THE REFLECTION MAP WITH DISCONTINUITIES

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December 1998

Present version: November 3, 2000

Mathematics of Operations Research 26 (2001) 447–484.

Abstract

We study the multi-dimensional reflection map on the spaces $D([0,T],\mathbb{R}^k)$ and $D([0,\infty),\mathbb{R}^k)$ of right-continuous $\mathbb{R}^k$-valued functions on $[0,T]$ or $[0,\infty)$ with left limits, endowed with variants of the Skorohod(1956) M_1 topology. The reflection map was used with the continuous mapping theorem by Harrison and Reiman (1981) and Reiman (1984) to establish heavy-traffic limit theorems with reflected Brownian motion limit processes for vector-valued queue-length, waiting-time and workload stochastic processes in single-class open queueing networks. Since Brownian motion and reflected Brownian motion have continuous sample paths, the topology of uniform convergence over bounded intervals could be used for those results. Variants of the M_1 topologies are needed to obtain alternative discontinuous limits approached gradually by the converging processes, as occurs in stochastic fluid networks with bursty exogenous input processes, e.g., with on-off sources having heavy-tailed on periods or off-periods (having infinite variance). We show that the reflection map is continuous at limits without simultaneous jumps of opposite sign in the coordinate functions, provided that the product M_1 topology is used. As a consequence, the reflection map is continuous with the product M_1 topology at all functions that have discontinuities in only one coordinate at a time. That continuity property also holds for more general reflection maps and is sufficient to support limit theorems for stochastic processes in most applications. We apply the continuity of the reflection map to obtain limits for buffer-content stochastic processes in stochastic fluid networks.

Keywords: reflection map, heavy-traffic limit theorems, queueing networks, stochastic fluid networks, Lévy processes, the function space D, functional limit theorems, invariance principles, weak convergence, Skorohod topologies, Skorohod M_1 topology, continuous mapping theorem.

Contents

1. Introduction

This paper is motivated by the desire to better understand and manage the performance of evolving communication networks. Network design and control is complicated by the "bursty" traffic found on these networks. Network traffic measurements reveal features such as heavy-tailed probability distributions, long-range dependence and self-similarity; e.g., see Park and Willinger (2000).

Since communication networks and their components can often be modelled as queueing networks, insight may be gained by studying the impact of bursty traffic (modelled appropriately) on the performance of queueing networks. One way to see the impact of bursty traffic on the performance of queueing networks is to consider heavy-traffic limits. The burstiness can have a dramatic impact on the heavy-traffic limits, changing both the scaling and the limit process; e.g., see Konstantopoulos and Lin (1996, 1998) and Whitt (2000a,b). In fact, the limit process can have discontinuous sample paths even when the converging stochastic processes have continuous sample paths, but that presents technical challenges because such limits, obtained via the continuous-mapping approach with reflection maps on the function space $D \equiv D([0,T], \mathbb{R}^k)$ of right-continuous $\mathbb{R}^k$-valued functions on $[0,T]$ with left limits, require a nonstandard topology on the function space D.

In this paper we address some of those technical challenges. In particular, we establish convergence to stochastic processes with discontinuous sample paths, such as reflected Lévy processes, for sequences of appropriately scaled vector-valued buffer-content stochastic processes with continuous sample paths in general stochastic fluid networks (special queueing networks with random continuous flow, designed to model traffic in communication networks). To establish these limits, we use variants of the Skorohod (1956) M_1 topology on the function space D, because the standard Skorohod (1956) J_1 topology does not permit such limits.

In the present paper we show that the multi-dimensional reflection map is continuous on the spaces $D([0,T], \mathbb{R}^k)$ and $D([0,\infty), \mathbb{R}^k)$ with appropriate versions of the M_1 topology under appropriate regularity conditions. In Whitt (2000a) we established functional central limit theorems (FCLTs), using the M_1 topology on D, for sequences of appropriately scaled exogenous cumulative input processes with continuous sample paths, where the limit process has discontinuous sample paths. With the continuity of the reflection map established here, those results yield FCLTs for the buffer-content processes in the stochastic fluid networks, as we show in Section 10.

The burstiness of network traffic can be due to failures of network elements as well as to the behavior of transmitting sources. Thus there is also interest in the performance of queueing networks faced with service interruptions. In this paper, we also establish heavy-traffic FCLTs for queueing networks with rare long service interruptions. The long service interruptions could be due to server breakdowns (often called "vacations" in the literature) or unusually long service times, as occur with heavy-tailed service-time distributions. In fact, Chen and Whitt (1993) obtained such heavy-traffic FCLTs for single-class open queueing networks, extending corresponding heavy-traffic FCLTs for single-class open queueing networks without service interruptions by Harrison and Reiman (1981) and Reiman (1984), and heavy-traffic FCLTs for single queues with service interruptions in Kella and Whitt (1990). However, there is an error in the proof in Chen and Whitt that needs to be corrected.

The queueing network model considered by Chen and Whitt is standard except for the interruptions. To represent the interruptions, each station has a single server that is alternatively up and down. To represent long rare service interruptions, the up and down times are allowed to be of order $n \equiv (1-\rho)^{-2}$ and $\sqrt{n} \equiv (1-\rho)^{-1}$, respectively, when the traffic intensity is $\rho \equiv 1 - n^{-1/2}$. This scaling yields a limit process that is a multi-dimensional reflection of a multi-dimensional Brownian motion plus a multi-dimensional jump process.

The FCLTs were proved in Chen and Whitt by applying the continuous mapping theorem with the multi-dimensional reflection map, after establishing a FCLT for the basic net-input process. However, the jumps in the limit process approached gradually in the converging processes make it necessary to work with the M_1 topology. The need for the M_1 topology also arises in the one-dimensional (single-queue) case considered by Kella and Whitt (1990), but the one-dimensional reflection map into the buffer-content process is easily seen to be continuous, so that there is no difficulty there. (There is a difficulty, however, if we consider the map into the two-dimensional process representing both the buffer-content process and the nondecreasing regulator process; see Example 4.3.) The error occurs in Proposition 2.4 of Chen and Whitt, which asserts that continuity and Lipschitz properties in the uniform topology extend directly to the nonuniform J_1 and M_1 topologies. (A complete statement appears here in Section 4.) We will show that the general idea of Proposition 2.4 is correct and that the heavy-traffic FCLT with the M_1 topology in Theorem 4.1 of Chen and Whitt is also correct, provided that the topology on D is understood to be the product M_1 topology. However, the required argument is much more complicated.

We show that the reflection map from $D([0,T],\mathbb{R}^k)$ into $D([0,T],\mathbb{R}^{2k})$, and from $D([0,\infty),\mathbb{R}^k)$

into $D([0,\infty),\mathbb{R}^{2k})$, is continuous at limits without simultaneous jumps of opposite sign in the coordinate functions, provided that the product topology is used on the range. Thus the heavy-traffic FCLT in Theorem 4.1 of Chen and Whitt is correct provided that the space D for the queueing limit is understood to be endowed with the product topology. (The distinction between the standard topology and the product topology does not arise with the uniform topology, because the space $D([0,T],\mathbb{R}^k)$ with the uniform topology coincides with the space $D([0,T],\mathbb{R}^1)^k$ with the product topology, where each component space is given the uniform topology.) Even though the corrected heavy-traffic FCLT in Chen and Whitt is weaker than originally claimed, the conclusion is still strong; it does not alter many applications. Even convergence in the product M_1 topology implies convergence of all finite-dimensional distributions at all times that are almost surely continuity points of the limit process.

Indeed, the greatest limitation of the FCLT theorem in Chen and Whitt is the requirement that the scaled net-input processes converge in $D([0,T],\mathbb{R}^k)$ to a limit process which almost surely has discontinuities in only one coordinate at a time. That assumption (made in (4.11) there) clearly implies that the limit process almost surely has no simultaneous jumps of opposite sign in the coordinate functions. In stochastic fluid networks and their limits, that condition is satisfied if the exogenous input processes at the stations are mutually independent and if the component limit processes have no common fixed discontinuities.

We also show that the reflection map is Lipschitz on the subset D_s of functions in D without simultaneous jumps of opposite sign in the coordinate functions, provided that a strong (standard) M_1 metric is used on the domain and a product M_1 metric is used on the range. For the heavy-traffic FCLT in Chen and Whitt, and for the stochastic-fluid-network FCLT in Section 10, we only use the continuity on D at limits in D_s, but the extra Lipschitz property is also useful to establish bounds on the rates of convergence of stochastic processes; see Whitt (1974). Bounds on the rate of convergence for the net input process translate immediately into corresponding bounds for the rate of convergence of the normalized queueing processes, due to the Lipschitz property. Whitt (1974) showed that a Lipschitz mapping on an underlying metric space induces an associated Lipschitz mapping on the space of all probability measures on that space, using an appropriate metric on the space of all probability measures inducing the topology of weak convergence, e.g., the Prohorov metric, as on p. 237 of Billingsley (1968). Initial rates of convergence may be obtained from strong approximations, e.g., as in Csörgő and Révész (1981).

It should be noted that having a discontinuous limit does not by itself imply that we need an

M_1 topology. An M_1 topology is only needed when the jumps in the limit process are approached gradually in the converging processes. Many examples of discontinuous limits in the familiar J_1 topology are contained in Jacod and Shiryaev(1987). Discontinuous limits for queueing processes with the J_1 topology are contained in Whitt (2000b).

Here is how the rest of this paper is organized: In Section 2 we give a brief summary of D and its topologies. In Section 3 we define the reflection map and state the main results. In Section 4 we give counterexamples, showing the necessity of our new conditions. We prove the main results in Sections 5–7. In Section 5 we establish properties of the instantaneous reflection map from $\mathbb{R}^k$ to $\mathbb{R}^{2k}$, which is the reflection map operating at a single time point. In Section 6 we study reflections of parametric representations, as needed for the M_1 topologies. The key idea in obtaining positive results is to establish conditions under which the reflection of a parametric representation of a function in D is a parametric representation of the reflected function. In Section 7 we apply the results in Sections 5 and 6 to prove the theorems in Section 3.

In Section 8 we discuss extensions to the space $D([0,\infty),\mathbb{R}^k)$. In Section 9 we discuss the reflection map as a function of the reflection matrix as well as the net-input function. In Section 10 we establish limits for stochastic fluid networks, applying the reflection map with the continuous mapping theorem. The first limit is a continuity result, showing that the buffer-content process is a continuous function of the basic model data. The second limit is a heavy-traffic limit.

We primarily consider the standard reflection map introduced by Harrison and Reiman (1981), but more general reflection maps have been considered subsequently; see Dupuis and Ishii (1991), Williams (1987, 1995) and Dupuis and Ramanan (1999a,b). In Section 11 we also establish general conditions for more general reflection maps to be Lipschitz with appropriate versions of the M_1 topology. We apply the general results to treat the two-sided regulator, as in Chapter 2 of Harrison (1985) and Berger and Whitt (1992).

For other work involving queues with the M_1 topology, see Mandelbaum and Massey (1995), Harrison and Williams (1996), Kella and Whitt (1996), Puhalskii and Whitt (1997, 1998) and Konstantopoulos (1999). The need to work with the product M_1 topology was recognized in another context by Harrison and Williams (1996). For additional related references, see Chen and Yao (2000), Kushner (2001) and Whitt (2000a,b, 2001). We conclude this introduction by defining several special subsets of the function space $D \equiv D^k \equiv D([0,T],\mathbb{R}^k)$ that we will use in this paper. We indicate where the subset is first defined.

