Proceedings of the 28th Conference
on Decision and Control
Tampa, Florida • December 1989

ON AVERAGES SEEN BY ARRIVALS IN DISCRETE TIME

Armand Makowski
Department of Electrical Engineering
University of Maryland
College Park, MD 20742

Benjamin Melamed
AT&T Bell Laboratories
Room 3M-310
Holmdel, NJ 07733

Ward Whitt
AT&T Bell Laboratories
Room 2C-178
Murray Hill, NJ 07974

ABSTRACT

We study the limiting behavior of averages from an embedded stochastic process obtained by sampling a discrete-time stochastic process at points of an associated discrete-time stochastic point process. We determine when the limit of the averages from the embedded process coincides with the limit of the averages from the original process. In a certain stationary Markov framework, this happens if and only if the point process is a Bernoulli sequence with future points being independent of the state of the Markov process.

I. INTRODUCTION

A fundamental principle in queueing theory is PASTA (Poisson Arrivals See Time Averages) [6], [16]. The PASTA property has recently been generalized by Brémaud [1], [2], König and Schmidt [7], Melamed and Whitt [10], [11] and Stidham and El Taha [13]; they characterize what the arrivals see in general and develop necessary and sufficient conditions to have ASTA (Arrivals See Time Averages); see [10] for more discussion. New Anti-PASTA results have also been established by Walrand [14], Green and Melamed [4], Melamed and Whitt [11] and Wolff [17]; i.e., in certain circumstances the arrival process must be Poisson if arrivals see time averages. However, much of the interest in this recent work is in treating non-Poisson point processes. For example, the recent results apply to non-Poisson flows in Jackson queueing networks; indeed they encompass the Arrival Theorem for closed queueing networks [10, Example 4].

Here we present related discrete-time results, extending discrete-time PASTA [5], [15]. We give quick clean proofs based on the strong law of large numbers (SLLN) for martingale differences or, equivalently, the stability theorem for partial sums of random variables centered at conditional expectations [9, p. 53, *E*]; see Lemma 1 below. This nice approach was previously employed by Georgiadis [3]. Similar arguments can be employed in continuous time; see Remark 2 below.

Let N (N_+) be the set of nonnegative (positive) integers. Let $\{\mathcal{F}_n : n \in N\}$ be a filtration on an underlying probability space $(\Omega, \mathcal{F}, P)$; i.e., $\{\mathcal{F}_n\}$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$. Let $U \equiv \{U_n : n \in N\}$ and $A \equiv \{A_n : n \in N_+\}$ be bounded nonnegative real-valued $\mathcal{F}_n$-adapted stochastic processes on $(\Omega, \mathcal{F}, P)$; $\mathcal{F}_n$-adapted means that U_n and A_n are measurable functions on $(\Omega, \mathcal{F}_n)$ for each n. Let $\lambda_n = E(A_{n+1} | \mathcal{F}_n)$, $n \ge 0$; see [9, p. 7] for background on conditional expectation; we call $\lambda \equiv \{\lambda_n : n \in N\}$ the $(P, \mathcal{F}_n)$-stochastic intensity of A.

We are interested in the limiting behavior of

$$\hat{U}_n^A \equiv \sum_{k=1}^{n} A_k U_{k-1} / \sum_{k=1}^{n} A_k , \quad n \ge 1 , \tag{1}$$

with $\hat{U}_n^A = 0$ if $A_1 = \ldots = A_n = 0$, and its relation to

$$\bar{U}_n = n^{-1} \sum_{k=0}^{n-1} U_k , \quad \bar{U}_n^{\lambda} = n^{-1} \sum_{k=0}^{n-1} \lambda_k U_k , \quad \bar{U}_n^A = n^{-1} \sum_{k=1}^{n} A_k U_{k-1} ,$$

$$\bar{\lambda}_n = n^{-1} \sum_{k=0}^{n-1} \lambda_k , \quad \bar{A}_n = n^{-1} \sum_{k=1}^{n} A_k , \quad n \ge 1 . \tag{2}$$

