 | S_3 > x] P(S_3 > x) h_1(x) dx \tag{27}$$

and

$$E[(S_1 \vee S_3)] = \int_0^\infty [\bar{H}_1(x) + \bar{H}_3(x) - \bar{H}_1(x)\bar{H}_3(x)]dx. \tag{28}$$

Proof. From equation (17), we have

$$Y \stackrel{\mathrm{d}}{=} (S_1 \vee S_3) + S_2 = S_1 + S_2 + S_3 - (S_1 \wedge S_3), \tag{29}$$

where S_1 is independent of S_3 and $\wedge$ denotes the minimum, so that

$$E[Y] = E[S_1] + E[S_2] + E[S_3] - E[S_1 \wedge S_3]. \tag{30}$$

Since S_1 is independent of S_3, we can exploit the tail integral formula for each of the four means to obtain (25). Moving on to the variance, we can use the first expression in equation (29) with the extra assumption and write equation (26), with equations (27) and (28) following by direct argument. □

We now give explicit expressions for the moments of Y when S_1, S_2, and S_3 are mutually independent exponential random variables. We anticipate that, as in many production settings, the stage playing times actually should be considerably less variable than an exponential distribution under normal conditions. However, it is unclear what will be the impact of some of the more serious sources of variability discussed in section 1.3. Thus, we are including this case because it provides a tractable frame of reference. The formulas also can be useful as checks on simulations. We consider more realistic stage service-time distributions in the next section.

Corollary 2. (*moments of* Y *for independent exponential playing times*) *For the fully loaded par-4 hole in which* S_i *are mutually independent exponential random variables with means* μ_i^{-1}, $i = 1, 2, 3$, *then*

$$E[Y] = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} - \frac{1}{\mu_1 + \mu_3} \tag{31}$$

and

$$\mathrm{Var}(Y) = \frac{1}{\mu_2^2} + \frac{1}{(\mu_1 + \mu_3)^2} + \mathrm{Var}(Z), \tag{32}$$

where $Z \equiv (S_1 \vee S_3) - (S_1 \wedge S_3)$,

$$E[Z^2] = P(S_1 < S_3)E[S_3^2] + P(S_3 < S_1)E[S_1^2] = \left(\frac{\mu_1}{\mu_1 + \mu_3}\right)\frac{2}{\mu_3^2} + \left(\frac{\mu_3}{\mu_1 + \mu_3}\right)\frac{2}{\mu_1^2}$$

and

$$E[Z] = P(S_1 < S_3)E[S_3] + P(S_3 < S_1)E[S_1] = \left(\frac{\mu_1}{\mu_1 + \mu_3}\right)\frac{1}{\mu_1} + \left(\frac{\mu_3}{\mu_1 + \mu_3}\right)\frac{1}{\mu_3}.$$

If, in addition, $\mu_1 = \mu_3$, *then* $Z \stackrel{\mathrm{d}}{=} S_1 \stackrel{\mathrm{d}}{=} S_3$, *so that*

$$\mathrm{Var}(Y) = \frac{1}{\mu_2^2} + \frac{5}{4\mu_1^2}. \tag{33}$$

Proof. First, we obtain equation (31) directly from equation (29) and the property that the minimum of exponential variables is exponential with a rate equal to the sum of the rates. For equation (32), we use equation (26) and

$$(S_1 \vee S_3) = (S_1 \wedge S_3) + Z, \tag{34}$$

where, by the lack of memory property, these are independent with the displayed moments. For the special case in which $S_1 \stackrel{\mathrm{d}}{=} S_3$, those formulas simplify. In particular, we get

$$\mathrm{Var}(Y) = \mathrm{Var}(S_2) + \mathrm{Var}(S_1 \vee S_3) = E[S_2]^2 + \left(\frac{E[S_1]}{2}\right)^2 + \mathrm{Var}(S_1),$$

which implies equation (33). □

4. Models of the Stage Playing Times

In order to apply the model of a par-4 hole as developed in section 2 and determine the maximum throughput for the hole, we need to specify the stage playing time distributions. For this purpose, it is natural to rely on performance data from golf courses; that is, we would directly fit the distributions to data on group play. However, to gain insight, it may also be useful to develop relatively parsimonious models with only a few parameters; for example, this is useful to conduct simulation experiments and do further mathematical analysis. In this section, we show how that can be done.

We first introduce general parametric structure to reduce the number of parameters. Then, we consider triangular stage playing time distributions, similar to those used in Tiger and Salzer (2004). Finally, we develop a model to account for the extra variability caused by an exceptional long delay, as occurs with a lost ball. Rare long stage service times can have a dramatic impact, as shown by Riccio (2012, 2013).

4.1. Parametric Models with Special Structure

We introduce additional structure in the stage playing time distributions in order to obtain models with only a few parameters. In particular, in the spirit of p. 94 of Riccio (2012) or p. 32 of Riccio (2013), we assume that S_1, S_2, and S_3 are mutually independent random variables with

$$S_1 \stackrel{\mathrm{d}}{=} S_3, \tag{35}$$

so that one parameter can be the means $E[S_1] = E[S_3]$; that is, we have

$$m = E[S_1] = E[S_3] = \frac{E[S_2]}{r}. \tag{36}$$

Hence, there are only the two parameters m and r beyond the distributions of S_1 and S_2. Moreover, we can go further by letting S_2 have a distribution of the same type as S_1.

Example 1. (A concrete exponential example). We now consider a concrete example of the exponential stage playing times satisfying equations (35) and (36). To characterize the variability of Y, let c_Y^2 be its scv. For this example, we have

$$\begin{aligned} E[Y] &= \frac{(3+2r)m}{2}, \quad \mathrm{Var}(Y) = \frac{(5+4r^2)m^2}{4} \quad \text{and} \\ c_Y^2 &\equiv \frac{\mathrm{Var}(Y)}{E[Y]^2} = \frac{5+4r^2}{(3+2r)^2}. \end{aligned} \tag{37}$$

Following Riccio (2012, 2013) again, let $m = 6$ and $r = 1/2$. First, for $r = 1/2$, $E[Y] = 3m/2$, $\mathrm{Var}(Y) = 3m^2/2$ and $c_Y^2 = 3/8 = 0.375$. From equation (37), we see that, for $m = 6$, $E[Y] = 12$ and $\mathrm{Var}(Y) = 54$. Since the expected total playing time on an 18-hole course without any waiting would be $18\times 15 = 270$ minutes or 4.5 hours, the value $m = 6$ gives reasonable sojourn times on the golf course only in the case of deterministic stage playing times (which were used by Riccio 2012). For realistic random stage service times, a more realistic value evidently would be $m = 4$.

4.2. The Case of Independent Triangular Stage Playing Times

Aiming for a more realistic model than an exponential distribution, we now follow Tiger and Salzer (2004) and assume that S_i has a triangular distribution, but we simplify by assuming that it is symmetric. In particular, let

$$S_i \stackrel{\mathrm{d}}{=} m_i - a_i + 2a_i T, \tag{38}$$

where $T \equiv T[0, 1]$ is a (symmetric) triangular distribution on the interval [0, 1] with density

$$f_T(x) = 4t, \quad 0 \le t \le 0.5, \quad \text{and} \quad 4-4t, \quad 0.5 \le t \le 1. \tag{39}$$

so that $E[T] = 1/2$ and $\mathrm{Var}(T) = 1/24$. An asymmetric triangular distribution with higher probability on large values than small values would evidently be more realistic, but we use the more elementary symmetric model for its tractability. We achieve an asymmetric distribution in the next section by adding the possibility of extra delays, as caused by lost balls.