Section 3: $D_s \equiv \{x \in D$: either $x(t) \geq x(t-)$ or $x(t) \leq x(t-)$ for all t, with the inequality allowed to depend upon $t\}$,

Section 3: $D_1 \equiv \{x \in D : x^i(t) \neq x^i(t-)$ for at most one i for all t, with the coordinate i allowed to depend upon $t\}$,

Section 3: $D_+ \equiv \{x \in D : x^i(t) \geq x^i(t-)$ for all i and $t\}$,

Section 5: $D_c \equiv \{x \in D : x$ is piecewise-constant with only finitely many discontinuities$\}$,

Section 6: $D_l \equiv \{x \in D$: x has only finitely many discontinuities and x is piecewise linear in between discontinuities, with only finitely many discontinuities in the derivative$\}$,

Section 6: $D_{s,l} \equiv D_s \cap D_l$

Section 11: $D_{1,l} \equiv D_1 \cap D_l$ *and* $D_{c,1} \equiv D_c \cap D_1$

2. Background on D and its Topologies

In this section we give basic definitions related to the function space D; see Skorohod (1956) and Whitt (2002) for more discussion. Let $D \equiv D^k \equiv D([0,T],\mathbb{R}^k)$ be the set of all $\mathbb{R}^k$-valued functions $x \equiv (x^1,\ldots,x^k) \equiv \{x(t), 0 \leq t \leq T\}$ on $[0,T]$ that are right continuous at all $t \in [0,T)$ and have left limits at all $t \in (0,T]$. We assume that the functions are continuous at T, so that we regard 0 and T as continuity points. It is common to form functions in D from sequences $\{s_j : j \geq 1\}$ by letting

$$x_n(t) \equiv s_{\lfloor nt \rfloor}, \quad 0 \leq t \leq T, \quad n \geq 1 \ , \tag{2.1}$$

where $\lfloor nt \rfloor$ is the greatest integer less than or equal to nt. To have x_n in (2.1) continuous at T for all n, we can let T be irrational. Moreover, the continuity condition at T invariably causes no problems in limits. Limiting stochastic processes typically will have no fixed discontinuities; i.e., letting $Disc(x)$ denote the set of discontinuity points of a function x, we usually have

$$P(t \in Disc(X)) = 0 \quad \text{for all} \quad t \ ,$$

where X is the random element of D. (Alternatively, we could allow the functions in D to be discontinuous at T.) We discuss the related space $D([0,\infty),\mathbb{R}^k)$ with domain $[0,\infty)$ in Section 8.

For $a \equiv (a^1, \ldots, a^k) \in \mathbb{R}^k$, let $\|a\| = \sum_{i=1}^k |a^i|$ and, for $x \in D$, let $\|x\| = \sup_{0 \le t \le T}\{\|x(t)\|\}$. We will use componentwise order in $\mathbb{R}^k$; i.e., we say $a_1 \le a_2$ in $\mathbb{R}^k$ if $a_1^i \le a_2^i$ in $\mathbb{R}$ for each i, $1 \le i \le k$. For $a, b \in \mathbb{R}$, let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For $a, b \in \mathbb{R}^k$, let $[a, b]$ and $[[a, b]]$ be the standard and product segments, respectively, defined by

$$[a, b] = \{\alpha a + (1 - \alpha) b : \ 0 \le \alpha \le 1\} \tag{2.2}$$

and

$$[[a, b]] = [a^1, b^1] \times \cdots \times [a^k, b^k] \ . \tag{2.3}$$

Note that $[a^i, b^i]$ coincides with the standard closed interval $[a^i \wedge b^i, a^i \vee b^i]$.

For $x \in D$, let Γ_x and G_x be the thin and thick completed graphs of x, defined by

$$\Gamma_x = \{(z, t) \in \mathbb{R}^k \times [0, T] : z \in [x(t-), x(t)]\} \tag{2.4}$$

and

$$G_x = \{(z, t) \in \mathbb{R}^k \times [0, T] : z \in [[x(t-), x(t)]]\} \ . \tag{2.5}$$

Let $\le$ be an order relation defined on the graphs by having $(z_1, t_1) \le (z_2, t_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $\|x(t_1-) - z_1\| \le \|x(t_1-) - z_2\|$. The relation $\le$ is a total order on Γ_x and a partial order on G_x.

A strong (weak) parametric representation of x is a continuous nondecreasing (using the orders on the graphs just defined) function (u, r) mapping $[0, 1]$ onto Γ_x (into G_x) such that $r(0) = 0$ and $r(1) = T$. Let $\Pi_s(x)$ and $\Pi_w(x)$ be the sets of all strong and weak parametric representations of x. (Note that we do not require that the strong parametric representations be one-to-one maps, but we require more than r being nondecreasing. We do not allow "backtracking" as the parametric representation passes over the portion of the graph corresponding to a jump.)

Let

$$d_s(x_1, x_2) = \inf_{\substack{(u_j, r_j) \in \Pi_s(x_j) \\ j = 1,2}} \{\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \tag{2.6}$$

and

$$d_w(x_1, x_2) = \inf_{\substack{(u_j, r_j) \in \Pi_w(x_j) \\ j = 1,2}} \{\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \ . \tag{2.7}$$

Convergence $x_n \to x$ as $n \to \infty$ for a sequence or net $\{x_n\}$ is said to hold in the strong M_1 or SM_1 (weak M_1 or WM_1) topology if $d_s(x_n, x) \to 0$ ($d_w(x_n, x) \to 0$) as $n \to \infty$. (It turns out that the SM_1 topology is unchanged if we require that the strong parametric representations be one-to-one maps. However, the weaker monotonicity property we have used seems easier to work with.)

In Whitt (2002) it is shown that d_s in (2.6) is a metric, but d_w in (2.7) is not. It is also shown that $d_w(x_n, x) \to 0$ as $n \to \infty$ if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$, where d_p is a metric inducing the *product topology* such as

$$d_p(x_1, x_2) = \sum_{1 \le i \le k} d_s(x_1^i, x_2^i) \ . \tag{2.8}$$

Thus, the WM_1 topology on $D([0,T], \mathbb{R}^k)$ coincides with the product topology on the product space $D^k \equiv D^1 \times \cdots \times D^1$, where each coordinate D^1 is endowed with the SM_1 topology. We also have the inequality

$$d_p(x_1, x_2) \le k d_w(x_1, x_2) \tag{2.9}$$

for all $x_1, x_2 \in D$, but example 5.3.2 of Whitt (2002) shows that there is no inequality in the opposite direction. When $k = 1$, $d_w = d_s = d_p$, but when $k > 1$, the SM_1 topology is strictly stronger than the WM_1 topology. Since the metrics coincide when $k = 1$, we write d for d_s and M_1 for SM_1 when $k = 1$.

The metrics d_s and d_p each make D an incomplete separable metric space for which the Borel σ-field generated by the open subsets coincides with the Kolmogorov σ-field generated by the coordinate projections. Even though d_s and d_p are incomplete, they are topologically equivalent to complete metrics, so that D with each of these metrics is Polish (metrizable as a complete separable metric space).

An important distinction between the strong and weak M_1 topologies on D is that linear functions of the coordinates are continuous in the SM_1 topology, but not in the WM_1 topology. For $x \in D^k$ and $\eta \in \mathbb{R}^k$, let $\eta x = \sum_{i=1}^k \eta^i x^i \in D^1$. In Section 5.9 of Whitt (2002) it is shown that $x_n \to x$ as $n \to \infty$ in (D^k, SM_1) if and only if $\eta x_n \to \eta x$ as $n \to \infty$ in (D^1, M_1) for all $\eta \in \mathbb{R}^k$. We will use this property in our counterexamples in Section 4.

The standard J_1 topology on $D([0,T], \mathbb{R}^k)$ is induced by the metric

$$d_{J_1}(x_1, x_2) = \inf_{\lambda \in \Lambda} \{ \|x_1 \circ \lambda - x_2\| \vee \|\lambda - e\| \} \ , \tag{2.10}$$

where e is the identity map, i.e., $e(t) = t, 0 \le t \le T$, and Λ is the set of all increasing homeomorphisms of $[0,T]$. Just like the SM_1 and WM_1 topologies above, the standard J_1 topology on $D([0,T], \mathbb{R}^k)$ is stronger than the product topology on the product space $D([0,T], \mathbb{R}^1, J_1)^k$. Thus we could define analogous SJ_1 and WJ_1 topologies, but we only discuss SJ_1 and call it J_1. Theorem 5.4.3 of Whitt (2002) shows that

$$d_s(x_1, x_2) \le d_{J_1}(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in D. \tag{2.11}$$

Weaker topologies than SM_1 and WM_1 on $D([0,T],\mathbb{R}^k)$ are the SM_2 and WM_2 topologies. The SM_2 topology is induced by the Hausdorff metric, denoted by m_s, on the space of thin graphs Γ_x, while the WM_2 topology is the product topology, which is induced by the associated product metric, defined as in (2.8) with m_s playing the role of d_s there. Whitt (2002) shows that the SM_2 topology is also induced by a metric defined as in (2.6), where only r is required to be nondecreasing instead of the entire parametric representation (u,r). With the SM_2 topology, the parametric representations are allowed to "backtrack" as they pass over the portion of the graph corresponding to a jump. In several places in the literature, including Chen and Whitt (1993), the SM_1 topology is not defined correctly, because only r is required to be nondecreasing.

Convergence in the WM_2 topology (and thus in the SM_2, WM_1, SM_1 and J_1 topologies) implies local uniform convergence at each continuity point of a limit. It also implies convergence in the L_1 metric on D, defined by

$$\delta(x_1,x_2)=\int_0^T \|x_1(t)-x_2(t)\|dt\ . \tag{2.12}$$

We call the topology induced by the metric δ on D the L_1 topology.

Even though there are many possible topologies on D, there is only one relevant σ-field. It is significant that the Borel σ-fields on D generated by the SM_1, WM_1, J_1, SM_2, WM_2 and L_1 topologies all coincide with the Kolmogorov σ-field generated by the coordinate projections. Thus continuity of a map from D^k to D^l with one of these topologies on the domain and another on the range immediately implies measurability of the map for other combinations of the topologies on the domain and range. However, note that the Borel σ-field associated with the uniform topology on D is strictly larger than the Kolmogorov σ-field. Thus the uniform topology on D causes measurability problems; see Section 18 of Billingsley (1968). When the uniform topology can be used, it is possible to use non-Borel σ-fields; see Pollard (1984).

3. The Main Results

The (standard) multi-dimensional reflection mapping was developed by Harrison and Reiman (1981). There had been quite a bit of previous work on the one-dimensional reflection map, including Skorohod (1961) and Beneš (1963). Iglehart and Whitt (1970a,b) applied the one-dimensional reflection map in order to obtain FCLTs for acyclic queueing networks. Prior to Harrison and Reiman(1981), other work on multi-dimensional reflection had primarily assumed smooth boundaries. (For related early work, see Tanaka (1979) and Lions and Sznitman (1984).) For more on the reflection map, see Reiman (1984), Chen and Mandelbaum (1991a-c), Chen and Whitt (1993) and Chen and Yao (2000).

Informally, the reflection map transforms an $\mathbb{R}^k$-valued net-input function x into an $\mathbb{R}^k$-valued content function (using queueing terminology) z and an $\mathbb{R}^k$-valued regulator function y. There is some freedom in the choice of initial conditions. Harrison and Reiman require that $x(0) \geq 0$, by which we mean that $x^i(0) \geq 0$ for $1 \leq i \leq k$, which implies that $z(0) = x(0)$ and $y(0) = 0$. (We let 0 and 1 represent the vectors $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$, respectively. It will be clear from the context that they should be elements of $\mathbb{R}^k$.) Instead we allow $x^i(0) < 0$ for one or more i, so that we have to allow $y^i(0) > 0$ for some i. Thus there may be an instantaneous reflection at time 0. We study the instantaneous reflection map in Section 5.