Here is what we have in mind: We think of U_n being $1_{\{X_n \in B\}}$ where 1_C is the indicator function of a set C, $X \equiv \{X_n : n \in N\}$ is an $\mathcal{F}_n$-adapted stochastic process on $(\Omega, \mathcal{F}, P)$ and B is a measurable subset of the state space of X. We think of A being a discrete-time point process, i.e., A_n is $\{0, 1\}$-valued, so that $A_n = 1$ corresponds to a point (arrival) at time n. We regard U_{n-1} as the state "seen" by this arrival at time n. With this interpretation, $\hat{U}_n^A$ is the average value of U "seen" by the arrivals up to time n, and ASTA holds if $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \bar{U}_n$ w.p.1. However, we do not restrict attention to this framework until §4, where we obtain our Anti-PASTA result.

We use the following well known basic martingale result.

Lemma 1. *$M_n^1 \equiv n(\bar{U}_n^A - \bar{U}_n^{\lambda})$ and $M_n^2 \equiv n(\bar{A}_n - \bar{\lambda}_n)$ are zero-mean $\mathcal{F}_n$-martingales with bounded increments $M_n^i - M_{n-1}^i$ for $i = 1, 2$, so that $(\bar{U}_n^A - \bar{U}_n^{\lambda}) \to 0$ and $(\bar{A}_n - \bar{\lambda}_n) \to 0$ w.p.1 as $n \to \infty$.*

Proof. Note that $M_n^1 - M_{n-1}^1 = U_{n-1} A_n - E(U_{n-1} A_n | \mathcal{F}_{n-1})$, and so is bounded and $\mathcal{F}_n$-measurable. Since $E(M_n^1 - M_{n-1}^1 | \mathcal{F}_{n-1}) = 0$, $\{M_n^1 : n \ge 1\}$ is a martingale. By the SLLN for martingale differences [9, p. 53, *E*, with $b_n = n$], $(\bar{U}_n^A - \bar{U}_n^{\lambda}) \to 0$ w.p.1 as $n \to \infty$. A similar argument applies to $\{M_n^2 : n \ge 1\}$. ■

Remark 1. Although the boundedness of U and A is a natural sufficient condition for Lemma 1, this condition can be relaxed. For the SLLNs above, it suffices to have $\sum_{n=1}^{\infty} n^{-2} \operatorname{Var}(U_{n-1} A_n) < \infty$ and $\sum_{n=1}^{\infty} n^{-2} \operatorname{Var}(A_n) < \infty$ [9, p. 53]. Allowing unbounded A is useful for treating batch arrivals [3], [5], [15].

Remark 2. For continuous-time versions of the results here, we can work with a bounded left-continuous process $\{U(t) : t \ge 0\}$ and a right-continuous counting process $\{N(t) : t \ge 0\}$ with stochastic intensity $\{\lambda(t) : t \ge 0\}$ such that $E[N(t)^2] < \infty$ for all $t \ge 0$. Since $M(t) \equiv \int_0^t U(s)\, dN(s) - \int_0^t U(s) \lambda(s)\, ds$ is the stochastic integral of U with respect to a square integrable martingale, $\{M(t) : t \ge 0\}$ is itself a square integrable martingale with quadratic variation process $\langle M \rangle_t = \int_0^t U(s)^2 \lambda(s)\, ds$ [8, §5.1, 5.4, 18.1]. Hence, the SLLN can be applied, just as for Lemma 1. To control the differences, we can assume that, for some $u > 0$,

$$E[M(t+u) - M(t)]^2 = E(\langle M \rangle_{t+u} - \langle M \rangle_t) = \int_t^{t+u} E[U(s)^2 \lambda(s)] ds < K$$

for all $t \ge 0$; e.g., it suffices to have $E\lambda(t)$ bounded. Variants of this continuous-time argument were given by P. Brémaud (personal communication, 1988) and more recently by Rosenkrantz and Simha [12]. ■

We assume that

$$\liminf_{n\to\infty} \bar{A}_n > 0 \quad \text{w.p.1} . \tag{3}$$

From Lemma 1, we have $\liminf_{n\to\infty} \bar{\lambda}_n = \liminf_{n\to\infty} \bar{A}_n > 0$ w.p.1 and, since $E\bar{A}_n = E\bar{\lambda}_n$, from Fatou's lemma it follows that $\liminf_{n\to\infty} E\bar{A}_n = \liminf_{n\to\infty} E\bar{\lambda}_n > 0$.