Definition (38) is tantamount to assuming that S_i has a triangular distribution on the interval $[m_i - a_i, m_i + a_i]$, so that $E[S_i] = m_i$ and $\mathrm{Var}(S_i) = a_i^2/6$. We further simplify by assuming that equation (36) holds and

$$a_1 = a_2 = a_3 = a, \tag{40}$$

so that there are only the three parameters m, r, and a. With that simplifying assumption, we have

$$S_1 \vee S_3 = m - a + 2a(T_1 \vee T_2), \tag{41}$$

where T_1 and T_2 are two i.i.d. triangular random variables on [0, 1].

Since

$$P(T \le t) = 2t^2, \quad 0 \le t \le 1/2, \quad \text{and}$$
$$P(T \le t) = 1 - 2(1-t)^2, \quad 1/2 \le t \le 1,$$

we have

$$\begin{aligned} P(T_1 \vee T_3 \le t) &= P(T \le t)^2 = 4t^4 \quad \text{and} \\ & f_{T_1 \vee T_3}(t) = 16t^3, \quad 0 \le t \le 1/2, \\ P(T_1 \vee T_3 \le t) &= (1 - 2(1-t)^2)^2 \quad \text{and} \\ & f_{T_1 \vee T_3}(t) = 8[-1 + 5t - 6t^2 + 2t^3], \\ & 1/2 \le t \le 1, \end{aligned} \tag{42}$$

so that $E[T_1 \vee T_3] = 37/60$ and $\mathrm{Var}(T_1 \vee T_3) = 101/3600$. Hence,

$$E[S_1 \vee S_3] = m + \frac{7a}{30} \quad \text{and} \quad \mathrm{Var}(S_1 \vee S_3) = \frac{101a^2}{900} \tag{43}$$

and

$$E[Y] = (1+r)m + \frac{7a}{30} \quad \text{and} \quad \mathrm{Var}(Y) = \frac{251a^2}{900}. \tag{44}$$

EXAMPLE 2. (TRIANGULAR ANALOG OF EXAMPLE 1). Just as in Example 1, suppose that $m = 6$ and $r = 1/2$. If $a = 3$ (which is as large as possible), then $E[Y] = 9.7$, $\mathrm{Var}(Y) = 2.51$ and $c_Y^2 = 0.02667$; if $a = 1$, then $E[Y] = 9.233$, $\mathrm{Var}(Y) = 0.2789$ and $c_Y^2 = 0.003271$. Notice that the variability as measured by c_Y^2 is much less than for the exponential distribution.

4.3. Modification for Occasional Lost Balls

The triangular distribution captures the variability we expect to have in stage playing times under normal circumstances. The variability is significantly less than the exponential distribution. However, there can be unexpected delays, such as are caused by a lost ball, which makes the time much longer than it would be otherwise. To avoid excessive delays, golf courses often impose an upper limit on the time spent looking for a lost ball, such as 5 minutes, but there could be more than one lost ball on any given hole.

To model these rare events in a relatively simple way, we consider random extra delays at an upper limit. For simplicity, we assume that a lost ball can only occur during the first stage, including the tee shot and the following walk, so we only modify the distribution of S_1. We first let this upper limit be the constant value L minutes. We then assume that such unexpected events occur for each group on the first stage of each hole with probability p. So, we introduce the two extra parameters p and L.

Thus, given any of the models for S_i discussed above, this modification leads to a new distribution for S_1. Let the new random time for group play on stage 1 be $\bar{S}_1$, and let $\bar{Y}$, be the new random time between successive times to clear the green, which is still defined by equation (15), but with $\bar{S}_1$ replacing S_1. Now we have

$$P(\bar{S}_1 = S_1) = 1 - p \quad \text{and} \quad P(\bar{S}_1 = L) = p. \tag{45}$$

Then the first two moments of $\bar{S}_1$ are

$$\begin{aligned} E[\bar{S}_1] &= (1-p)E[S_1] + pL \quad \text{and} \\ E[\bar{S}_1^2] &= (1-p)E[S_1^2] + pL^2, \end{aligned} \tag{46}$$

so that

$$\mathrm{Var}(\bar{S}_1) = (1-p)[\sigma_{S_1}^2 + p(L - E[S_1])^2]. \tag{47}$$

Then

$$\bar{Y} = (\bar{S}_1 \vee S_3) + S_2.$$

However, for bounded stage playing times such as occur with the triangular distribution, we can go further. If, in addition to equations (35) and (36), we have

$$P(S_3 \le L) = 1, \tag{48}$$

then we have

$$\begin{aligned} E[\bar{Y}] &= p(L + rm) + (1-p)E[Y] \quad \text{and} \\ E[\bar{Y}^2] &= p(L + rm)^2 + (1-p)E[Y^2]. \end{aligned} \tag{49}$$

Hence, we can combine the lost-ball feature with the model in section 4.2 to obtain a tractable model. If we add condition equation (40), then we obtain tractable models depending on the parameter 5-tuple (m, r, a, p, L). We can thus incorporate the rare lost ball with the usual low variability of the triangular distribution to obtain a final estimate of the pace of play. In particular, we can combine equations (49) and (44) to obtain the first two moments of $\bar{Y}$ for the *tri + LB* model with parameter 5-tuple (m, r, a, p, L):

$$\begin{aligned} E[\bar{Y}] &= p(L + rm) + (1-p)\left((1+r)m + \frac{7a}{30}\right) \\ E[\bar{Y}^2] &= p(L + rm)^2 + (1-p)\left(\frac{251a^2}{900} + \left((1+r)m + \frac{7a}{30}\right)^2\right). \end{aligned} \tag{50}$$

EXAMPLE 3. (EXAMPLE 2 REVISITED WITH THE TRIANGULAR DISTRIBUTION AND LOST BALLS). For example, Let S_i have triangular distributions as in section 4.2 with parameters m, r and a. Since $E[S_1] = m$ and $\mathrm{Var}(S_1) = a^2/6$.

$$\begin{aligned} E[\bar{S}_1] &= (1-p)m + pL \quad \text{and} \\ E[\bar{S}_1^2] &= (1-p)\left(m^2 + \frac{a^2}{6}\right) + pL^2, \end{aligned} \tag{51}$$

Suppose that we use the parameters $m = 6$ and $r = 1/2$ for the triangular distribution, as in Example 2, and let $p = 0.05$ and $L = 12$ for the lost balls. Then,

$$\begin{aligned} E[\bar{S}_1] &= 0.95(6) + (0.05)(12) = 6.3 \quad \text{and} \\ E[\bar{S}_1^2] &= (0.95)\left(36 + \frac{9}{6}\right) + (0.05)(144) = 42.825 \end{aligned} \tag{52}$$

so that

$$\mathrm{Var}(\bar{S}_1) = 42.825 - (6.3)^2 = 42.825 - 39.69 = 3.135 \tag{53}$$

Since $\mathrm{Var}(S_1) = 1.50$, the variance of S_1 increased by more than a factor of 2.