The reflection map $R \equiv (\psi, \phi) : D([0,T], \mathbb{R}^k) \to D([0,T], \mathbb{R}^{2k})$ associated with a substochastic matrix Q (nonnegative with row sums less than or equal to 1) such that $Q^n \to 0$ as $n \to \infty$, where Q^n is the n-fold product of Q with itself, maps x into $(y, z) \equiv (\psi(x), \phi(x))$ such that

$$z = x + (I - Q^t)y \geq 0 \ , \tag{3.1}$$

$$y \text{ is nondecreasing with } y(0) \geq 0 \ , \tag{3.2}$$

and

$$y \text{ is the minimal function satisfying (3.1) and (3.2)} \ . \tag{3.3}$$

where Q^t is the transpose of Q. We call $Q \equiv (Q_{i,j})$ the reflection matrix. It is also natural to call Q the routing matrix, because in applications the reflection map arises naturally when a proportion $Q_{i,j}$ of all output from queue i is routed to queue j; see Section 10. Existence and uniqueness of the reflection map were established by Harrison and Reiman (1981), Reiman (1984) and Chen and Mandelbaum (1991c). They also showed that the minimal element y is the componentwise minimum: With $D_\uparrow$ the subset of nondecreasing nonnegative functions in D and

$$\Psi(x) \equiv \{w \in D_\uparrow : x + (I - Q^t)w \geq 0\} \ , \tag{3.4}$$

$$y = \inf \Psi(x) \ , \tag{3.5}$$

i.e.,

$$y^i(t) = \inf\{w^i(t) \in \mathbb{R} : w \in \Psi(x)\}, \quad 1 \leq i \leq k, \quad 0 \leq t \leq T \ . \tag{3.6}$$

They also showed that the pair (y, z) is characterized by the complementarity property: Under (3.1) and (3.2), (3.3) is equivalent to

$$\int_0^T z^i(t) dy^i(t) = 0, \quad 1 \leq i \leq k \ . \tag{3.7}$$

For $k = 1$ and $Q = 0$, the reflection map has a relatively simple form, i.e.,

$$\phi(x)(t) \equiv z(t) = x(t) - \inf\{x(s) \wedge 0 : 0 \leq s \leq t\} \tag{3.8}$$

and

$$\psi(x)(t) \equiv y(t) = -\inf\{x(s) \wedge 0 : 0 \leq s \leq t\}\ . \tag{3.9}$$

It is possible to consider more general reflection maps. First, the matrix $I - Q^t$ in (3.1) can be replaced by a completely-S matrix; see Williams (1995) and references therein. Second, the content-portion of the reflection map ϕ can map into other sets besides the nonnegative orthant $\mathbb{R}_+^k \equiv [0,\infty)^k$; see Williams (1987), Dupuis and Ishii (1991) and Dupuis and Ramanan (1999a,b). We briefly consider such generalizations in Section 11.

We are interested in continuity and Lipschitz properties of the reflection map. As reviewed in Chen and Whitt (1993), $R \equiv (\psi, \phi)$ is Lipschitz continuous with the uniform metric. Chen and Whitt give tight bounds on the Lipschitz constant which in general depends on the routing matrix Q. The following is Proposition 2.3 of Chen and Whitt (1993). To express it we use the matrix norm

$$\|A^t\| \equiv \max_j \sum_{i=1}^k |A_{i,j}^t| \tag{3.10}$$

for real $k \times k$ matrices A. Since Q^t is a $k \times k$ column-stochastic matrix, $\|Q^t\| \leq 1$ and $\|(Q^t)^k\| < 1$.

Theorem 3.1 *For all $x_1, x_2 \in D$,*

$$\|\psi(x_1) - \psi(x_2)\| \leq \|(I - Q^t)^{-1}\| \cdot \|x_1 - x_2\| \leq \frac{k}{1-\gamma}\|x_1 - x_2\| \tag{3.11}$$

and

$$\|\phi(x_1) - \phi(x_2)\| \leq (1 + \|I - Q^t\| \cdot \|(I - Q^t)^{-1}\|)\|x_1 - x_2\| \leq \left(1 + \frac{2k}{1-\gamma}\right)\|x_1 - x_2\|\,, \tag{3.12}$$

where

$$\gamma \equiv \|(Q^t)^k\| < 1\ . \tag{3.13}$$

Remark 3.1 The upper bounds in Theorem 3.1 are minimized by making $Q_{i,j} = 0$ for all i, j. Let K^* be the infimum of K such that

$$\|R(x_1) - R(x_2)\| \leq K\|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in D\ . \tag{3.14}$$

We call K^* the Lipschitz constant. The bounds yield $K^* \leq 2$ when $Q_{i,j} = 0$ for all i, j, but the example in Remark 2.1 of Chen and Whitt (1993) shows that $K^* = 2$ in that case. That example has $k = 1$, $Q = 0$, $x_1(t) = 0$, $0 \le t \le 1$, and $x_2 = -I_{[1/3,1/2)} + I_{[1/2,1]}$ in $D([0,1],\mathbb{R})$. Then $y_1 = z_1 = x_1$, but $y_2 = I_{[1/3,1]}$ and $z_2 = 2I_{[1/2,1]}$, so that $\|z_1 - z_2\| = 2$.

Example 3.1 To see that there is no universal bound on the Lipschitz constant K^* independent of Q, let $x_1(t) = 0$, $0 \le t \le 2$, and $x_2 = -I_{[1,2]}$ in $D([0,2],\mathbb{R})$, so that $\|x_1 - x_2\| = 1$. Let $Q = 1 - \epsilon$, so that (3.1) becomes

$$z = x + \epsilon y \ . \tag{3.15}$$

Then $z_2 = z_1 = y_1 = x_1$, but $y_2 = \epsilon^{-1} I_{[1,2]}$, so that $\|y_1 - y_2\| = \epsilon^{-1}$.

Example 3.2 To see that the Lipschitz constant for the component map ϕ can be arbitrarily large as well, consider the two-dimensional example with $Q_{1,1} = 1 - \epsilon$, $Q_{2,1} = 1$ and $Q_{2,2} = Q_{1,2} = 1/2$, so that

$$(I - Q^t) = \begin{pmatrix} \epsilon & -1 \\ -1/2 & 1/2 \end{pmatrix} \ . \tag{3.16}$$

Let $x_1^1 = -I_{[1,2]}$, $x_2^1(t) = 0$, $0 \le t \le 2$, and $x_1^2 = x_2^2 = \epsilon^{-1} I_{[0,2]}$ in $D([0,2],\mathbb{R}^2)$. Then $\|x_1 - x_2\| = 1$, but $z_1^1(t) = z_2^1(t) = 0$, $0 \le t \le 2$, $z_1^2 = \epsilon^{-1} I_{[0,1)}$ and $z_2^2 = \epsilon^{-1} I_{[0,2]}$, so that $\|z_1 - z_2\| = \epsilon^{-1}$.

The following are our main results, which we prove in Sections 5-7. Our first result establishes continuity of the reflection map R (for an arbitrary reflection matrix Q) as a map from (D, SM_1) to (D, L_1) and, under a restriction on the limit, as a map from (D, WM_1) to (D, WM_1). We will give examples to show the necessity of the conditions.

Let D_s be the subset of functions in D without simultaneous jumps of opposite sign in the coordinate functions; i.e., $x \in D_s$ if, for all $t \in (0,T)$, either $x(t) - x(t-) \le 0$ or $x(t) - x(t-) \ge 0$, with the sign allowed to depend upon t. The subset D_s is a closed subset of D in the J_1 topology and thus a measurable subset of D with the SM_1 and WM_1 topologies (since the Borel σ-fields coincide). The space D_s is the first of several subsets of D that we consider. We introduced all the subsets of D at the end of Section 1.

Theorem 3.2 *Suppose that $x_n \to x$ in (D, SM_1).*

(a) Then

$$R(x_n)(t_n) \to R(x)(t) \quad in \quad \mathbb{R}^{2k} \tag{3.17}$$

for each $t \in Disc(x)^c$ and sequence $\{t_n : n \ge 1\}$ with $t_n \to t$,

$$\sup_{n \ge 1} \|R(x_n)\| < \infty \ , \tag{3.18}$$

$$R(x_n) \to R(x) \quad in \quad (D, L_1) \tag{3.19}$$

and

$$\psi(x_n) \to \psi(x) \quad in \quad (D, WM_1)\ . \tag{3.20}$$

(b) If in addition $x \in D_s$, then

$$\phi(x_n) \to \phi(x) \quad in \quad (D, WM_1)\ , \tag{3.21}$$

so that

$$R(x_n) \to R(x) \quad in \quad (D, WM_1)\ . \tag{3.22}$$

Under the extra condition in part (b), the mode of convergence on the domain actually can be weakened.

Theorem 3.3 *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then* (3.22) *holds.*

Remark 3.2 Interestingly, when $x \notin D_s$, the limit of $\phi(x_n)(t_n)$ for $t_n \to t \in Disc(x)$ can fall outside the product segment $[[\phi(x)(t-), \phi(x)(t)]]$ defined in (2.3); see Example 4.6 below. Thus the asymptotic fluctuations in $\phi(x_n)$ can be greater than the fluctuations in $\phi(x)$. The behavior here is analogous to the Gibbs phenomenon associated with Fourier series; see Remark 5.1 of Abate and Whitt (1992) and references cited there.

Example 5.3.1 of Whitt (2002) shows that convergence $x_n \to x$ can hold in (D, WM_1) but not in (D, SM_1) even when $x \in D_s$. Thus Theorems 3.2 (a) and 3.3 cover distinct cases. An important special case of both occurs when $x \in D_1$, where D_1 is the subset of x in D with discontinuities in only one coordinate at a time; i.e., $x \in D_1$ if $t \in Disc(x^i)$ for at most one i when $t \in Disc(x)$, with the coordinate i allowed to depend upon t. In Section 5.8 of Whitt (2002) it is shown that WM_1 convergence $x_n \to x$ is equivalent to SM_1 convergence when $x \in D_1$.

Just as with D_s above, D_1 is a closed subset of (D, J_1) and thus a Borel measurable subset of (D, SM_1). Since $D_1 \subseteq D_s$, the following corollary to Theorem 3.3 is immediate.

Corollary 3.1 *If $x_n \to x$ in (D, WM_1) and $x \in D_1$, then $R(x_n) \to R(x)$ in (D, WM_1).*

We can obtain stronger Lipschitz properties on special subsets. Let D_+ be the subset of x in D with only nonnegative jumps, i.e., for which $x^i(t) - x^i(t-) \geq 0$ for all i and t. As with D_s and D_1 above, D_+ is a closed subset of (D, J_1) and thus a measurable subset of (D, SM_1).

Theorem 3.4 *There is a constant K (the same as associated with the uniform norm in* (3.14)) *such that*

$$d_s(R(x_1), R(x_2)) \le K d_s(x_1, x_2) \tag{3.23}$$

for all $x_1, x_2 \in D_+$, and

$$d_p(R(x_1), R(x_2)) \le d_w(R(x_1), R(x_2)) \le K d_w(x_1, x_2) \le K d_s(x_1, x_2) \tag{3.24}$$

for all $x_1, x_2 \in D_s$.

We can actually do somewhat better than in Theorems 3.2 and 3.3 when the limit is in D_+.

Theorem 3.5 *If*

$$x_n \to x \quad in \quad (D, SM_1) \ , \tag{3.25}$$

where $x \in D_+$, then

$$R(x_n) \to R(x) \quad in \quad (D, SM_1) \ . \tag{3.26}$$

4. Preliminary Results and Counterexamples

In this section we establish a few preliminary results and give counterexamples showing the necessity of the conditions in Theorems 3.2–3.5. We first explain the error in Chen and Whitt (1993). In Proposition 2.4 of Chen and Whitt (1993) a result is asserted that would imply that (ψ, ϕ) is Lipschitz with the J_1 and SM_1 metrics, but this assertion is incorrect. To restate the erroneous proposition, let (D, U) denote D with the uniform metric and so forth.

Proposition 2.4 of Chen and Whitt (1993). *If $f : (D, U) \to (D, U)$ is continuous* (Lipschitz with modulus K), *then $f : (D, J_1) \to (D, J_1)$ and $f : (D, M_1) \to (D, M_1)$ are continuous* (*Lipschitz with modulus* $(K \vee 1)$).

Upon a little reflection, it is evident (as pointed out by Nimrod Bayer and Anatolii Puhalskii in personal communications) that this general assertion is false even if $k = 1$, as can be seen from the following example provided by Bayer.