II. SAMPLE PATH RESULTS

We first show that arrivals always see weighted time averages provided that the weighted time averages are well defined. When we talk about limits, we mean finite limits.

Theorem 2. *$\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \overline{U}_n^\lambda / \lim_{n\to\infty} \overline{\lambda}_n$ w.p.1 provided that two of the three limits exist w.p.1.*

Proof. Note that

$$\hat{U}_n^A = \frac{\overline{U}_n^A}{\overline{A}_n} = \frac{(\overline{U}_n^A - \overline{U}_n^\lambda) + \overline{U}_n^\lambda}{(\overline{A}_n - \overline{\lambda}_n) + \overline{\lambda}_n}, \quad n \geq 1, \tag{4}$$

and apply Lemma 1. By (3), the denominator cannot approach zero. ∎

Let $\mathrm{cov}(Y, Z)$ be the covariance of random variables Y and Z. The following is the discrete-time analog of the important covariance formula in [1], [10], [13]. Note that the two conditions below are not equivalent.

Corollary. *If either $\{(\lambda_n, U_n): n \in N\}$ or $\{(A_{n+1}, U_n): n \in N\}$ is stationary and ergodic, then*

$$\lim_{n\to\infty} \hat{U}_n^A = \frac{E\lambda_0 U_0}{E\lambda_0} = E\,U_0 + \frac{\mathrm{cov}(\lambda_0, U_0)}{E\lambda_0}$$

$$= \lim_{n\to\infty} \overline{U}_n + \frac{\mathrm{cov}(\lambda_0, U_0)}{E\lambda_0} \quad \text{w.p.1}.$$

Proof. The first condition implies that $\overline{U}_n \to E\,U_0$, $\overline{U}_n^\lambda \to E\lambda_0 U_0$ and $\overline{\lambda}_n \to E\lambda_0$ w.p.1 as $n \to \infty$. Then apply Theorem 2. The second condition implies that $\overline{U}_n \to E\,U_0$, $\overline{U}_n^A \to EA_1 U_0$ and $\overline{A}_n \to EA_1$, but $EA_1 U_0 = EE(A_1 U_0 | \mathcal{F}_0) = E\lambda_0 U_0$ and $EA_1 = EE(A_1 | \mathcal{F}_0) = E\lambda_0$. Now use the first relation in (4). ∎

As observed in [5], a discrete-time analog of PASTA [14] occurs when λ is deterministic and constant. Then A is a Bernoulli process, i.e., A_n is independent of $\mathcal{F}_{n-1}$ and $P(A_n = 1) = p$ for all $n \geq 1$.

Theorem 3. (Discrete-time PASTA) *If $\lambda_n = E\lambda_0$ w.p.1, $n \geq 0$, then $\hat{U}_n^A$ converges w.p.1 if and only if $\overline{U}_n$ converges w.p.1; if the convergence holds, then $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \overline{U}_n$ w.p.1.*

Proof. Apply Theorem 2, noting that $\overline{U}_n^\lambda = (E\lambda_0)\overline{U}_n$ and $\overline{\lambda}_n = E\lambda_0$ w.p.1, $n \geq 0$. ∎

However, from the Corollary to Theorem 2, we see that in a stationary framework we have ASTA (i.e., $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \overline{U}_n$) if and only if $\mathrm{cov}(\lambda_0, U_0) = 0$, which does not require that $\lambda_0 = E\lambda_0$ w.p.1 as in Theorem 3. ($\mathrm{cov}(\lambda_0, U_0) = 0$ coincides with the lack of bias assumption, LBA, in [10].) We now provide necessary and sufficient conditions for ASTA.