The next step is to determine the distribution of $\bar{Y}$. Notice that

$$P(\bar{Y} = 12 + S_2) = 0.05 = 1 - P(\bar{Y} = Y). \qquad (54)$$

Hence,

$$\begin{aligned} E[\bar{Y}] &= (0.05)(15) + (0.95)(9.7) \\ &= 0.75 + 9.215 = 9.965 \quad \text{and} \\ E[\bar{Y}^2] &= (0.05)(225) + (0.95)(96.6) \\ &= 11.25 + 91.71 = 102.96 \end{aligned} \qquad (55)$$

so that

$$\begin{aligned} \mathrm{Var}(\bar{Y}) &= 102.96 - (9.965)^2 = 102.96 - 99.30 \\ &= 3.66 \quad \text{and} \quad c^2_{\bar{Y}} = 3.66/(9.965)^2 \\ &= 3.66/99.30 = 0.03865 \end{aligned} \qquad (56)$$

As expected, $c^2_{\bar{Y}}$ is considerably greater than $c^2_Y = 0.02667$, by a factor of about 1.5, but $c^2_{\bar{Y}}$ is still about 10 times smaller than for the exponential distribution. It is thus natural to regard the exponential distribution as a crude upper bound.

We show histograms of Y (based on $n = 10^5$ groups) when the stage playing times have the triangular distribution with $(m, r, a) = (6, 0.5, 3)$ in Figure 1 and that triangular distribution modified to account for lost balls with $(p, L) = (0.05, 12)$ in Figure 2. The lost balls clearly produce a heavier upper tail, but within a reasonable range, because Y remains bounded above by $(\bar{S}_1 \vee S_3) + S_2 \leq 12 + (3 + 3) = 18$ (compared to $((S_1 \vee S_3) + S_2 \leq ([6 + 3] \vee [6 + 3]) + (3 + 3) = 15$ for the triangular distribution).

Figure 1 Histogram of the Distribution of Y when the Stages have a Triangular Distribution with (*m*, *a*) = (6, 3), having mean 9.71 and variance 2.51

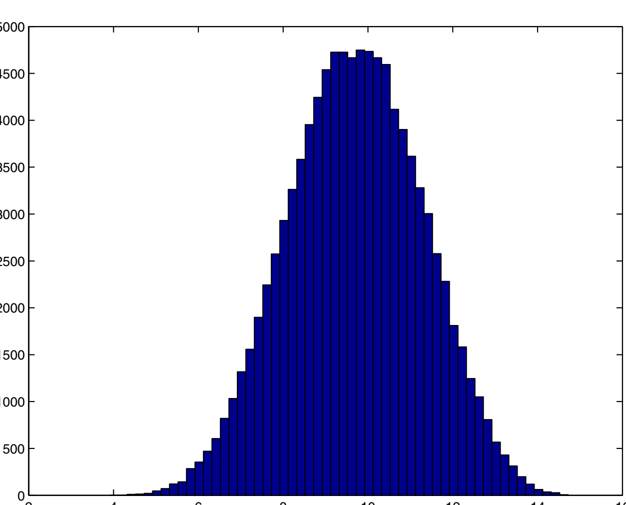
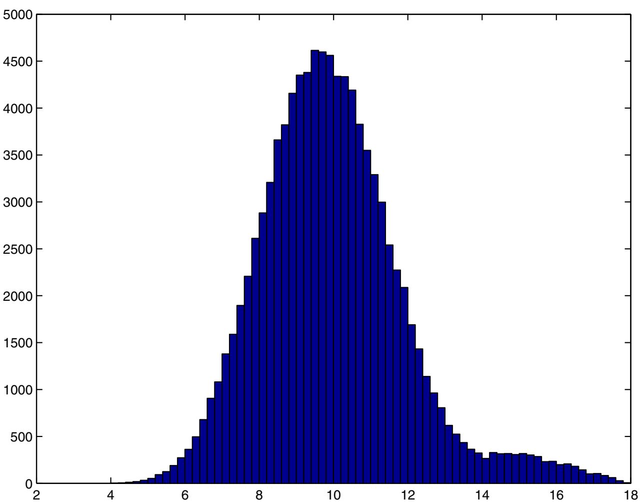


Figure 2 Histogram of the Distribution of Y when the Stages have a Triangular Distribution Modified to Account for Lost Balls with (*p*, *L*) = (0.05, 12), having mean 9.97 and variance 3.81



5. A Par-3 Hole, with and without Wave-up

There are three steps for group play on a par-3 hole, with or without wave-up:

$$T \rightarrow W \rightarrow G.$$

The first step T is hitting shots off the tee; the second step W is walking to the green, possibly including approach shots; and the third step G is putting on the green. In this case, we identify the stages with steps, but speak of stages, to be consistent with par-4.

A par-3 hole without wave-up is relatively simple, because only one group can be on the course at that hole at any one time. If we have stage playing times $S_{i,n}$ for group n as before, then the total time for group n to play the hole is $X_n = S_{1,n} + S_{2,n} + S_{3,n}$. And these group playing times are also the cycle times in this case.

We now compare a par-3 hole to a par-4 hole, assuming identical stage playing times. We provide conditions under which the capacity of a par-4 hole is greater than the capacity of a par-3 hole and quantify the difference. Let a superscript denote the hole type.

Corollary 3. *Suppose that the sequence of stage playing time vectors $\{(S_{1,n}, S_{2,n}, S_{3,n}) : n \geq 1\}$ is i.i.d. and distributed as (S_1, S_2, S_3), where S_i is strictly positive. Suppose that these are used on both par-3 and par-4 holes. Then,*

$$E[X^{(3)}] - E[Y^{(4)}] = E[S_1 + S_3] - E[(S_1 \vee S_3)] > 0. \quad (57)$$

Corollary 3 is consistent with experience indicating that the par-3 holes often tend to be bottleneck holes. That is the motivation for the wave-up rule discussed next.

5.1. A Par-3 Hole with Wave-Up

The wave-up rule stipulates that, after a group has hit its tee shots and walked up to their balls near the green, they should wait before clearing the green until the following group hits its tee shots, provided that the following group has already arrived and is ready to play. If the following group has not yet arrived at the hole, then the current group immediately starts stage 3. The following group then cannot start play on the hole until after the current group completes stage 3 and departs. The wave-up rule is intended to reduce the expected time between successive groups clearing the green, and thus increase the capacity of par-3 holes. We show how to quantify that benefit.

We now develop the recursion for the par-3 hole with wave-up. Let A_n be the time that group n arrives at the hole and is ready to play. Let B_n be the time that group n starts playing on this hole, that is, the instant when one of the group goes into the tee box. Let T_n be the time that group n completes stage 1, the tee shots; let W_n be the time that group n completes stage 2, its walk to the green and its chip shots; and let G_n be the time that group n completes stage 3, and clears the green.

The wave-up rule makes the formulas for B_n and G_n in terms of the other variables somewhat complicated. At time $W_n \vee G_{n-1}$, group $n-1$ has cleared the green and group n has completed stage 2, so that group n is ready to play stage 3. However, group $n+1$ may impose a constraint. At time $W_n \vee G_{n-1}$, group n can start stage 3 (to play on the green) only if either (i) group $n+1$ has not yet arrived at the hole and is not ready to tee off or if (ii) group $n+1$ has completed its tee shots. Otherwise, group n starts stage 3 at time T_{n+1}. Thus, we introduce the event E_n, defined by

$$E_n \equiv \{A_n \leq W_{n-1} \vee G_{n-2} < T_n\}, n \geq 1, \qquad (58)$$

and let E_n^c be its complement, $n \geq 1$. If group n is the last scheduled group, then let $A_{n+1} \equiv \infty$ (or a very large value), so that the event E_{n+1} ever occurs.