Example 4.1 Let $f(x)(t) = x(1)$ for $t \ge 1$ and 0 otherwise. Clearly $f : D([0,2], \mathbb{R}, U) \to D([0,2], \mathbb{R}, U)$ is Lipschitz. Let $x(t) = I_{[1,2]}(t)$ and $x_n(t) = I_{[1+n^{-1},2]}(t)$ for $n \ge 2$. Clearly $x_n \to x$ in $D([0,2], \mathbb{R}, J_1)$, but $f(x_n)(t) = 0$ for all t and n, while $f(x)(t) = x(t) = I_{[1,2]}(t)$. Hence this function f is not even continuous as a map from (D, J_1) to (D, L_1). ■

Example 4.1 shows that, at minimum, the function f in Proposition 2.4 of Chen and Whitt (1993) needs to have additional properties before such a conclusion can be reached. However, the reflection map $R \equiv (\psi, \phi)$ does have important properties that make it possible to establish the desired Lipschitz inheritance properties with appropriate qualifications. (Such properties were established by Harrison and Reiman (1981) and Chen and Mandelbaum (1991a-c).)

Let $x \circ \gamma$ be the composition of x and γ, i.e., $(x \circ \gamma)(t) = x(\gamma(t))$.

Lemma 4.1 *Let $\eta \in \mathbb{R}^k$, $\beta > 0$ and γ be a nondecreasing right-continuous function mapping $[0, T_1]$ onto $[0, T]$. If $x \in D([0,T], \mathbb{R}^k)$ and $\eta + \beta x(0) \geq 0$, then $\eta + \beta(x \circ \gamma) \in D([0, T_1], \mathbb{R}^k)$ and*

$$R(\eta + \beta(x \circ \gamma)) = \beta R\left(\frac{\eta}{\beta} + x\right) \circ \gamma \ . \tag{4.1}$$

Proof. First note that the nondecreasing right-continuity property of γ makes $x \circ \gamma$ right continuous with left limits: If $t_n \downarrow t$, then $\gamma(t_n) \downarrow \gamma(t)$ and $x(\gamma_n(t_n)) \to x(\gamma(t))$; if $t_n \uparrow t$, then $\gamma(t_n) \uparrow \gamma(t-)$ and $x(\gamma_n(t_n)) \to x(\gamma(t-)) = (x \circ \gamma)(t-)$. Next note that $\beta(z \circ \gamma)$ and $\beta(y \circ \gamma)$ satisfy (3.1)–(3.7) when we replace x by $\beta(x \circ \gamma)$. ■

The invariance under the time transformation γ in Lemma 4.1 makes the Lipschitz property for the reflection map correct for the J_1 metric, using the proof given in Chen and Whitt (1993).

Theorem 4.1 *Let $R : D^k \to D^{2k}$ be the reflection map defined in (3.1)-(3.3). There is a constant K such that*

$$d_{J_1}(R(x_1), R(x_2)) \leq K d_{J_1}(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in D([0,T], \mathbb{R}^k) \ . \tag{4.2}$$

Moreover, the Lipschitz constant is the Lipschitz constant K^* associated with the uniform norm in (3.10).

Proof. Using the argument in Proposition 2.4 of Chen and Whitt, applied specifically to R,

$$\begin{aligned} d_{J_1}(R(x_1), R(x_2)) &= \inf_{\lambda \in \Lambda} \{\|R(x_1) \circ \lambda - R(x_2)\| \vee \|\lambda - e\|\} \\ &= \inf_{\lambda \in \Lambda} \{\|R(x_1 \circ \lambda) - R(x_2)\| \vee \|\lambda - e\|\} \\ &\leq \inf_{\lambda \in \Lambda} \{K\|x_1 \circ \lambda - x_2\| \vee \|\lambda - e\|\} \\ &\leq (K \vee 1) d_{J_1}(x_1, x_2) = K d_{J_1}(x_1, x_2) \ , \end{aligned}$$

by the known Lipschitz property of R in $(D, \|\cdot\|)$. ■

We can also apply Theorem 4.1 to deduce measurability of the reflection map. This measurability is needed to apply the continuous mapping theorem on D even when the limiting stochastic process has continuous sample paths, e.g., as in Reiman (1984). The Lipschitz property in Theorem 3.1 implies that the reflection map is measurable on D using the Borel σ-field generated by the uniform topology on both the domain and range, but that Borel σ-field is strictly larger than the Kolmogorov σ-field, so (contrary to the claim on p. 9 of Chen and Mandelbaum (1991c)) Theorem 3.1 does not imply the following result.

Corollary 4.1 *The reflection map $R : D^k \to D^{2k}$ is measurable, using the Kolmogorov σ-field on the domain and range.*

For $k = 1$, the Lipschitz property of ϕ with the M_1 metric d also follows by the argument of Chen and Whitt (1993), as has been known for some time.

Theorem 4.2 *Let $\phi : D^1 \to D^1$ be the component of the reflection map in* (3.8). *There is a constant K such that*

$$d(\phi(x_1), \phi(x_2)) \leq K d(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in D^1 \ , \tag{4.3}$$

where d is the M_1 metric. Moreover, the Lipschitz constant is the Lipschitz constant associated with the uniform norm.

Proof. For $k = 1$, it is easy to see that for each $x \in D^1$, $(\phi(u), r)$ is a parametric representation of $\phi(x)$ whenever (u, r) is a parametric representation of x. (It is a consequence of Theorem 6.2 below.) Hence, if K is the Lipschitz constant for the uniform norm, with $K \geq 1$ by Theorem 3.1, then

$$\begin{aligned} d(\phi(x_1,), \phi(x_2)) &= \inf_{\substack{(u_i', r_i) \in \Pi(\phi(x_i)) \\ i=1,2}} \{\|u_1' - u_2'\| \vee \|r_1 - r_2\|\} \\ &\leq \inf_{\substack{(u_i, r_i) \in \Pi(x_i) \\ i=1,2}} \{\|\phi(u_1) - \phi(u_2)\| \vee \|r_1 - r_2\|\} \\ &\leq \inf_{(u_i, r_i) \in \Pi(x_i)} \{K\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \\ &\leq K d(x_1, x_2) \ . \quad \blacksquare \end{aligned}$$

However, other positive results for the M topologies evidently are harder to obtain. Indeed, the following elementary example from Konstantopoulos (1999) shows that ϕ is not continuous in the M_2 topology when $k = 1$. (As with M_1, for $k = 1$ the SM_2 and WM_2 topologies coincide.)

Example 4.2 To see that ϕ is not continuous on $D([0,2],\mathbb{R})$ in the M_2 topology, let $x = -I_{[1,2]}$ and

$$x_n(0) = x_n(1-3n^{-1}) = 0,\ x_n(1-2n^{-1}) = -1,\ x_n(1-n^{-1}) = 0, x_n(1) = x_n(2) = -1, \qquad (4.4)$$

with x_n defined by linear interpolation elsewhere. Then $x_n \to x$ as $n \to \infty$, but $z(t) = \phi(x)(t) = 0$, $0 \le t \le 2$, while $z_n(1-n^{-1}) = 1$, so that $z_n \not\to z$ as $n \to \infty$. ■

We now show that the reflection map is actually not continuous on $D([0,T],\mathbb{R}^1)$ with the SM_1 topology. (This would not be a counterexample if we restricted attention to the component ϕ mapping x into z in (3.1) or, more generally, the WM_1 topology were used on the range.)

Example 4.3 To show that $R \equiv (\psi,\phi) : (D([0,2],\mathbb{R}^1), SM_1) \to (D([0,2],\mathbb{R}^2), SM_1)$ is *not* continuous for $i = 1,2$, let

$$x_n(t) = 1 - 2n(t-1)I_{[1,1+n^{-1})}(t) - 2I_{[1+n^{-1},2]}(t) \qquad (4.5)$$

and

$$x(t) = 1 - 2I_{[1,2]}(t), \quad 0 \le t \le 2\ . \qquad (4.6)$$

It is easy to see that $d(x_n,x) \to 0$ as $n \to \infty$,

$$z_n(t) = 1 - 2n(t-1)I_{[1,1+(2n)^{-1})}(t)\ , \qquad (4.7)$$

$$y_n(t) = 2n(t-(1+(2n)^{-1}))I_{[1+(2n)^{-1},1+n^{-1})}(t) + I_{[1+n^{-1},2]}(t)\ , \qquad (4.8)$$

$$z(t) = I_{[0,1)}(t) \quad \text{and} \quad y(t) = I_{[1,2]}(t)\ . \qquad (4.9)$$

We use the fact that any linear function of the coordinate functions, such as addition or subtraction, is continuous in the SM_1 topology; see Section 5.9 of Whitt (2002). Note that $z(t) + y(t) = 1$, $0 \le t \le 2$, while

$$z_n(t) + y_n(t) = 1 - 2n(t-1)I_{[1,1+(2n)^{-1}]}(t) + 2n(t-(1+(2n)^{-1})I_{[1+(2n)^{-1},1+n^{-1}]}(t) \qquad (4.10)$$

so that $d(z_n + y_n, z + y) \not\to 0$ as $n \to \infty$, which implies that $(x_n,y_n) \not\to (z,y)$ as $n \to \infty$ in $D([0,T],\mathbb{R}^2)$ with the SM_1 metric. However, we do have $d(z_n,z) \to 0$ and $d(y_n,y) \to 0$ as $n \to \infty$, so the maps from x to y and z separately are continuous. ■

Example 4.3 suggests that the difficulty might only be in simultaneously considering both maps ψ and ϕ. We show that this is not the case by giving a counterexample with ϕ alone (but again in two dimensions).

Example 4.4 We now show that $\phi : (D([0,2],\mathbb{R}^2), SM_1) \to (D([0,2],\mathbb{R}^2), SM_1)$ is not continuous. We use the trivial reflection map corresponding to two separate queues, for which Q is the 2×2 matrix of 0's. Let x_n^1 be as in Example 4.3, i.e.,

$$x_n^1(t) = 1 - 2n(t-1)I_{[1,1+n^{-1})}(t) - 2I_{[1+n^{-1},2]}(t) \tag{4.11}$$

and let

$$x_n^2(t) = 2 - 3n(t-1)I_{[1,1+n^{-1})}(t) - 3I_{[1+n^{-1},2]}(t) \; . \tag{4.12}$$

It is easy to see that $d_s((x_n^1, x_n^2), (x^1, x^2)) \to 0$ as $n \to \infty$, where

$$x^1(t) = 1 - 2I_{[1,2]}(t) \quad \text{and} \quad x^2(t) = 2 - 3I_{[1,2]}(t) \; . \tag{4.13}$$

(The same functions r_n and r can be used in the parametric representations of the two coordinates.) Clearly $\phi((x^1, x^2)) = (z^1, z^2)$, where

$$z^1(t) = I_{[0,1)}(t) \quad \text{and} \quad z^2(t) = 2I_{[0,1)}(t) \; , \tag{4.14}$$

while

$$z_n^1(t) = 1 - 2n(t-1)I_{[1,1+(1/2n))}(t) \tag{4.15}$$

$$z_n^2(t) = 2 - 3n(t-1)I_{[1,1+(2/3n))}(t) \; . \tag{4.16}$$

Note that $2z^1(t) - z^2(t) = 0$, $0 \le t \le 2$, while

$$2z_n^1(1+(2n)^{-1}) - z_n^2(1+(2n)^{-1}) = -z_n^2(1+(2n)^{-1}) = -1/2 \quad \text{for all} \quad n \; . \tag{4.17}$$

Hence $d_s(2z_n^1 - z_n^2, 2z^1 - z^2) \not\to 0$ so that $d_s((z_n^1, z_n^2), (z^1, z^2)) \not\to 0$ as $n \to \infty$. However, in this example, ϕ is continuous if we use the WM_1 topology on the range. ■

We now show that the reflection map is not continuous if the WM_1 topology is used on the domain. Examples 4.3–4.5 show that we need to have *both* the strong topology on the domain and the weak topology on the range.