Theorem 4. *Suppose that one of $\overline{U}_n$ or $\hat{U}_n^A$ converges w.p.1 as $n \to \infty$. The following are then equivalent:*

(i) $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \overline{U}_n$ *w.p.1 (ASTA);*

(ii) $\lim_{n\to\infty} (\overline{U}_n^A - \overline{A}_n \overline{U}_n) = 0$ *w.p.1;*

(iii) $\lim_{n\to\infty} (\overline{U}_n^\lambda - \overline{\lambda}_n \overline{U}_n) = 0$ *w.p.1.*

Proof. To relate (i) and (ii), note that

$$\hat{U}_n^A = \frac{\overline{U}_n^A}{\overline{A}_n} = \frac{\overline{U}_n^A - \overline{A}_n \overline{U}_n}{\overline{A}_n} + \overline{U}_n \tag{5}$$

and apply (3). To relate (ii) and (iii), note that $(\overline{U}_n^A - \overline{A}_n \overline{U}_n) - (\overline{U}_n^\lambda - \overline{\lambda}_n \overline{U}_n) = (\overline{U}_n^A - \overline{U}_n^\lambda) - (\overline{A}_n - \overline{\lambda}_n)\overline{U}_n \to 0$ w.p.1 by Lemma 1. ∎

We call condition (ii) in Theorem 4 the *operational condition*, because we are more likely to apply it when analyzing data; we call (iii) the *theoretical condition*, because we are more likely to apply it when analyzing models.

III. EXPECTED VALUE RESULTS

As in [10], [11], we can also work with expected values. To motivate this approach, suppose for the moment that $\overline{U}_n \to \overline{u}$, $\overline{U}_n^\lambda \to \overline{u}^\lambda$, $\overline{U}_n^A \to \overline{u}^A$, $\overline{\lambda}_n \to \overline{\lambda}$ and $\overline{A}_n \to \overline{a}$ w.p.1 as $n \to \infty$, where $\overline{u}$, $\overline{u}^\lambda$, $\overline{u}^A$, $\overline{\lambda}$ and $\overline{a}$ are deterministic. By Lemma 1, $\overline{u}^A = \overline{u}^\lambda$ and $\overline{a} = \overline{\lambda}$. By Theorem 2, $\hat{U}_n^A \to \hat{u}^A$ w.p.1 where $\hat{u}^A = \overline{u}^\lambda / \overline{\lambda}$. Since $\overline{U}_n$, $\overline{U}_n^\lambda$, $\overline{U}_n^A$, $\overline{\lambda}_n$ and $\overline{A}_n$ are bounded, $E\overline{U}_n \to \overline{u}$, $E\overline{U}_n^\lambda \to \overline{u}^\lambda$, $E\overline{U}_n^A \to \overline{u}^\lambda$, $E\overline{\lambda}_n \to \overline{\lambda}$ and $E\overline{A}_n \to \overline{\lambda}$ as $n \to \infty$ too. Now we show that $E\hat{U}_n^A \to \hat{u}^A$ too under the assumptions above.

Lemma 5. *If $0 \leq U_n \leq B$ for all n, then $0 \leq \hat{U}_n^A \leq B$ for all n.*

Proof. If $A_1 = \ldots = A_n = 0$, then $\hat{U}_n^A = 0$ by convention. Otherwise,

$$\hat{U}_n^A = \sum_{k=0}^{n-1} \left(A_{k+1} / \sum_{k=0}^{n-1} A_{k+1}\right) U_k, \quad n \geq 1,$$

so that $\hat{U}_n^A$ is a convex combination of U_k, $0 \leq k \leq n-1$. ∎

Corollary. *If $\hat{U}_n^A \to \hat{u}^A$ w.p.1 as $n \to \infty$, then $E\hat{U}_n^A \to \hat{u}^A$ as $n \to \infty$.*

Since $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} E\hat{U}_n^A = \lim_{n\to\infty} E\overline{U}_n^A / E\overline{A}_n$ w.p.1 under the conditions above, interest centers on the asymptotic behavior of $E\overline{U}_n^A / E\overline{A}_n$.