Thus the wave-up rule is specified by the following four-part recursion:

$$\begin{aligned} B_n &\equiv (W_{n-1} \vee G_{n-2})1_{E_n} + (A_n \vee G_{n-1})1_{E_n^c},\ n \geq 2, B_1 \equiv A_1, \\ T_n &\equiv B_n + S_{1,n}, W_n \equiv T_n + S_{2,n},\ n \geq 1, \text{and} \\ G_n &\equiv [(W_n \vee G_{n-1})1_{E_{n+1}^c} + T_{n+1}1_{E_{n+1}}] + S_{3,n} \\ &\equiv (W_n \vee G_{n-1}) + S_{1,n+1}1_{E_{n+1}} + S_{3,n}. \end{aligned} \qquad (59)$$

As initial conditions, assuming that the system starts empty, we set $W_0 \equiv G_0 \equiv G_{-1} \equiv 0$. If n is the last group, then, instead,

$$G_n \equiv (W_n \vee G_{n-1}) + S_{3,n}.$$

Note that the expression for B_n involves G_{n-2}, because only two groups can be playing on the hole at the same time. Observe that the event E_n in (58) actually simplifies. By the first line of (59),

$$T_n = B_n + S_{1,n} \geq W_{n-1} + S_{1,n} > W_{n-1},$$

so that $E_n = \{A_n \leq W_{n-1} \vee G_{n-2}\}$. Note that care is needed in treating the last group to play, if n is the last group, then A_{n+1} is made large, so that E_{n+1} never occurs.

5.2 A Fully Loaded Par-3 Hole with Wave-up

For a fully loaded hole, the recursion in (59) simplifies, because $A_n = 0$ for all n, except for the last group. In particular, except for the last group, the event E_n in (58) occurs for all n and the recursion in (59) becomes

$$\begin{aligned} B_n &\equiv W_{n-1} \vee G_{n-2}, \quad T_n \equiv B_n + S_{1,n}, \\ W_n &\equiv T_n + S_{2,n} \quad \text{and} \quad G_n \equiv T_{n+1} + S_{3,n}, \quad n \geq 1. \end{aligned} \qquad (60)$$

Again, as initial conditions, assuming that the system starts empty, we set $W_0 \equiv G_0 \equiv G_{-1} \equiv 0$. If n is the last group, then, instead, $G_n \equiv W_n \vee G_{n-1} + S_{3,n}$.

We now show that this recursion simplifies, so that we can identify the maximum throughput. Indeed, we find that the random cycle time of a par-3 hole with wave-up has the same structure as for a par-4 hole, but with the stage playing times playing different roles. The following result parallels Theorem 2, with T_n playing the role of F_n before.

THEOREM 6 (*representation for T_n as a partial sum*) *For the fully loaded par-3 hole with wave-up, the recursion for T_n in equation (60) can be expressed as*

$$T_n = T_{n-1} + Y_n, \quad n \geq 1, \quad \text{so that} \quad T_n = \sum_{k=1}^{n} Y_k, \quad n \geq 1, \text{where} \qquad (61)$$

$$Y_n \equiv (S_{2,n-1} \vee S_{3,n-2}) + S_{1,n}, \quad n \geq 1, \qquad (62)$$

with $T_0 = 0 \equiv S_{2,0} \equiv S_{3,0} \equiv S_{3,-1} \equiv 0$. The playing time is $X_n \equiv G_n - B_n = Y_{n+1} + S_{1,n} + S_{3,n}$.

PROOF. From the first steps of equation (60), we get $B_1 = 0$, $T_1 = S_{1,1}$, $B_2 = W_1 = S_{1,1} + S_{2,1}$, and $T_2 = S_{1,1} + S_{2,1} + S_{1,2} = T_1 + S_{2,1} + S_{1,2}$, so that $G_1 = T_2 + S_{3,1}$ and $T_3 = T_2 + (S_{2,2} \vee S_{3,1}) + S_{1,3}$. We then can imply mathematical induction. We get $T_n = (W_{n-1} \vee G_{n-2}) +$

$S_{1,n}$. We then insert the expression for G_{n-2} to get $T_n = (W_{n-1} \vee [T_{n-1}+S_{3,n-2}])+S_{1,n}$ and the expression for W_{n-1} to get

$$T_n = ([T_{n-1} + S_{2,n-1}] \vee [T_{n-1} + S_{3,n-2}]) + S_{1,n}, \quad (63)$$

from which equation (62) follows. The playing time X_n is obtaining by combining $G_n = T_{n+1}+S_{3,n}$, $T_n = B_n+S_{1,n}$ and $T_{n+1}-T_n = Y_{n+1}$. □

We thus can apply Theorem 6 to obtain the following analog of Theorem 1.

THEOREM 7. (*Maximum throughput*) *Consider the fully loaded par-3 model with wave-up in which the sequence of stage playing time random vectors* $\{(S_{1,n}, S_{2,n}, S_{3,n}): n \geq 1\}$ *is i.i.d. each distributed as the random vector* (S_1, S_2, S_3), *whose components are strictly positive with finite means. Then*

$$\bar{C}_n \to E[Y] \quad and \quad \Theta_n \to \frac{1}{E[Y]} \equiv \theta^* \quad as \quad n \to \infty \ w.p.1 \quad (64)$$

where Y *is a generic random variable distributed as* Y_n *in equation (62), that is,*

$$Y \stackrel{d}{=} (S_2 \vee S_3) + S_1, \quad (65)$$

The average playing time converge to $E[X]=E[Y]+E[S_3]+E[S_1]$.

PROOF. Given Theorem 1 and the extra condition, we can apply the LLN to deduce the stated limit for $n^{-1}T_n$. The relation equation (62) implies that the sequence $\{Y_n\}$ is two-dependent. Hence, the proof is a minor variant of the proof of Theorem 1. We obtain the other limits because $T_n - W_n = S_{2,n}$ and $G_n - T_{n+1} = S_{3,n}$. Here, we use the fact that $n^{-1}S_{j,n} \to 0$ w.p.1 as $n \to \infty$. The reasoning applies to the playing times X_n, except that they are 3-dependent instead of 2-dependent. □

Theorems 6 and 7 show that the fully loaded par-3 hole with wave-up has essentially the same mathematical structure as a fully loaded par-4 hole, except that different random variables appear in the expression for Y_n in equation (62) and Y in equation (65) than appeared in equations (17) and (15). We see that both the fully loaded par-4 and the fully loaded par-3 hole with wave-up can easily be analyzed in detail.

We can also compare the par-4 hole to the par-3 hole with wave-up. Under extra conditions, we see a reversal of the ordering in Corollary 3; the maximum throughput on a par-4 hole becomes larger than on a par-3 with wave-up. We say that S_1 is stochastically greater than or equal to S_2 and write $S_1 \geq_{st} S_2$ if $P(S_1 > t) \geq P(S_2 > t)$ for all t.