Example 4.5 We show that neither ψ nor ϕ need be continuous when the WM_1 topology is used on the domain, without imposing extra conditions. Consider $D([0,2],\mathbb{R}^2)$ and let $x^1 = I_{[1,2]}$, $x^2 = -2I_{[1,2]}$ and

$$Q = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \; . \tag{4.18}$$

Then the reflection map yields $y^1(t) = \psi^1(x)(t) = z^i(t) = \phi^i(x)(t) = 0$, $0 \leq t \leq 2$, for $i = 1, 2$ and $y^2 = \psi^2(x) = 2I_{[1,2]}$. Let the converging functions be $x_n^1 = I_{[1+n^{-1},2]}$ and $x_n^2 = -2I_{[1-n^{-1},2]}$ for $n \geq 1$. It is easy to see that $x_n \to x$ as $n \to \infty$ in WM_1 but that $x_n \not\to x$ as $n \to \infty$ in SM_1, because $(2x_n^1 + x_n^2)(1) = -2$, while $(2x^1 + x^2)(t) = 0$, $0 \leq t \leq 2$. The reflection map applied to x_n works on the jumps at times $1 - n^{-1}$ and $1 + n^{-1}$ separately, yielding $y_n^1 = (4/3)I_{[1-n^{-1},2]}$, $y_n^2 = (8/3)I_{[1-n^{-1},2]}$, $z_n^1 = I_{[1+n^{-1},2]}$ and $z_n^2(t) = 0$, $0 \leq t \leq 2$. Clearly $z_n^1 \not\to z^1$ and $y_n^i \not\to y^i$ as $n \to \infty$ for $i = 1, 2$ for any reasonable topology on the range. In particular, conclusions (3.17) and (3.19) – (3.22) all fail in this example.

Moreover, when we choose suitable parametric representations $(u_n, r_n) \in \Pi_w(x_n)$ and $(u, r) \in \Pi_w(x)$ to achieve $x_n \to x$ in WM_1, $(R(u), r)$ is not a parametric representation for $R(x)$. To be clear about this, we give an example: We let all the functions u_n, r_n, u and r be piecewise-linear. We define the functions at the discontinuity points of the derivative. We understand that the functions are extended to $[0, 1]$ by linear interpolation. Let

$$
\begin{aligned}
&r(0) = 0,\ r(0.2) = r(0.8) = 1,\ r(1) = 2\ ,\\
&u^1(0) = u^1(0.4) = 0,\ u^1(0.8) = u^1(1) = 1\ ,\\
&u^2(0) = u^2(0.2) = 0,\ u^2(0.4) = u^2(1) = -2\ ,\\
&r_n(0) = 0,\ r_n(0.2(1 - n^{-1})) = r_n(0.2(2 - n^{-1})) = 1 - n^{-1}\ ,\\
&r_n(0.2(2 + n^{-1})) = r_n(0.2(4 + n^{-1})) = 1 + n^{-1},\ r_n(1) = 2\ ,\\
&u_n^1(0) = u_n^1(0.2(2 + n^{-1})) = 0,\ u_n^1(0.2(4 + n^{-1}) = u_n^1(1) = 1\ ,\\
&u_n^2(0) = u_n^2(0.2(1 - n^{-1}) = 0,\ u_n^2(0.2(2 - n^{-1}) = u_n^2(1) = -2\ .
\end{aligned}
$$

This construction yields $(u_n, r_n) \in \Pi_w(x_n)$, $n \geq 1$, $(u, r) \in \Pi_w(x)$, but $(\phi^1(u), r) \notin \Pi(\phi^1(x))$, because $\phi^1(x)(t) = 0$, $0 \leq t \leq 2$, while $\phi^1(u)(1) = 1$. Note that $(u, r) \in \Pi_w(x)$, but $(u, r) \notin \Pi_s(x)$. ■

We now show that we need not have $R(x_n) \to R(x)$ in (D, WM_1) when $x_n \to x$ in (D, SM_1) without having the extra regularity conditions $x \in D_s$. A difficulty can occur when $x^i(t) - x^i(t-) > 0$ for some coordinate i, while $x^j(t) - x^j(t-) < 0$ for another coordinate j.

Example 4.6 We now show the need for the condition $x \in D_s$ in Theorem 3.2 (b). In our limit $x \equiv (x^1, x^2)$, x^1 has a jump down and x^2 has a jump up at $t = 1$. Our example is the simple network corresponding to two queues in series. Let $x \equiv (x^1, x^2)$ and $x_n \equiv (x_n^1, x_n^2)$, $n \geq 1$, be elements of $D([0,2],\mathbb{R}^2)$ defined by

$$\begin{array}{ll} x^1(0) = x^1(1-) = 1, & x^1(1) = x^1(2) = -3 \\ x^2(0) = x^2(1-) = 1, & x^2(1) = x^2(2) = 2 \\ x^1_n(0) = x^1_n(1) = 1, & x^1_n(1+n^{-1}) = x^1_n(2) = -3 \\ x^2_n(0) = x^2_n(1) = 1, & x^2_n(1+n^{-1}) = x^2_n(2) = 2 \ , \end{array}$$

with the remaining values determined by linear interpolation. Let the substochastic matrix generating the reflection be

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{so that} \quad I - Q^t = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} .$$

Then $z^1 = z^2 = I_{[0,1)}$, $y^1 = 3I_{[1,2]}, y^2 = I_{[1,2]}$ and

$$\begin{array}{lll} z^1_n(0) = z^1_n(1) = 1, & z^1_n(1+(4n)^{-1}) = z^1_n(2) = 0 & \\ z^2_n(0) = z^2_n(1) = 1, & z^2_n(1+(4n)^{-1}) = 5/4, & z^2_n(1+2(3n)^{-1}) = z^2_n(2) = 0 \end{array}$$

with the remaining values determined by linear interpolation. Since $z^2_n(1+(4n)^{-1}) = 5/4$ for all n and $z^2(t) \le 1$ for all t, z^2_n fails to converge to z^2 in any of the Skorohod topologies. We remark that the graphs $G_{\phi(x_n)}$ of $\phi(x_n)$ do converge in the Hausdorff metric to the graph $G_{\phi(x)}$ of $\phi(x)$ augmented by the set $\{1\} \times [1, 5/4]$. This example motivates considering larger spaces of functions than D; see the final chapter in Whitt (2002).

5. The Instantaneous Reflection Map

In this section we review and establish properties of the instantaneous reflection map, which is applied at time 0 if $x^i(0) < 0$ for some i and which can be used to characterize the behavior of the full reflection map at discontinuity points. Indeed, we can use the instantaneous reflection map to define the full reflection map on D. See Chen and Mandelbaum (1991a-c) for related material.

Let the instantaneous reflection map be $R_0 \equiv (\phi_0, \psi_0) : \mathbb{R}^k \to \mathbb{R}^{2k}$, where $\psi_0 : \mathbb{R}^k \to \mathbb{R}^k$ is defined by

$$\psi_0(u) \equiv \inf\{v \in \mathbb{R}^k_+ : u + (I - Q^t)v \ge 0\} \ , \tag{5.1}$$

where $u_1 \le u_2$ in $\mathbb{R}^k$ if $u^i_1 \le u^i_2$ in $\mathbb{R}$, $1 \le i \le k$. The instantaneous reflection map is also known as the linear complementarity problem, which has a long history; see Cottle, Pang and Stone (1992). It turns out that the infimum in (5.1) is attained (so that we can refer to the minimum) and there are useful expressions for it. Given the solution to (5.1), we can define the other component of the instantaneous reflection map $\phi_0 : \mathbb{R}^k \to \mathbb{R}^k$ by

$$\phi_0(u) = u + (I - Q^t)\psi_0(u) \ . \tag{5.2}$$

We are motivated to consider the instantaneous reflection map i n (5.1) and (5.2) because it describes the queue content associated with a potential instantaneous net input u. For example, we might have an instantaneous input vector $u_1 \geq 0$ and potential instantaneous output vector u_2 which is routed to other queues by the stochastic matrix Q, so that the overall potential instantaneous net input is

$$u = u_1 - u_2 + u_2 Q^t \ . \tag{5.3}$$

However, if the potential output u_2 exceeds the available supply, then we may have to disallow some of the ouput u_2. That can be accomplished by adding a minimal $(I - Q^t)v$ to u in (5.3), which gives (5.1). In fact, as we will show, the instantaneous reflection map in (5.1) and (5.2) is well defined for any $u \in \mathbb{R}^k$, not just for u of the form (5.3).

We exploit the assumptions about the matrix Q through the following well known lemma.

Lemma 5.1 *If Q is a substochastic matrix with $Q^n \to 0$ as $n \to \infty$, then $I - Q$ and $I - Q^t$ are nonsingular with nonnegative inverses*

$$(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n \tag{5.4}$$

and

$$(I - Q^t)^{-1} = \sum_{n=0}^{\infty} (Q^t)^n = ((I - Q)^{-1})^t \ . \tag{5.5}$$

We first establish upper and lower bounds on $\psi_0(u)$. For $u \in \mathbb{R}^k$, let

$$u^+ \equiv u \vee 0 \equiv (u^1 \vee 0, \ldots, u^k \vee 0) \quad \text{and} \quad u^- \equiv u \wedge 0 \equiv (u^1 \wedge 0, \ldots, u^k \wedge 0) \ .$$

Lemma 5.2 *For any $u \in \mathbb{R}^k$,*

$$0 \leq -(u^-) \leq \psi_0(u) \leq -(I - Q^t)^{-1} u^- \ . \tag{5.6}$$

Proof. Let $v = -(I - Q^t)^{-1} u^-$ and note that

$$u + (I - Q^t)v = u - (I - Q^t)(I - Q^t)^{-1} u^- = u - u^- = u^+ \geq 0 \ .$$

Then, by the definition of ψ_0 in (5.1), $\psi_0(u) \leq v$, which establishes the upper bound. By (5.2),

$$\phi_0(u) = u + (I - Q^t)\psi_0(u) \geq 0.$$

Since $\psi_0(u) \geq 0$ and $Q \geq 0$,

$$\psi_0(u) \geq -u + Q^t \psi_0(u) \geq -u \ ,$$

which implies the lower bound. ∎

We now establish an additivity property of ψ_0.

Lemma 5.3 *If $0 \le v_0 \le \psi_0(u)$ in $\mathbb{R}^k$, then*

$$\psi_0(u) = \psi_0(u + (I - Q^t)v_0) + v_0 \ . \tag{5.7}$$

Proof. By (5.1),

$$\begin{aligned}
\psi_0(u) &= \min\{v \in \mathbb{R}^k_+ : u + (I - Q^t)v \ge 0\} \\
&= \min\{v \in \mathbb{R}^k_+ : u + (I - Q^t)v_0 + (I - Q^t)(v - v_0) \ge 0\} \\
&= v_0 + \min\{v' \in \mathbb{R}^k_+ : u + (I - Q^t)v_0 + (I - Q^t)v' \ge 0\} \\
&= v_0 + \psi_0(u + (I - Q^t)v_0) \ ,
\end{aligned}$$

using the condition in the penultimate step. ∎

We now characterize the instantaneous reflection map in terms of a map applied to the positive and negative parts of the vector u. In particular, for any $u \in \mathbb{R}^k$, let

$$T(u) = u^+ + Q^t u^- \tag{5.8}$$

and let T^k be the k-fold iterate of the map T, i.e., $T^k(u) = T(T^{k-1}(u))$ for $k \ge 1$ with $T^0(u) \equiv u$. Note that T is a nonlinear function from $\mathbb{R}^k$ to $\mathbb{R}^k$. The following result is essentially Lemma 1 in Kella and Whitt (1996).