Theorem 6. *$\lim_{n\to\infty} E\overline{U}_n^A / E\overline{A}_n = \lim_{n\to\infty} E\overline{U}_n^\lambda / E\overline{\lambda}_n$ provided that one of the limits exist.*

Proof. By Lemma 1, $E\overline{U}_n^A = E\overline{U}_n^\lambda$ and $E\overline{A}_n = E\overline{\lambda}_n$. ∎

Corollary. *(a) If $n^{-1} \sum_{k=0}^{n-1} \mathrm{cov}(\lambda_k, U_k) \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \frac{E\,\overline{U}_n^A}{E\,\overline{A}_n} = \frac{\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} E\lambda_k E U_k}{\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} E\lambda_k} \tag{6}$$

provided two of the three limits exist.

(b) If, in addition, either $E\lambda_n \to \overline{\lambda}$ or $EU_n \to \overline{u}$ as $n \to \infty$, then $\lim_{n\to\infty} (E\overline{U}_n^A / E\overline{A}_n) = \lim_{n\to\infty} E\overline{U}_n$.

Proof. For (a), note that

$$E\overline{U}_n^\lambda = n^{-1} \sum_{k=0}^{n-1} \mathrm{cov}(\lambda_k, U_k) + n^{-1} \sum_{k=0}^{n-1} E\lambda_k E U_k$$

and apply Theorem 6 recalling that $\liminf_{n\to\infty} E\overline{\lambda}_n > 0$ w.p.1 as a result of (3). For (b), note that $\overline{\lambda} > 0$ by (3) and apply (6). ∎

The following is the obvious expectation version of Theorem 4.

Theorem 7. *Suppose that one of $E\overline{U}_n^A / E\overline{A}_n$ and $E\overline{U}_n$ converges as $n \to \infty$. The following are then equivalent:*

(i) $\lim_{n\to\infty} \dfrac{E\overline{U}_n^A}{E\overline{A}_n} = \lim_{n\to\infty} E\overline{U}_n$*;*

(ii) $\lim_{n\to\infty} (E\overline{U}_n^\lambda - E\overline{\lambda}_n E\overline{U}_n) = 0$*;*

(iii) $\lim_{n\to\infty} (E\overline{U}_n^A - E\overline{A}_n E\overline{U}_n) = 0$.

Proof. By Lemma 1 and (3), (i) holds if and only if $\lim_{n\to\infty} (E\overline{U}_n^\lambda - E\overline{\lambda}_n E\overline{U}_n) / E\overline{\lambda}_n = 0$, which is equivalent to (ii) by virtue of (3). By Lemma 1 again, (ii) is equivalent to (iii). ∎

We combine previous results to further characterize the limiting behavior of $\hat{U}_n^A$.

Theorem 8. *(a) If $\overline{U}_n^A \to \overline{u}^A$ and $\overline{A}_n \to \overline{a}$ w.p.1 as $n \to \infty$ where $\overline{u}^A$ and $\overline{a}$ are deterministic, then necessarily*

$$\overline{u}^A = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} E\lambda_k U_k, \quad \overline{a} = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} E\lambda_k$$

and

$$\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} E\hat{U}_n^A = \frac{\overline{u}^A}{\overline{a}},$$

(b) If, in addition, $n^{-1}\sum_{k=0}^{n-1}\mathrm{cov}(\lambda_k, U_k) \to 0$ *and either* $E\lambda_n \to \bar{\lambda}$ *or* $EU_n \to u^*$ *as* $n\to\infty$ *for some* u^*, *then* $\lim_{n\to\infty} E\bar{U}_n = \lim_{n\to\infty} E\hat{U}_n^A = \bar{u}^A/\bar{a}$.

(c) If, in addition to (a) and (b), $\bar{U}_n \to \bar{u}$ *w.p.1 as* $n\to\infty$ *where* $\bar{u}$ *is deterministic, then* $\bar{u} = \bar{u}^A/\bar{a}$.