COROLLARY 4. *Suppose that the three sequences of stage playing time vectors* $\{S_{i,n}: n \geq 1\}$, $1 \leq i \leq 3$ *are independent and each is i.i.d. and distributed as* S_i. *Suppose that these are used on both a par-4 hole and a par-3 hole with wave-up. If* $S_1 \geq_{st} S_2$ *(as seems natural), then*

$$E[Y^{(3)}] \geq E[Y^{(4)}] \quad \text{so that} \quad \theta^{3*} \leq \theta^{4*}. \quad (66)$$

On the other hand, if $S_1 \leq_{st} S_2$, *then*

$$E[Y^{(3)}] \leq E[Y^{(4)}] \quad \text{so that} \quad \theta^{4*} \leq \theta^{3*}. \quad (67)$$

PROOF. We only prove equation (66), because the proof for (67) is essentially the same. Note that

$$(S_1 \vee S_3) - S_1 = (S_3 - S_1)^+ \quad \text{and}$$
$$(S_2 \vee S_3) - S_2 = (S_3 - S_2)^+.$$

Use the assumed stochastic order to construct random variables $\tilde{S}_1$ and $\tilde{S}_2$ such that $P(\tilde{S}_1 \geq \tilde{S}_2) = 1$ with $\tilde{S}_i$ distributed the same as S_i, i = 1, 2; see Prop. 9.2.2 of Ross (1996). That implies that $P((S_3 - \tilde{S}_1)^+ \leq (S_3 - \tilde{S}_2)^+) = 1$, so that $E[(S_3 - S_1)^+] \leq E[(S_3 - S_2)^+]$, which in turn implies that $E[S_1 \vee S_3] - E[S_1] \leq E[S_2 \vee S_3] - E[S_2]$, which implies equation (66) by just adding $E[S_1] + E[S_2]$ to both sides. □

6. A Par-5 Hole

In contrast, a par-5 hole is more complicated, because now three groups can be on the course simultaneously. For a par-5 hole, we identify seven steps instead of the five steps for a par-4 hole. There now are two fairway shots instead of only one and three walking steps instead of only two. These seven steps can be grouped into five stages, as opposed to three for a par-4 hole:

$$(T, W_1) \to F_1 \to W_2 \to F_2 \to (W_3, G)$$

Assuming an empty system initially, the first group can do all the stages, one after another without constraint. However, for $n \geq 2$, group n cannot start stage 1 until *both* group n arrives at the tee and group $n-1$ has completed stage 2, that is, has completed its fairway shots (completed F_1). Similarly, for $n \geq 2$, group n cannot start stage 2 until *both* group n arrives at stage 2 and group $n-1$ has completed stage 4, that is, has cleared the second fairway shot (completed F_2). After completing stage 2, each group may go right on to stage 3. For $n \geq 2$, group n cannot start stage 4 until *both* group n arrives at stage 4 and group $n-1$ has completed stage 5, that is, has cleared the green (completed (W_3, G)). After completing stage 4, each group may go right on to stage 5.

As before, let A_n be the arrival time of the nth group at the tee of this hole on the golf course. Let $S_{i,n}$ be the time required for group n to complete stage i, $1 \leq i \leq 5$. Let B_n be the time that group n starts playing on this hole, that is, the instant when one of the group goes into the tee box; let T_n be the time that group n completes stage 1, including the tee and the following walk; let $F_{1,n}$ be the time that group n completes stage 2, their first shots on the fairway; let $W_{2,n}$ be the time that group n completes stage 3, their walk from the first fairway shots to their second ones; let $F_{2,n}$ be the time that group n completes stage 4, their second shots on the fairway; and let G_n be the time that group n completes step 5, and clears the green. Clearly, G_n is the group-n departure or completion time, while B_n is the group-n start time.

The mathematical model that relates the model primitives to the performance consists of the following six-part recursion:

$$\begin{aligned} B_n &\equiv A_n \vee F_{1,n-1}, \quad T_n \equiv B_n + S_{1,n}, \\ F_{1,n} &\equiv (T_n \vee F_{2,n-1}) + S_{2,n}, \quad W_{2,n} \equiv F_{1,n} + S_{3,n}, \\ F_{2,n} &\equiv (W_{2,n} \vee G_{n-1}) + S_{4,n} \quad \text{and} \quad G_n \equiv F_{2,n} + S_{5,n}, \end{aligned} \tag{68}$$

$n \geq 1$. As initial conditions, assuming that the system starts empty, we set $F_{1,0} \equiv F_{2,0} \equiv G_0 \equiv 0$.

6.1. A Fully Loaded Par-5 Hole

Paralleling section 3 , we now consider a fully-loaded par-5 hole; that is, all groups are at the hole at time 0 ready to play. Under this fully loaded condition, the recursion in equation (68) reduces to

$$\begin{aligned} B_n &\equiv F_{1,n-1}, \quad T_n \equiv B_n + S_{1,n} = F_{1,n-1} + S_{1,n}, \\ F_{1,n} &\equiv (T_n \vee F_{2,n-1}) + S_{2,n}, \quad W_{2,n} \equiv F_{1,n} + S_{3,n}, \\ F_{2,n} &\equiv (W_{2,n} \vee G_{n-1}) + S_{4,n} \quad \text{and} \quad G_n \equiv F_{2,n} + S_{5,n}, \end{aligned} \tag{69}$$

$n \geq 1$. Again, as initial conditions, assuming that the system starts empty, we set $F_{1,0} \equiv F_{2,0} \equiv G_0 \equiv 0$.

The fully loaded model can be analyzed more directly than the original model with random arrivals, but the structure evidently is much more complicated than for a par-4 hole. We present three preliminary theorems that expose structure and then apply them to characterize the maximum throughput in Theorem 11 below. The preliminary theorems can be regarded as components of the statement and proof of the main result.

THEOREM 8. (*two-dimensional recursion*) *For the fully loaded par-5 hole starting out empty, the sequence of fairway-clearing vectors* $\{(F_{1,n}, F_{2,n}) : n \geq 1\}$ *can be represented as a two-dimensional recursion driven by the vectors* $(S_{1,n}, S_{2,n}, S_{3,n}, S_{4,n}, S_{5,n-1})$ *in the form*

$$\begin{aligned} F_{1,n} &= [(F_{1,n-1} + S_{1,n}) \vee F_{2,n-1}] + S_{2,n}, \\ F_{2,n} &= [(F_{1,n} + S_{3,n}) \vee (F_{2,n-1} + S_{5,n-1})] + S_{4,n}, \end{aligned} \tag{70}$$

$n \geq 2$, *where* $F_{1,1} = S_{1,1} + S_{2,1}$ *and* $F_{2,1} = F_{1,1} + S_{3,1} + S_{4,1} = S_{1,1} + S_{2,1} + S_{3,1} + S_{4,1}$.

PROOF. First, note that the initial values $F_{1,1}$ and $F_{2,1}$ are valid starting empty. Then note that, for $n \geq 2$, the recursion in equation (69) can be expressed as

$$\begin{aligned} F_{1,n} &= (T_n \vee F_{2,n-1}) + S_{2,n} \\ &= [(F_{1,n-1} + S_{1,n}) \vee F_{2,n-1}] + S_{2,n}, \end{aligned} \tag{71}$$

as given in the first line of equation (70). Then,

$$\begin{aligned} F_{2,n} &= (W_{2,n} \vee G_{n-1}) + S_{4,n} \\ &= [(F_{1,n} + S_{3,n}) \vee G_{n-1}] + S_{4,n}, \end{aligned} \tag{72}$$

implying the second line of equation (70). □

We now will develop a single recursion for each of the two differences:

$$D_n \equiv F_{2,n} - F_{1,n} \quad \text{and} \quad V_n \equiv F_{1,n} - F_{2,n-1}, \quad n \geq 2. \tag{73}$$

where these sequences are initialized by $D_1 = S_{3,1} + S_{4,1}$ and $V_1 = F_{1,1} = S_{1,1} + S_{2,1}$.

We will then exploit the relation

$$F_{2,n} - F_{2,n-1} = D_n + V_n, \quad n \geq 2. \tag{74}$$

We also have the exceptional first time, $F_{2,1} = S_{1,1} + S_{2,1} + S_{3,1} + S_{4,1}$.