Theorem 5.1 *Let $u_n \equiv T(u_{n-1})$ for T in (5.8) and $u_0 \equiv u$. Then, for any $u \in \mathbb{R}^k$,*

$$u^+_{n-1} \ge u^+_n \ge 0 \tag{5.9}$$

and

$$0 \ge u^-_n \ge (Q^t)^n u^-_0 \quad \textit{for all} \quad n \ , \tag{5.10}$$

so that

$$u^-_n \to 0, \quad u_n \to u_\infty \ge 0 \quad \textit{and} \quad \psi_0(u_n) \to 0 \quad \textit{as} \quad n \to \infty \, . \tag{5.11}$$

For each $n \ge 1$,

$$\psi_0(u) = -\sum_{k=0}^{n-1} u^-_k + \psi_0(u_n) \tag{5.12}$$

and

$$u_n = u - (I - Q^t) \sum_{k=0}^{n-1} u_k^- \ , \tag{5.13}$$

so that ψ_0 *in* (5.1) *is well defined with*

$$\psi_0(u) = -\sum_{k=0}^{\infty} u_k^- \equiv -\sum_{k=0}^{\infty} T^k(u)^- \tag{5.14}$$

and

$$\phi_0(u) = u_\infty \equiv \lim_{n\to\infty} T^n(u) \ . \tag{5.15}$$

Proof. Since $u_n = u_{n-1}^+ + Q^t u_{n-1}^-$ by (5.8), $Q^t u_{n-1}^- \le u_n \le u_{n-1}^+$, which implies (5.9) and $Q^t u_{n-1}^- \le u_n^- \le 0$. By induction, these inequalities imply (5.10). Since $(Q^t)^n \to 0$ as $n \to \infty$, (5.9) and (5.10) imply the first two limits in (5.11). By Lemma 5.2,

$$\psi_0(u_n) \le -(I - Q^t)^{-1} u_n^- \ . \tag{5.16}$$

Since $u_n^- \to 0$, (5.15) implies the last limit in (5.11). Formula (5.12) follows from Lemmas 5.2 and 5.3 by induction. From (5.8), $u_n - u_{n-1} = -(I - Q^t)u_{n-1}^-$, from which (5.13) follows by induction. Since $\psi_0(u_n) \to 0$, (5.12) implies (5.14), where the sum is finite. Moreover, (5.11)–(5.14) imply that $u_\infty = u + (I - Q^t)\psi_0(u)$, which in turn implies (5.15). ∎

We can apply Theorem 5.1 to deduce the complementarity property.

Corollary 5.1 *For any* $u \in \mathbb{R}^k$*,*

$$\phi_0^i(u)\psi_0^i(u) = 0 \quad \textit{for all} \quad i \ . \tag{5.17}$$

Proof. If $\phi_0^i(u) > 0$, then $u_n^i > 0$ for all n by (5.9), which implies that $(u_n^-)^i = 0$ for all n and $\psi_0^i(u) = 0$ by (5.14). On the other hand, if $\psi_0^i(u) > 0$, then $u_k^i < 0$ for some k by (5.14), which implies that $(u_k^+)^i = 0$ for some k, so that $u_\infty^i = 0$ by (5.9). ∎

Theorem 5.1 implies the following important monotonicity property.

Corollary 5.2 *If* $u_1 \le u_2$ *in* $\mathbb{R}^k$*, then*

$$\phi_0(u_1) \le \phi_0(u_2) \quad \textit{and} \quad \psi_0(u_1) \ge \psi_0(u_2) \ .$$

We now apply the instantaneous reflection map to give a constructive definition of the reflection map R on D. For that purpose, let D_c be the subset of piecewise-constant functions in D

with only finitely many discontinuities. Properties of D can often be established and/or better understood by focusing on D_c. Restriction to D_c can capture properties of D because functions in D can be approximated arbitrarily closely by functions in D_c, as shown by Lemma 1 on p. 110 of Billingsley (1968), modified to allow the functions to have range $\mathbb{R}^k$ instead of $\mathbb{R}^1$. (Given the result for $\mathbb{R}^1$-valued functions, we can obtain the result for $\mathbb{R}^k$-valued functions by treating the coordinates separately and taking the union of the k sets of discontinuity points.) We restate this key approximation lemma.

Lemma 5.4 *For any $x \in D$, there exist $x_n \in D_c$, $n \geq 1$, such that $\|x_n - x\| \to 0$ as $n \to \infty$.*

It is easy to define the reflection map on D_c using the instantaneous reflection map on $\mathbb{R}^k$. For any x in D_c, the set of discontinuity points is $Disc(x) = \{t_1, \ldots, t_m\}$ for some positive integer m and time points $t_0 \equiv 0 < t_1 < \cdots < t_m < T$. Clearly we should have

$$\psi(x)(t_i) \equiv y(t_i) = \psi_0(z(t_{i-1}) + x(t_i) - x(t_{i-1})) + y(t_{i-1}) \tag{5.18}$$

and

$$\phi(x)(t_i) \equiv z(t_i) = \phi_0(z(t_{i-1}) + x(t_i) - x(t_{i-1})) \tag{5.19}$$

for $0 \leq i \leq m$, where $z(t_{-1}) \equiv y(t_{-1}) \equiv x(t_{-1}) \equiv 0$. Thus we can make (5.18) and (5.19) the definition on D_c. We can then *define* $R(x)$ for $x \in D$ by

$$R(x) \equiv \lim_{n\to\infty} R(x_n) \tag{5.20}$$

for $x_n \in D_c$ with $\|x_n - x\| \to 0$.

Theorem 5.2 *For all $x \in D_c$,* (5.18) *and* (5.19) *coincide with* (3.1)–(3.3). *For all $x \in D$, the limit in* (5.20) *exists and is unique. Moreover, R is Lipschitz as a map from $(D, ||\cdot||)$ to $(D, ||\cdot||)$ and satisfies properties* (3.1)–(3.7).

Proof. By induction, it follows that (5.18) and (5.19) are equivalent to (3.1)–(3.3) on D_c: By (5.18), (5.1) and the inductive assumption,

$$\begin{aligned} y(t_i) &= \psi_0(z(t_{i-1}) + x(t_i) - x(t_{i-1})) + y(t_{i-1}) \\ &= \min\{v \in \mathbb{R}^k_+ : z(t_{i-1}) + x(t_i) - x(t_{i-1}) + (I - Q^t)v \geq 0\} + y(t_{i-1}) \\ &= \min\{v \in \mathbb{R}^k_+ : x(t_i) + (I - Q^t)y(t_{i-1}) + (I - Q^t)v \geq 0\} + y(t_{i-1}) \\ &= \min\{v \geq y(t_{i-1}) : x(t_i) + (I - Q^t)v \geq 0\} \ , \end{aligned} \tag{5.21}$$

which corresponds to (3.1)–(3.3). Corollary 5.1 implies (3.7) on D_c. Hence, for $x \in D$ given, choose $x_n \in D_c$ with $\|x_n - x\| \to 0$. Since $\|x_n - x\| \to 0$ for $x_n \in D_c$, $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. As noted above, we can deduce the Lipschitz property of R on D_c by applying Theorem 3.1. By that Lipschitz property on D_c, $\|R(x_n) - R(x_m)\| \leq K\|x_n - x_m\| \to 0$. Since $(D, \|\cdot\|)$ is a complete metric space, there exists $(y, z) \in D$ such that $\|R(x_n) - (y, z)\| \to 0$. To show uniqueness, suppose that $\|x_{jn} - x\| \to 0$ for $j = 1, 2$. Then $\|x_{1n} - x_{2n}\| \to 0$ and $\|R(x_{1n}) - R(x_{2n})\| \leq K\|x_{1n} - x_{2n}\| \to 0$, so that the limits necessarily coincide. Given that $x_n \in D_c$, so that (x_n, y_n, z_n) satisfy (3.1)–(3.7) with $\|(x_n, y_n, z_n) - (x, y, z)\| \to 0$, it follows that (x, y, z) satisfies (3.1)–(3.7) too. (If (3.7) were to be violated for (z^i, y^i) for some i, then it follows that (3.7) would necessarily be violated by (z^i_n, y^i_n) for some n, because there would exist an interval $[a, b]$ in $[0, T]$ such that $z^i(t) \geq \epsilon > 0$ for $a \leq t \leq b$ and $y^i(b) > y^i(a)$.) Alternatively, since there exists a unique solution to (3.1)–(3.7), it must coincide with the one obtained via the limit (5.20). To directly verify the Lipschitz property given the Lipschitz property on D_c, for any $x_1, x_2 \in D$, let x_{1n}, $x_{2n} \in D_c$ with $\|x_{1n} - x_1\| \to 0$ and $\|x_{2n} - x_2\| \to 0$. Then, for any $\epsilon > 0$, there is an n_0 such that

$$\begin{aligned}
\|R(x_1) - R(x_2)\| &\leq \|R(x_1) - R(x_{1n})\| + \|R(x_{1n}) - R(x_{2n})\| + \|R(x_{2n}) - R(x_2)\| \\
&\leq K(\|x_1 - x_{1n}\| + \|x_{1n} - x_{2n}\| + \|x_{2n} - x_2\|) \\
&\leq K\|x_1 - x_2\| + 2K(\|x_1 - x_{1n}\| + \|x_{2n} - x_2\|) \\
&\leq K\|x_1 - x_2\| + \epsilon
\end{aligned}$$

for all $n \geq n_0$. Since ϵ was arbitrary, the Lipschitz property is established. ∎

From the above or directly from (3.1)–(3.7), we can establish basic additivity properties of the reflection map. First, following Harrison and Reiman (1981) and Chen and Mandelbaum (1991c), let ϕ_t denote the component of the reflection map on $D([0, t], \mathbb{R}^k)$ and, for any t_1, $0 < t_1 < T$, let

$$\delta_{t_1}(x)(t) = x(t_1 + t) - x(t_1), \quad 0 \leq t \leq T - t_1 \ . \tag{5.22}$$

Lemma 5.5 *For any $x \in D([0, T], \mathbb{R}^k)$ such that $T - t_1 \in Disc(x)^c$, $0 < t_1 < T$ and $0 \leq t \leq T - t_1$,*

$$\phi_T(x)(t + t_1) = \phi_{T-t_1}(\phi_T(x)(t_1) + \delta_{t_1}(x))(t) \ . \tag{5.23}$$

From Lemma 5.5 or from Theorem 5.2 and (5.18)–(5.19), we have the following result.

Lemma 5.6 *For any $x \in D$ and t, $0 < t < T$,*

$$\psi(x)(t) \equiv y(t) = \psi_0(z(t-) + x(t) - x(t-)) + y(t-)$$

and

$$\phi(x)(t) \equiv z(t) = \phi_0(z(t-) + x(t) - x(t-)) \ .$$

We can apply Lemma 5.6 to relate the set of discontinuity points of $R(x)$ to the set of discontinuity points of x, which we denote by $Disc(x)$.

Corollary 5.3 *For any* $x \in D$,

$$Disc(R(x)) = Disc(x) \ . \tag{5.24}$$

Proof. By Lemma 5.6, we can write

$$z(t) - z(t-) = x(t) - x(t-) + (I - Q^t)(y(t) - y(t-)) \ , \tag{5.25}$$

where $y^i(t) - y^i(t-)$ is minimal, $1 \leq i \leq k$. If $x(t) - x(t-) = 0$ (where here 0 is the zero vector), then necessarily $y(t) - y(t-) = 0$, which then forces $z(t) - z(t-) = 0$. On the other hand, if $x(t) - x(t-) \neq 0$, then we cannot have both $z(t) - z(t-) = 0$ and $y(t) - y(t-) = 0$, so we must have $t \in Disc(R(x))$. ∎

We obtain our strongest results for the case in which no coordinate of x has a negative jump, i.e., when $x \in D_+$.

Corollary 5.4 *For any* $x \in D_+$, *we have* $\psi(x) \in C$, $\phi(x) \in D_+$ *and*

$$\phi(x)(t) - \phi(x)(t-) = x(t) - x(t-) \ . \tag{5.26}$$

Finally, we can apply Lemma 5.6 and Corollary 5.2 to determine how reflections of parametric representations perform. This is the key new result in this section.

Lemma 5.7 *Suppose that* $x \in D$, $t \in Disc(x)$ *and* $0 \leq \alpha \leq 1$.

(a) If $x(t) \geq x(t-)$, *then*

$$\hat{\psi}(x,t,\alpha) \equiv \psi_0(z(t-) + \alpha[x(t) - x(t-)]) + y(t-) = \hat{\psi}(x,t,0) = y(t-) \tag{5.27}$$

and

$$\hat{\phi}(x,t,\alpha) \equiv \phi_0(z(t-) + \alpha[x(t) - x(t-)]) = \hat{\phi}(x,t,0) + \alpha[x(t) - x(t-)] \tag{5.28}$$

for $0 \leq \alpha \leq 1$.

(b) If $x(t) \leq x(t-)$ *and* $0 \leq \alpha_1 < \alpha_2 \leq 1$, *then*

$$\hat{\psi}(x,t,\alpha_1) \leq \hat{\psi}(x,t,\alpha_2) \tag{5.29}$$

and

$$\hat{\phi}(x,t,\alpha_1) \geq \hat{\phi}(x,t,\alpha_2) \tag{5.30}$$

for $\hat{\psi}$ in (5.27) and $\hat{\phi}$ in (5.28).