Proof. Part (a) follows easily from (3) and (4): By Lemma 1, $\bar{U}_n^{\lambda} \to \bar{u}^{\lambda}$ and $\bar{\lambda}_n \to \bar{\lambda}$ w.p.1 as $n\to\infty$ with $\bar{u}^{\lambda} = \bar{u}^A$ and $\bar{\lambda} = \bar{a}$. Consequently, $E\hat{U}_n^{\lambda} \to \bar{u}^A$ and $E\bar{\lambda}_n \to \bar{a}$ as $n\to\infty$ and, by Theorem 2, $\hat{U}_n^A \to \bar{u}^A/\bar{a}$ w.p.1 as $n\to\infty$. Moreover, $E\hat{U}_n^A \to \bar{u}^A/\bar{a}$ as $n\to\infty$ as well, by the Corollary to Lemma 5. For part (b), apply the Corollary to Theorem 6. Finally, (c) is easy because the limit of $E\bar{U}_n$ is established in (b). ■

Example 1. To see that $\bar{U}_n$ need not converge under the assumptions of Theorem 8(a) and (b), let $A_{2k-1} = 0$ and $A_{2k} = U_{2k-1} = 1$ for $k \geq 1$ w.p.1. then $\hat{U}_n^A \to \bar{u}^A = 1/2$ and $\bar{A}_n \to \bar{a} = 1/2$ as $n\to\infty$ w.p.1. Since A is deterministic, $\lambda_k = A_{k+1}$ and $\mathrm{cov}(\lambda_k, U_k) = 0$ for all k. Finally, choose U_{2k} such that $EU_{2k} = 1$ but $\bar{U}_n$ does not converge as $n\to\infty$. For example, let $U_{2k} = 1 + (-1)^{[\log_2 2k]}U_0$ where $P(U_0 = 1) = P(U_0 = -1) = 1/2$ and $[x]$ is the integer part of x. ■

IV. ANTI-PASTA

Henceforth we consider the $\mathcal{F}_n$-adapted process X mentioned in § 1 and assume that $U_n = f(X_n)$, $n \geq 1$, for some bounded measurable real-valued function f on the state space of X. From Theorem 8(b) and the Corollaries to Theorems 2 and 6, we know that ASTA follows from $\mathrm{cov}(\lambda_n, U_n) = 0$ for all $n \geq 1$ plus other regularity conditions. The following is essentially [10, Theorem 4].

Theorem 9. $\mathrm{cov}(\lambda_n, f(X_n)) = 0$ *for all bounded measurable f if and only if* $E(\lambda_n | X_n) = E\lambda_n$ *w.p.1.*

Proof. Note that $Ef(X_n)\lambda_n = Ef(X_n)E\lambda_n$ for all bounded measurable f if and only if $E\lambda_n$ is a version of the conditional expectation $E(\lambda_n|X_n)$; [9, p. 7]. ■

Let $\mathcal{F}_n^A$ and $\mathcal{F}_n^X$ be the σ-fields generated by $\{A_1, \ldots, A_n\}$ and $\{X_0, \ldots, X_n\}$ respectively. The following is the discrete-time version of [11, Theorem 5]. For related Anti-PASTA results, see [14], [4], [17] and references cited therein.

Theorem 10. *Suppose that X is a stationary ergodic Markov process,* $\mathcal{F}_n = \mathcal{F}_n^X$, $U_n = f(X_n)$ *and A has the form* $A_n = g(X_{n-1}, X_n)$ *for some* $\{0, 1\}$*-valued measurable function g for all* $n \geq 1$. *Then the following are equivalent:*

(i) $\lim_{n\to\infty} \hat{U}_n^A = \lim_{n\to\infty} \bar{U}_n$ *w.p.1 for all bounded measurable f (ASTA);*

(ii) $\mathrm{cov}(\lambda_0, f(X_0)) = 0$ *for all bounded measurable f;*

(iii) $E(\lambda_0|X_0) = E\lambda_0$ *w.p.1;*

(iv) $\lambda_0 = E\lambda_0$ *w.p.1;*

(v) A_{n+1} *is independent of* $\mathcal{F}_n$ *for all* $n \geq 0$ *and A is an i.i.d Bernoulli sequence.*