THEOREM 9. (*one-dimensional recursion for the difference* D_n) *For the fully loaded par-5 hole starting empty,*

$$D_n = [S_{3,n} \vee (S_{5,n-1} - S_{2,n} - (S_{1,n} - D_{n-1})^+)] + S_{4,n}, \quad n \geq 2, \tag{75}$$

and $D_1 = S_{3,1} + S_{4,1}$ *for* D_n *in equation (73). Hence,* D_n *is a nondecreasing nonnegative function of* D_{n-1}, $S_{3,n}$, $S_{4,n}$, $S_{5,n-1}$, $-S_{1,n}$, *and* $-S_{2,n}$ *with* $D_n \leq S_{3,n} + S_{5,n-1} + S_{4,n}$, $n \geq 2$, *so that*

$$E[D_n] \leq E[S_{3,n}] + E[S_{5,n-1}] + E[S_{4,n}], \quad n \geq 1. \tag{76}$$

Moreover, $D_n + V_n$ is an increasing convex function of $(S_{1,n}, S_{2,n}, S_{3,n}, S_{4,n}, S_{5,n})$.

PROOF. We subtract $F_{1,n}$ from the second equation in equation (70) to get

$$\begin{aligned} D_n &\equiv F_{2,n} - F_{1,n} = [S_{3,n} \vee (F_{2,n-1} - F_{1,n} + S_{5,n-1})] + S_{4,n} \\ &= [S_{3,n} \vee (-V_n + S_{5,n-1})] + S_{4,n}, \quad n \geq 2. \end{aligned} \tag{77}$$

We then subtract $F_{2,n-1}$ from the first equation in equation (70) to get

$$\begin{aligned} V_n &\equiv F_{1,n} - F_{2,n-1} = [(F_{1,n-1} - F_{2,n-1} + S_{1,n}) \vee 0] + S_{2,n} \\ &= [(-D_{n-1} + S_{1,n}) \vee 0] + S_{2,n} \\ &= (S_{1,n} - D_{n-1})^+ + S_{2,n}, \end{aligned} \tag{78}$$

where $(x)^+ \equiv (x \vee 0)$. We then substitute equation (78) into equation (77) to get

$$D_n = \left[S_{3,n} \vee (-(S_{1,n} - D_{n-1})^+ - S_{2,n} + S_{5,n-1})\right] + S_{4,n}, \quad n \geq 2, \tag{79}$$

which implies equation (75). To see that D_n is a nondecreasing function of D_{n-1}, notice that, if D_{n-1} increases then $(S_{1,n} - D_{n-1})^+$ necessarily decreases or stays the same, but then $-(S_{1,n} - D_{n-1})^+$ necessarily increases or stays the same. Finally, the upper bound follows from, first, replacing all negative terms by 0 and, second, by applying the elementary inequality $(a \vee b) \leq (a + b)$ for nonnegative a and b. The final conclusion follows by combining equations (77) and (78). □

THEOREM 10. (*one-dimensional recursion for the difference V_n*) *For the fully loaded par-5 hole starting empty,*

$$V_n = (S_{1,n} - D_{n-1})^+ + S_{2,n}, \quad n \geq 2, \tag{80}$$

with $V_1 = F_{1,1} = S_{1,1} + S_{2,1}$ for V_n in equation (73). Hence, V_n is a nondecreasing nonnegative function of $S_{1,n}$, $S_{2,n}$, and $-D_{n-1}$ The sequence $\{V_n : n \geq 1\}$ can also be represented directly as as the one-dimensional recursion

$$\begin{aligned} V_n &\equiv F_{1,n} - F_{2,n-1} = S_{2,n} + (S_{1,n} - D_{n-1})^+ \\ &= S_{2,n} + \left(S_{1,n} - S_{4,n-1} - [S_{3,n-1} \vee (S_{5,n-2} - V_{n-1})]\right)^+, \; n \geq 3, \\ V_2 &\equiv (S_{1,2} - D_1)^+ + S_{2,2} \\ &= (S_{1,2} - S_{3,1} - S_{4,1})^+ + S_{2,2}, \end{aligned} \tag{81}$$

and $V_1 = S_{1,1} + S_{2,1}$. Hence, V_n is a nondecreasing nonnegative function of $S_{1,n}$, $S_{2,n}$, V_{n-1}, $-S_{3,n-1}$, $-S_{4,n-1}$, and $-S_{5,n-2}$, with

$$V_n \leq S_{1,n} + S_{2,n}, \quad \text{so that} \quad E[V_n] \leq E[S_{1,n}] + E[S_{2,n}], \quad n \geq 2. \tag{82}$$

PROOF. The reasoning is similar to the proof of Theorem 9. We start with equation (78) and then insert equation (77), both from the proof of Theorem 9. After elementary algebra, we obtain equation (81). The remaining relations then follow easily. □

We can now apply Theorems 9 and 10, under additional conditions, to determine the maximum throughput. Let $\Rightarrow$ denote convergence in distribution. Let $\stackrel{d}{=}$ denote equality in distribution.

THEOREM 11. (*maximum throughput*) *For the fully loaded par-5 hole starting empty, if the stage playing times come from 5 independent sequences of i.i.d. random variables with finite means, then*

$$D_n \Rightarrow D \quad \text{and} \quad E[D_n] \uparrow E[D] < E[S_3] + E[S_5] + E[S_4] < \infty \quad \text{as} \quad n \to \infty, \tag{83}$$

where the limiting random variable D has a distribution satisfying the stochastic equation

$$D \stackrel{d}{=} [S_3 \vee (S_5 - S_2 - (S_1 - D)^+)] + S_4, \tag{84}$$

with all the random variables on the right side being mutually independent, and

$$V_n \Rightarrow V \quad \text{and} \quad E[V_n] \uparrow E[V] < E[S_1] + E[S_2] < \infty \quad \text{as} \quad n \to \infty, \tag{85}$$

where the limiting random variable can be expressed in terms of the limit D satisfying equation (84) above by

$$V \stackrel{d}{=} S_2 + (S_1 - D)^+. \tag{86}$$

In addition, V satisfies the stochastic equation

$$V \stackrel{d}{=} [S_2 + (S_1 - S_4 - [S_3 \vee (S_5 - V)])^+, \tag{87}$$

with all the random variables on the right side being mutually independent. As a consequence, if in addition the random variables S_2 and S_4 have positive density functions, then

$$\bar{F}_{2,n} \equiv \frac{F_{2,n}}{n} = \frac{1}{n}\sum_{k=1}^{n}(F_{2,k} - F_{2,k-1}) = \frac{1}{n}\sum_{k=1}^{n}(D_k + V_k) \Rightarrow E[D] + E[V] \tag{88}$$

so that

$$\Theta_n \Rightarrow \theta^* = \frac{1}{E[D] + E[V]} \quad \text{as} \quad n \to \infty, \tag{89}$$

where V is characterized directly in equation (87) and in terms of D in equation (86) and D is the unique solution to the stochastic equation (84).