6. Reflections of Parametric Representations

In order to establish continuity and stronger Lipschitz properties of the reflection map R in (3.1)–(3.3) with the M_1 topologies, we would like to have $(R(u), r)$ be a parametric representation of $R(x)$ when (u, r) is a parametric representation of x. We now obtain positive results in this direction. (Proofs appears at the end of the section.)

Theorem 6.1 *Suppose that $x \in D$, $(u,r) \in \Pi_s(x)$ and $r^{-1}(t) = [s_-(t), s_+(t)]$.*

(a) If $t \in Disc(x)^c$, then

$$R(u)(s) = R(x)(t) \quad \textit{for} \quad s_-(t) \leq s \leq s_+(t)\ . \tag{6.1}$$

(b) If $t \in Disc(x)$, then

$$R(u)(s_-(t)) = R(x)(t-) \quad \textit{and} \quad R(u)(s_+(t)) = R(x)(t)\ . \tag{6.2}$$

(c) If $t \in Disc(x)$ and $x(t) \geq x(t-)$, then

$$\phi(u)(s) = \phi(x)(t-) + \left(\frac{u^j(s) - u^j(s_-(t))}{u^j(s_+(t)) - u^j(s_-(t))}\right)[x(t) - x(t-)] \tag{6.3}$$

for any j, $1 \leq j \leq k$, and

$$\psi(u)(s) = \psi(x)(t-) = \psi(x)(t) \quad \textit{for} \quad s_-(t) \leq s \leq s_+(t)\ , \tag{6.4}$$

so that

$$R(u)(s) \in [R(x)(t-), R(x)(t)] \quad \textit{for} \quad s_-(t) \leq s \leq s_+(t)\ . \tag{6.5}$$

(d) If $t \in Disc(x)$ and $x(t) \leq x(t-)$, then $\phi^i(u)$ and $\psi^i(u)$ are monotone in $[s_-(t), s_+(t)]$ for each i, so that

$$R(u)(s) \in [[R(x)(t-), R(x)(t)]] \quad \textit{for} \quad s_-(t) \leq s \leq s_+(t)\ . \tag{6.6}$$

We can draw the desired conclusion that $(R(u), r)$ is a parametric representation of $R(x)$ if we can apply parts (c) and (d) of Theorem 6.1 to all jumps. Recall that D_+ (D_s) is the subset of D for which condition (c) (condition (c) or (d)) holds at all discontinuity points of x.

Theorem 6.2 *Suppose that $x \in D$ and $(u,r) \in \Pi_s(x)$.*

(a) If $x \in D_+$, then $(R(u),r) \in \Pi_s(R(x))$.

(b) If $x \in D_s$, then $(R(u),r) \in \Pi_w(R(x))$.

We also have an analogs of Theorems 6.1 and 6.2 for the case $x \in D_s$ and $(u,r) \in \Pi_w(x)$.

Theorem 6.3 *If $x \in D_s$ and $(u,r) \in \Pi_w(x)$, then $(R(u),r) \in \Pi_w(R(x))$.*

As a basis for proving Theorem 6.1, we exploit piecewise-constant approximations.

Lemma 6.1 *For any $x \in D_c$, $(u,r) \in \Pi_s(x)$ and $r^{-1}(t) = [s_-(t), s_+(t)]$,*

$$R(u)(s_-(t)) = R(x)(t-) \quad \textit{and} \quad R(u)(s_+(t)) = R(x)(t) \ . \tag{6.7}$$

In order to prove Lemma 6.1, we establish several other lemmas. First, the following property of the reflection map applied to a single jump at time t is an easy consequence of the definition of the reflection map. We consider the reflection map applied to the jump in two parts. Given the linear relationship in (3.1), it suffices to focus on only one of ψ or ϕ.

Lemma 6.2 *For any $b_1, b_2 \in \mathbb{R}^k$, $0 < \beta < 1$ and $0 < t \le T$,*

$$\phi(b_1 + b_2 I_{[t,T]})(u) = \phi(\phi(b_1 + \beta b_2 I_{[t,T]})(t) + (1-\beta) b_2 I_{[t,T]})(u) \quad \textit{for} \quad t \le u \le T \ .$$

Lemma 6.3 *For any $b_1, b_2 \in \mathbb{R}^k$ and right-continuous nondecreasing nonnegative real-valued function α on $[0,T]$ with $\alpha(0) = 0$,*

$$\phi(b_1 + \alpha b_2)(t) = \phi(b_1 + \alpha(t) b_2 I_{[0,T]})(t), \quad 0 \le t \le T \ . \tag{6.8}$$

Proof. Represent α as the uniform limit of nondecreasing nonnegative functions α_n in D_c. Then $\|\phi(b_1 + \alpha_n b_2) - \phi(b_1 + \alpha b_2)\| \to 0$ as $n \to \infty$ by the known continuity of ϕ in the uniform metric. Hence it suffices to assume that $\alpha \in D_c$. We then establish (6.8) by recursively considering the successive discontinuity points of α, using Lemmas 6.2 and 5.5. ∎

Proof of Lemma 6.1. Any $x \in D_c$ can be represented as

$$x = \sum_{j=0}^{m} b_j I_{[t_j,T]} \tag{6.9}$$

for $0 = t_0 < t_1 < \cdots < t_m \le T$ and $b_j \in \mathbb{R}^k$ for $0 \le j \le m$. Thus t_j is the j^{th} discontinuity point of x. Let $[s_-(t_j), s_+(t_j)] = r^{-1}(t_j)$ for each j. Since $(u, r) \in \Pi_s(x)$ instead of just $\Pi_w(x)$, u can be expressed as

$$u = \sum_{j=0}^{m} \alpha_j b_j \ , \tag{6.10}$$

where $\alpha_0(s) = 1$ for all s and, for $j \ge 1$, $\alpha_j : [0,1] \to [0,1]$ is continuous and nondecreasing with $\alpha_j(s) = 0$, $s \le s_-(t_j)$ and $\alpha_j(s) = 1$, $s \ge s_+(t_j)$. We can now consider successive intervals $[s_-(t_j), s_+(t_j)]$ recursively exploiting Lemma 6.3. First, for any s with $0 \le s \le s_-(t_1)$.

$$\phi(u)(s) = \phi(b_0 I_{[0,1]})(s) = \phi(x)(0) = \phi_0(x(0)) \ . \tag{6.11}$$

Now assume that (6.7) holds for all $j \le m-1$ and consider $s \in [s_-(t_m), s_+(t_m)]$. By the induction hypothesis and Lemmas 5.5 and 6.3,

$$\phi(u)(s) = \phi(\phi(x)(t_{m-1}) + \alpha_m b_m I_{[s_-(t_m),1]})(s) = \phi(\phi(x)(t_{m-1}) + \alpha_m(s) b_m I_{[s_-(t_m),1]})(s) \ , \tag{6.12}$$

so that (6.7) holds for t_m. ■

We now show that it is essential in Lemma 6.1 to have $(u, r) \in \Pi_s(x)$ instead of just $(u, r) \in \Pi_w(x)$. We also show that we cannot improve upon Lemma 6.1 to conclude that $(R(u), r) \in \Pi_w(R(x))$ when $(u, r) \in \Pi_s(x)$.

Example 6.1 To demonstrate the points above, let $x \in D_c$ and R be defined by

$$x^1 = I_{[0,1)} - 3I_{[1,2]}, \quad x^2 = I_{[0,1)} + 2I_{[1,2]} \ ,$$
$$Q = \begin{pmatrix} 0 & 1 \\ 0 & .9 \end{pmatrix} \ , \quad \text{so that} \quad I - Q^t = \begin{pmatrix} 1 & 0 \\ -1 & .1 \end{pmatrix} \ . \tag{6.13}$$

Then $z^1 = z^2 = I_{[0,1)}$, $y^1 = 3I_{[1,2]}$ and $y^2 = 10I_{[1,2]}$. To see that the conclusion of Lemma 6.1 fails when we only have $(u, r) \in \Pi_w(x)$, let a parametric representation (u, r) in $\Pi_w(x)$ be defined by

$$\begin{array}{l} r(0) = 0, \quad r(1/3) = r(2/3) = 1, \quad r(1) = 2 \\ u^1(0) = u^1(1/3) = 1, \quad u^1(1/2) = u^1(1) = -3 \\ u^2(0) = u^2(1/2) = 1, \quad u^2(2/3) = u^2(1) = 2 \end{array} \tag{6.14}$$

with r, u^1 and u^2 defined by linear interpolation elsewhere. Notice that $[s_-(1), s_+(1)] = [1/3, 2/3]$, $\phi^2(u)(1/2) = 0$ and $\phi^2(u)(2/3) = 1 > 0 = z^2(1)$. Moreover, $\phi^2(u)(s) = 1$ on $[2/3, 1]$.

Next, to see that we need not have $(R(u), r) \in \Pi_w(R(x))$ when $(u, r) \in \Pi_s(x)$, let r be defined in (6.14) and let the parametric representation (u, r) in $\Pi_s(x)$ be defined by

$$u^1(0) = u^1(1/3) = 1, \ u^1(2/3) = u^1(1) = -3$$
$$u^2(0) = u^2(1/3) = 1, \ u^2(2/3) = u^2(1) = 2 \tag{6.15}$$

with r, u^1, u^2 defined at other points by linear interpolation. Clearly $(u,r) \in \Pi_s(x)$. Note that $u^i(s) \geq 0$ for all $s \leq 5/12$. Then $r(5/12) = 1$, $u^1(5/12) = 0$ and $u^2(5/12) = 5/4$. Clearly $\phi(u)(5/12) = u(5/12) = (0, 5/4)$, which is not in $[[(0,0),(1,1)]]$, the weak range of $z = \phi(x)$. Further analysis shows that $\phi^1(u)(s) = 0$ for $s \geq 5/12$, while $\phi^2(u) = u^2$ on $[0, 1/3]$, $\phi^2(u)(5/12) = 5/4$, $\phi^2(u)(5/9) = \phi^2(u)(1) = 0$, with $\phi^2(u)$ defined elsewhere by linear interpolation. Similarly, ψ has slope $(12, 0)$ over $(5/12, 5/9)$ and slope $(12, 90)$ over $(5/9, 2/3)$, so that $\psi(u)(5/12) = (0,0)$, $\psi(u)(5/9) = (5/3, 0)$, $\psi(u)(2/3) = \psi(u)(1) = (3, 10)$ and ψ is defined by linear interpolation elsewhere. ∎

Proof of Theorem 6.1. (a) Since $t \in Disc(x)^c$, $u(s) = x(t)$ for $s_-(t) \leq s \leq s_+(t)$. Given $x \in D$ with $t \in Disc(x)^c$, it is possible to choose $x_n \in D_c$ such that $t \in Disc(x_n)^c$ for all n and $\|x_n - x\| \to 0$, by a slight strengthening of Lemma 5.4. By characterization (i) of M_1 convergence in Theorem 5.6.1 of Whitt (2002), given $(u,r) \in \Pi_s(x)$, we can find $(u_n, r_n) \in \Pi_s(x_n)$ such that

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \quad \text{as} \quad n \to \infty .$$

Since R is continuous in the uniform topology, $\|R(u_n) - R(u)\| \to 0$ and $\|R(x_n) - R(x)\| \to 0$ as $n \to \infty$. Let s_n be such that $r_n(s_n) = t$. Since $x_n \in D_c$ and $t \in Disc(x_n)^c$, $R(u_n)(s_n) = R(x_n)(t)$ by Lemma 6.1. Since $0 \leq s_n \leq 1$, $\{s_n\}$ has a convergent subsequence $\{s_{n_k}\}$. Let s' be the limit of that convergent subsequence. Since $r_{n_k}(s_{n_k}) = t$ for all n_k, we necessarily have $s' \in [s_-(t), s_+(t)]$. Since $\|R(u_n) - R(u)\| \to 0$, $R(x_{n_k})(t) = R(u_{n_k})(s_{n_k}) \to R(u)(s')$. Since we have already seen that $R(x_n)(t) \to R(x)(t)$, we must have $R(u)(s') = R(x)(t)$. Since $R(u)$ is constant on $[s_-(t), s_+(t)]$, we must have $R(u)(s) = R(x)(t)$ for all s with $s_-(t) \leq s \leq s_+(t)$.