Proof. Since X is stationary, $\{(A_{n+1}, U_n)\colon n \geq 0\} = \{(g(X_n, X_{n+1}), f(X_n))\colon n \geq 0\}$ is stationary too. By the Corollary to Theorem 2, ASTA holds if and only if $\mathrm{cov}(\lambda_0, f(X_0)) = 0$. By Theorem 9, $\mathrm{cov}(\lambda_0, f(X_0)) = 0$ for all bounded measurable f if and only if $E(\lambda_0|X_0) = E\lambda_0$. Since $\mathcal{F}_n = \mathcal{F}_n^X$ and X is Markov,

$$\lambda_n = E(g(X_n, X_{n+1})|\mathcal{F}_n) = E(g(X_n, X_{n+1})|X_n), \quad n \geq 0,$$

and therefore $\lambda_n = E(\lambda_n|X_n)$. In particular, $\lambda_0 = E(\lambda_0|X_0)$, so that (iii) and (iv) are equivalent. Recall that by stationarity (iv) is equivalent to $\lambda_n = E\lambda_n = p$ for all n w.p.1. Finally, since $\lambda_n = E(A_{n+1}|\mathcal{F}_n) = P(A_{n+1} = 1|\mathcal{F}_n)$ w.p.1, (iv) and (v) are equivalent; i.e., $P(A_{n+1} = 1|\mathcal{F}_n) = p$ w.p.1 for all n if and only if A_{n+1} is independent of $\mathcal{F}_n$ for all $n \geq 0$ and A is an i.i.d sequence. A variant of Wolff's proof in [17] could also be used to show that (iv) implies (v). ■

Acknowledgments. This work was done while Armand Makowski was visiting as a consultant to AT&T Bell Laboratories. He is grateful to Debasis Mitra and his colleagues for their hospitality. The authors are also grateful to the authors of unpublished papers for showing them their work.

REFERENCES

[1] P. Brémaud, "Characteristics of queueing systems observed at events and the connection between stochastic intensity and Palm probability," *Queueing Systems*, vol. 5, 1989.

[2] P. Brémaud, "Necessary and sufficient conditions for the equality of event averages and time averages," *J. Appl. Prob.*, vol. 27, 1990.

[3] L. Georgiadis, "Relations between arrival and time averages of a process in discrete-time systems with conditional lack of anticipation," Electrical Engineering Department, University of Virginia, Charlottesville, 1987.

[4] L. Green and B. Melamed, "An Anti-PASTA result for Markovian systems," *Opns. Res.*, vol. 38, 1990.

[5] S. Halfin, "Batch delays versus customer delays," *Bell System Tech. J.*, vol. 62, pp. 2011-2015, 1983.

[6] D. König and V. Schmidt, "Imbedded and non-imbedded stationary characteristics of queueing systems with varying service rate and point processes," *J. Appl. Prob.*, vol. 17, pp. 753-767, 1980.

[7] D. König and V. Schmidt, "EPSTA: The coincidence of time-stationary and customer-stationary distributions," *Queueing Systems*, vol. 5, 1989.

[8] R. S. Lipster and A. N. Shiryayev, *Statistics of Random Processes*, I *and* II, New York, NY: Springer Verlag, 1977, 1978.

[9] M. Loève, *Probability Theory II*, fourth ed., New York, NY: Springer-Verlag, 1978.

[10] B. Melamed and W. Whitt, "On arrivals that see time averages," *Opns. Res.*, vol. 38, 1990.

[11] B. Melamed and W. Whitt, "On arrivals that see time averages: a martingale approach," *J. Appl. Prob.*, vol. 27, 1990.

[12] W. A. Rosenkrantz and R. Simha, "Poisson arrivals see time averages: a stochastic integral approach," Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA, 1989.

[13] S. Stidham, Jr. and M. El Taha, "Sample-path analysis of processes with imbedded point processes," *Queueing Systems*, vol. 5, 1989.

[14] J. Walrand, *An Introduction to Queueing Networks*, Englewood Cliffs, NJ: Prentice Hall, 1988.

[15] W. Whitt, "Comparing batch delays and customer delays," *Bell System Tech. J.*, vol. 62, pp. 2001-2009, 1983.

[16] R. W. Wolff, "Poisson arrivals see time averages," *Opns. Res.*, vol. 30, pp. 223-231, 1982.

[17] R. W. Wolff, "A note on PASTA and Anti-PASTA for continuous-time Markov chains," *Opns. Res.*, vol. 38, 1990.