PROOF. Under the assumptions, the sequence $\{D_n: n \geq 1\}$ is a stochastically nondecreasing stochastically bounded Markov chain, that is,

$$0 \equiv D_0 \leq D_1 \leq_{st} D_2 \leq_{st} D_3 \leq_{st} \cdots \tag{90}$$

and

$$0 = E[D_0] \leq E[D_1] \leq E[D_2] \leq E[D_3] \leq \ldots \leq E[S_3] + E[S_5] + E[S_4] < \infty \tag{91}$$

by so that the limit in equation (83) holds. A proper limit exists in equation (83) because the mean is uniformly bounded by equation (91). The same argument applies to the sequence $\{V_n{:}n{\geq}1\}$. The equations (84) and (87) follow by taking limits in equations (75) and (81), exploiting the continuity of the right side. The limit equation (88) follows from the LLN applied to the sequences $\{D_k\}$ and $\{V_k\}$ separately. The extra positive density condition before equation (88) implies that the Markov chains can be regarded as Harris recurrent, so that a coupling argument can be applied to identify an embedded renewal process, justifying the LLN, as in Athreya and Ney (1978) and Lindvall (1992). Then, the limit in equation (89) follows from the convergence-together theorem, Theorem 11.4.7 of Whitt (2002). □

In Theorem 11, we made a stronger assumptions about the stage playing times than we did in Theorems 1 and Theorem 7 for par-4 and par-3 with wave-up. We conjecture that the conclusion remains true if only the stage playing time vectors are i.i.d., but that remains to be proved.

Theorem 11 characterizes the maximum throughput for a par-5 hole, and should prove useful in establishing additional properties, but it does not provide an explicit formula. However, the required means $E[D]$ and $E[V]$ in equation (89) can be calculated by iterating the one-dimensional recursions in (75) and (81), using the averages

$$\bar{D}_n \equiv \frac{1}{n}\sum_{k=1}^{n} D_k \quad \text{and} \quad \bar{V}_n \equiv \frac{1}{n}\sum_{k=1}^{n} V_k \tag{92}$$

for suitably large n. This is an efficient simulation, requiring only that we generate the stage playing time vectors $(S_{1,k}, \ldots, S_{5,k})$, $1 \leq k \leq n$, and then apply the recursions. We illustrate in section 6.3.

We conclude by stating a result for the playing times on a par 5 hole.

COROLLARY 5. (*playing time*) *For the fully-loaded par-5 hole, under the conditions of Theorem* 11, *the playing time is* $X_n \equiv G_n - B_n = (D_n + V_n) + D_{n-1} + S_{5,n}$ *so that its average for n groups converges as* $n \to \infty$ *to*

$$E[X] = E[Y] + E[D] + E[S_5] = 2E[D] + E[V] + E[S_5]. \tag{93}$$

PROOF. From equations (69), (73) and (74)

$$\begin{aligned} X_n \equiv G_n - B_n &= F_{2,n} + S_{5,n} - F_{1,n-1} \\ &= (F_{2,n} - F_{2,n-1}) + (F_{2,n-1} - F_{1,n-1}) + S_{5,n} \\ &= (D_n + V_n) + D_{n-1} + S_{5,n}. \end{aligned} \tag{94}$$

□

6.2. A Possible Simplification

We now develop a simplification of the recursion for a par-5 hole under an additional assumption. Since it can be shown that $D_n \equiv F_{2,n} - F_{1,n} \geq S_{3,n} + S_{4,n}$, it may be reasonable to assume as an approximation that $D_{n-1} \geq S_{1,n}$. Then the explicit formulas in Theorems 9 and Theorem 10 simplify. The elementary proof is omitted.

COROLLARY 6. (*simplification*) *If, in addition to the assumptions of Theorem* 11, $D_{n-1} \geq S_{1,n}$, *then* $(S_{1,n} - D_{1,n-1})^+ = 0$ *and the exact formula for* D_n *in equation* (75), V_n *in equation* (80) *and* $F_{2,n} - F_{2,n-1}$ *simply, producing*

$$\begin{aligned} &D_n = [S_{3,n} \vee (S_{5,n} - S_{2,n})] + S_{4,n},\ V_n = S_2 \text{ and} \\ &F_{2,n} - F_{2,n-1} = Z_n \equiv [S_{3,n} \vee (S_{5,n} - S_{2,n})] + S_{4,n} + S_{2,n},\ n \geq 2. \end{aligned} \tag{95}$$

Hence, the reciprocal of the maximum throughput rate can be expressed as

$$\frac{1}{\theta^*} = E[Z], \quad \text{where} \quad Z \stackrel{d}{=} [S_3 \vee (S_5 - S_2)] + S_4 + S_2 \tag{96}$$

so that

$$\begin{aligned} E[Z] &= E[S_3] + E[S_4] + E[S_5] - E[S_3 \wedge (S_5 - S_2)] \\ &\geq E[S_4] + E[S_5], \end{aligned} \tag{97}$$

where

$$E[S_3 \wedge (S_5 - S_2)] = \int_0^\infty P(S_3 > x)P(S_5 - S_2 > x)dx. \tag{98}$$

6.3. A Simulation Example for a Par-5 Hole

Simulation experiments of a par-5 hole were conducted to numerically verify that the different recursions in section 6.1 are consistent. In particular, we used simulation to estimate the cycle time $E[C_\infty]$. The first recursion is the general recursion in (68) but with a very high arrival rate to make the hole overloaded. This was compared to the direct recursions for the fully loaded par-5 model provided in: (i) equation (69), (ii) equation (70) with $G_n = F_{2,n} + S_{5,n}$, (iii) equations (75) (80), and (74) plus $G_n = F_{2,n} + S_{5,n}$, and (iv) equations (75), (81) and (74) plus $G_n = F_{2,n} + S_{5,n}$. the simulations confirmed that these all agree.

For the example, we let the stage service times be mutually independent, with the times $S_{j,n}$ being i.i.d. for each j, $1 \le j \le 5$. The stage service-time distributions were variants of the models used to study the par-4 hole. In particular, as in section 4.2 , a symmetric triangular distribution was used for each stage, with the possibility of a lost ball in the first stage (which includes driving from the tee). For the basic triangular distributions, we let $a_i = a$ for $1 \le i \le 5$, and we let the mean values be related by

$$m_1 = m_5 = m \quad \text{and} \quad m_i = rm, \quad 2 \le i \le 4. \tag{99}$$

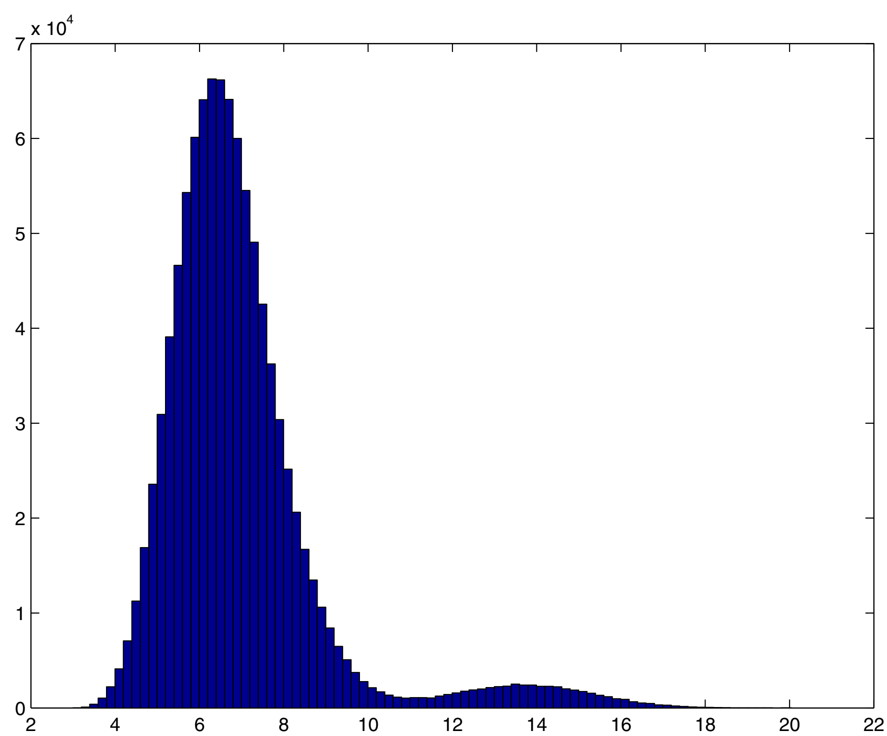
Thus the vector of triangular random variables has the parameter triple (m, r, a); we used $(m, r, a) = (4, 0.5, 1.5)$.

As in section 4.3, the possibility of an occasional lost ball was included in the first stage. For the model with parameters $(m, r, a) = (4,0.5, 1.5)$ and $(p, L) = (0.05, 12)$, the mean and variance of the random cycle time C_∞ were estimated based on a sample of size 10^6 determined by 500 independent replications of 10,000 groups and using the last 2000 observations. The estimated mean was $E[C_\infty] = 6.98$ and the estimated squared coefficient of variation (scv, variance divided by the square of the mean) was $c^2_{C_\infty} = 0.079$. We also found that the playing time had mean $E[X_n] = E[G_n - B_n] = 15.29$ and variance $\mathrm{Var}(X_n) = \mathrm{Var}(G_n - B_n) = 4.03$, and so scv 0.0172. Since a par-5 hole is the longest hole, the expected playing time will usually be longer than for the other holes, but nevertheless the cycle time may be shorter. Again, this is consistent with experience.

It is also natural to consider whether the simplifying assumption in Corollary 6 that $D_{n-1} \ge S_{1,n}$ is reasonable or not. Unfortunately, we find that it is not, given the lost ball feature. It is likely to hold, at least approximately, if there are no lost balls. But necessarily the reverse inequality $S_{1,n} > D_{n-1}$ will hold whenever there is a lost ball.

We show the histogram of the random cycle times $C_n \equiv G_n - G_{n-1}$ when the stage playing times have the triangular distribution with $(m, r, a) = (4,0.5, 1.5)$ modified to account for lost balls with $(p, L) = (0.05, 12)$ in Figure 3. The lost balls produce a heavier upper tail, but within a reasonable range,

Figure 3 Histogram of the Distribution the Cycle Time $C_n \equiv G_n - G_{n-1}$ when the Stages have the Triangular Distributions with $(m, r, a) = (4, 0.5, 1.5)$, with S_1 Modified to Account for Lost Balls with $(p, L) = (0.05, 12)$. The mean is 6.98 and the variance is 3.85



because, by Theorem 9 and Theorem 10, C_n remains bounded above by $\bar{S}_{1,n} + S_{2,n} + S_{3,n} + S_{4,n} + S_{5,n-1} + S_{5,n} \leq 28.5$ (compared to 16.5 for the triangular distribution).

7. Conclusions

We have developed stochastic models of group play on each of the standard holes of a golf course. These models are similar to simulation models constructed by Kimes and Schruben (2002), Tiger and Salzer (2004), and Riccio (2012, 2013), but we innovate by combining the *steps* of group play into the essential *stages* for representing the precedence constraints associated with more than one group simultaneously playing on the hole. For example, the five steps of group play on a par-4 hole depicted in equation (1) are converted to the three stages depicted in equation (2). Thus, the mathematical model data for each hole are the group arrival times A_n and the stage playing times $S_{i,n}$ for all groups n. The mathematical models are concisely expressed as recursions. For par-4, par-3 with wave-up, and par-5, these recursions are given in equations (3), (59) and (68), respectively. This simplification is valuable for exposing the essential mathematical structure. It is not essential for simulation, although it also might be helpful there.

We have applied the stochastic models to determine the capacity of each hole. To do so, we considered fully loaded versions of the holes, in which new groups are always ready to tee off at the first opportunity. For the fully loaded models, the recursions simplify. The respective recursions for the fully loaded models are given in equations (12) (60), and (69). The maximum possible throughputs (the capacities) of these holes are given in equations (14), (64), and (89), respectively.

The capacity of the hole has a relatively simple concise expression in terms of the stage playing times for par-4 and par-3 with or without wave-up. Thus, the formulas provide a strong basis for comparing these holes, as illustrated by Corollaries 3 and 4. They clearly can be valuable in course design, for example, if the objective is to make the capacities of the holes nearly equal.

The story for par-5 holes is more complicated. For par-5, the capacity is expressed in terms of the steady-state means of two one-dimensional Markov processes in Theorem 11. These two component means can be calculated as the long-run averages of the one-dimensional recursions, as indicated in equation (92). The simple simulation in section 6.3 shows that the capacity can be readily computed, even though a convenient explicit formula is not available. Corollary 6 also gives an explicit expression of the throughput for a par-5 hole under a simplifying assumption, which seems that it might often be reasonable, but that assumption is not satisfied by the example in section 6.3, which includes the possibility of a lost ball on stage 1.

We introduce tractable models of the stage playing times in section 4. These are symmetric triangular distributions modified by the possibility of a lost ball in the first stage (when driving from the tee). These are alternatives to the assymmetric triangular distributions used by Tiger and Salzer (2004). Figure 2 shows that the new distribution is also skewed, like the asymmetric triangular distribution. As a step toward exposing the essential structure, we introduced simplified models of the stage playing times with fewer parameters. For example, for the three stage playing times on a par-4 hole, there are a total of 5 parameters (m, a, r, L, p) obtained by combining equations (35), (36), (38), (40), and (45). The parameter m is a basic mean value parameter, while r is a relative mean value parameter. The parameter a characterizes the spread of the basic triangular distribution, and thus can be thought of as a variability parameter, but it does not include the contribution of lost balls. Finally, L and p directly quantify the quantitative impact and likelihood of a lost ball. Example 3 shows that the capacity of a par-4 hole can be easily computed for this stage playing time model. The impact of the parameters can be seen there as well. Since we have not yet fit these models to data, this approach should be viewed as illustrating what can be done to obtain tractable models of the stage playing times.

These models of individual holes extend directly to models of play on an 18-hole golf course with any configuration of these holes, possibly with different parameters for each hole. These models can thus be applied directly to simulate the performance of groups playing on a golf course. As indicated in section 1.1, work is under way to develop analytical performance approximations of the performance on the entire course, exploiting the representations here; see Fu and Whitt (2013).

Evidently the methods developed here for modeling and analyzing the pace of play on golf courses has the potential for application to other production and service problems, because similar precedence constraints hold in other contexts. For example, such precedence constraints within an emergency department are illustrated by the activities network flow chart in Figure 8 of Armony et al. (2013).

There are many remaining directions for research. It would be nice to better understand the random cycle times on a par-5 hole, but our results in section 6.1 reveal limitations on what is possible. The analysis here does not cover all considerations. While a rare extremely long stage playing time, such as might occur with a lost ball, can be incorporated through the distribution of the stage playing times, we have yet to model the deleterious impact of an exceptionally slow group, that tends to be much slower than other groups on most holes. It remains to explore alternative control schemes for improving the pace of play. Finally, it is important to fit the models to data on group play over golf courses. We can then determine appropriate stage playing time distributions and verify that the model assumptions are reasonable.

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