(b) Since R maps D into D and C into C, $R(x)$ is right-continuous with left limits, while $R(u)$ is continuous. Given $t \in Disc(x)$, we can find $t_n \in Disc(x)^c$ with $t_n \uparrow t$. We can apply part (a) to obtain $R(u)(s_+(t_n)) = R(x)(t_n) \to R(x)(t-)$, but $s_+(t_n) \uparrow s_-(t)$, so that $R(u)(s_+(t_n)) \to R(u)(s_-(t))$. Hence, we have established the first claim: $R(u)(s_-(t)) = R(x)(t-)$. Similarly, we can find $t_n \in Disc(x)^c$ with $t_n \downarrow t$. Then we can apply part (a) again to obtain $R(u)(s_-(t_n)) = R(x)(t_n) \to R(x)(t)$. Since $s_-(t_n) \downarrow s_+(t)$, $R(u)(s_-(t_n)) \downarrow R(u)s_+(t))$. Hence $R(x)(t) = R(u)(s_+(t))$ as claimed.

(c) We can apply Lemma 5.7(a). Since the increment $x(t) - x(t-)$ is nonnegative in each component,

$$z(t) = z(t-) + x(t) - x(t-)$$

and $y(t) = y(t-)$. Similarly,

$$\phi(u)(s) = \phi(u)(s_-(t)) + u(s) - u(s_-(t))$$

and $\psi(u)(s) = \psi(u)(s_-(t))$ for $s_-(t) \le s \le s_+(t)$.

(d) We apply Lemma 5.7(b). Each coordinate $\phi^i(u)$ and $\psi^i(u)$ is monotone in s over $[s_-(t), s_+(t)]$, so that (6.6) holds.

Proof of Theorem 6.2. (a) We combine parts (a)–(c) of Theorem 6.1 to get $(R(u), r)(s) \in \Gamma_{R(x)}$ for all s. Since R maps C into C, $(R(u), r)$ is continuous. Also r is nondecreasing with $r(0) = 0$ and $r(1) = T$ because $(u, r) \in \Pi_s(x)$. Finally, $(R(u), r)$ maps $[0, 1]$ onto $\Gamma_{R(x)}$ and $(R(u), v)$ is nondecreasing with respect to the order on $\Gamma_{R(x)}$ because the increments of $R(u)$ coincide with the increments of u over each discontinuity in x because $x \in D_+$, and (u, r) has these properties.

(b) We incorporate part (d) of Theorem 6.1 to get $R(u)$ monotone over $[s_-(t), s_+(t)] = r^{-1}(t)$ for each $t \in Disc(x) = Disc(R(x))$. This allows us to conclude that $(R(u), r) \in \Pi_w(R(x))$. ■

We now turn to the proof of Theorem 6.3. For the proof, we find it convenient to use a different class of approximating functions. Let D_l be the subset of all functions in D that (i) have only finitely many jumps and (ii) are continuous and piecewise linear in between jumps with only finitely many changes of slope. Let $D_{s,l} = D_s \cap D_l$.

Analogous to Lemma 5.4, we have the following result.

Lemma 6.4 *For any $x \in D_s$, there exist $x_n \in D_{s,l}$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.*

Proof. For $x \in D_s$ and $\epsilon > 0$ given, apply Lemma 5.4 to find $x_1 \in D_c$ (with only finitely many discontinuities) such that $\|x - x_1\| < \epsilon/4$. The function x_1 can have discontinuities with simultaneous jumps of opposite sign, but the magnitude of the jumps in one of the two directions must be at most $\epsilon/2$. Form the desired function, say x_2, from x_1. Suppose that $\{t_1, \ldots, t_k\} = Disc(x_1)$. Suppose that x_1 has one or more negative jump at t_j, none of which has magnitude exceeding $\epsilon/2$. If x has a negative jump at t_j in coordinate i for some i, then replace x_1^i over $[t_{j-1}, t_j)$ by the linear function connecting $x_1^i(t_{j-1})$ and $x_1^i(t_j)$. Similarly, if x_1 has one or more positive jumps at some time t_j with all magnitudes less than $\epsilon/2$, then proceed as above. It is easy to see that $Disc(x_2) \subseteq Disc(x_1)$, $x_2 \in D_{s,l}$ and $\|x - x_2\| < \epsilon$. ■

We now show that limits of parametric representations are parametric representations when $\|x_n - x\| \to 0$.

Lemma 6.5 *If (i) $\|x_n - x\| \to 0$ as $n \to \infty$, (ii) $(u_n, r_n) \in \Pi_z(x_n)$ for each n, where $z = s$ or w, and (iii) $\|u_n - u\| \vee \|r_n - r\| \to 0$ as $n \to \infty$ where u and r are functions mapping $[0,1]$ into $\mathbb{R}^k$ and $\mathbb{R}^1$, respectively, then $(u, r) \in \Pi_z(x)$ for the same z.*

Proof. Since (u, r) is the uniform limit of the continuous functions (u_n, r_n), (u, r) is itself continuous. Since r is the limit of the nondecreasing functions r_n, r is itself nondecreasing. Since $r_n(0) = 0$ and $r_n(1) = T$ for all n, $r(0) = 0$ and $r(1) = T$. Since r is also nondecreasing and continuous, r maps $[0,1]$ onto $[0,T]$. Pick any s with $0 < s < 1$. Then $r(s) = t$ for some t, $0 \leq t \leq T$, and $r_n(s) = t_n \to t$ as $n \to \infty$. Suppose that $(u_n, r_n) \in \Pi_s(x_n)$ for all n. That means that

$$u_n(s) = \alpha_n(s) x_n(t_n) + (1 - \alpha_n(s)) x_n(t_n-)$$

for all n. Since $0 \leq \alpha_n(s) \leq 1$, there exists a convergent subsequence $\{\alpha_{n_k}(s)\}$ such that $\alpha_{n_k}(s) \to \alpha(s)$ as $n_k \to \infty$. At least one of the following three cases must prevail: (i) $t_{n_k} > t$ for infinitely many n_k, (ii) $t_{n_k} = t$ for infinitely many n_k and (iii) $t_{n_k} < t$ for infinitely many n_k. In case (i), we can choose a further subsequence $\{n_{k_j}\}$ so that $u_{n_{k_j}}(s) \to x(t)$; in case (ii), we can choose a further subsequence so that $u_{n_{k_j}}(s) \to \alpha(s)x(t) + [1 - \alpha(s)]x(t-)$; in case (iii) we can choose a further subsequence so that $u_{n_{k_j}}(s) \to x(t-)$. Since $u_n(s) \to u(s)$, the limit of the subsequence must be $u(s)$. Hence, $(u(s), r(s)) \in \Gamma_x$ for each s. Since (u, r) is continuous with $r(0) = 0$ and $r(1) = T$, (u, r) maps $[0,1]$ onto Γ_x. Since (u_n, r_n) is monotone as a function from $[0,1]$ to $(\Gamma_{x_n}, \leq)$ and $\|u_n - u\| \vee \|r_n - r\| \to 0$, (u, r) is monotone from $[0,1]$ to $(\Gamma_x, \leq)$. Hence, $(u, r) \in \Pi_s(x)$. Finally, suppose that $(u_n, r_n) \in \Pi_w(x_n)$ for all n. By the result above applied to the individual coordinates, $(u^i(s), r(s)) \in \Gamma_{x^i}$ and thus $(u^i, r) \in \Pi_s(x^i)$ for each i, which implies that $(u, r) \in \Pi_w(x)$. ■

Proof of Theorem 6.3. For $x \in D_s$, apply Lemma 6.4 to find $x_n \in D_{s,l}$ such that $\|x_n - x\| \to 0$. Suppose that $(u, r) \in \Pi_w(x)$. Then it is possible to find u_n such that $(u_n, r) \in \Pi_w(x_n)$ and $\|u_n - u\| \to 0$: To do so, let $u_n(s_-(t)) = x_n(t-)$ and $u_n(s_+(t)) = x_n(t)$, where $[s_-(t), s_+(t)] = r^{-1}(t)$ for each $t \in Disc(x_n)$. If $t \in Disc(x_n)^c$, let $u_n(s) = u_n(s_+(t))$ for $s_-(t) \leq s \leq s_+(t)$; if $t \in Disc(x_n)$, define u_n so that $\|u_n - u\| \to 0$. Given that $(u_n, r) \in \Pi_w(x_n)$, we can apply mathematical induction over the finitely many time points such that x_n has a jump or a change of slope to show that $(R(u_n), r) \in \Pi_w(R(x_n))$ for each n. We use Lemma 5.7 critically at this point. Finally, we apply Lemma 6.5 to deduce that $(R(u), r) \in \Pi_w(R(x))$. For that, we use the fact that $\|R(x_n) - R(x)\| \to 0$ and $\|R(u_n) - R(u)\| \to 0$.

7. Proofs of the Main Theorems

In this section we apply the results in Sections 5 and 6 to prove Theorems 3.2–3.5.

Proof of Theorem 3.2. (a) We first prove (3.17). Since $x_n \to x$ in (D, SM_1), we can find parametric representations $(u, r) \in \Pi_s(x)$ and $(u_n, r_n) \in \Pi_s(x_n)$ for $n \geq 1$ such that

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \ .$$

By Theorem 6.1(a), $R(u)(s) = R(x)(t)$ for any $s \in [s_-(t), s_+(t)] \equiv r^{-1}(t)$, since $t \in Disc(x)^c$. Moreover, by Corollary 5.3, $t \in Disc(R(x))^c$. For any sequence $\{t_n : n \geq 1\}$ with $t_n \to t$, we can find another sequence $\{t'_n : n \geq 1\}$ such that $t'_n \to t$, $t'_n \in Disc(x_n)^c$ and $\|R(x_n)(t'_n) - R(x_n)(t_n)\| \to 0$ as $n \to \infty$. (Here we exploit the fact that $R(x_n) \in D$ for each n.) Consequently, $R(x_n)(t_n) \to R(x)(t)$ if and only if $R(x_n)(t'_n) \to R(x)(t)$. By Theorem 6.1(a) again, $R(u_n)(s_n) = R(x)(t'_n)$ for any $s_n \in [s_-(t'_n), s_+(t'_n)] = r_n^{-1}(t'_n)$. Since $0 \leq s_n \leq 1$ for all n, any such sequence $\{s_n : n \geq 1\}$ has a convergent subsequence $\{s_{n_k} : k \geq 1\}$. Suppose that $s_{n_k} \to s'$ as $n_k \to \infty$. Since $t'_n \to t$ as $n \to \infty$ and $t'_{n_k} = r_{n_k}(s_{n_k}) \to r(s')$ as $n_k \to \infty$, we must have $s' \in [s_-(t), s_+(t)]$. Then, since $\|R(u_n) - R(u)\| \to 0$,

$$R(x_{n_k})(t'_{n_k}) = R(u_{n_k})(s_{n_k}) \to R(u)(s') = R(x)(t) \ . \tag{7.1}$$

Since every subsequence of $\{R(x_n)(t'_n) : n \geq 1\}$ must have a convergent subsequence with the same limit, we must have $R(x_n)(t'_n) \to R(x)(t)$ as $n \to \infty$, which we have shown implies that $R(x_n)(t_n) \to R(x)(t)$ as $n \to \infty$, as claimed in (3.17). Next we establish (3.18). For any $x \in D$, $\|x\| \equiv \sup_{0 \leq t \leq T} \|x(t)\| < \infty$. Since $d_s(x_n, x) \to 0$, $\|x_n\| \to \|x\|$ as $n \to \infty$. Hence, it suffices to show that there is a constant K such that

$$\|R(x)\| \leq K\|x\| \quad \text{for all} \quad x \in D \ , \tag{7.2}$$

but that follows from Theorem 3.1. We apply the bounded convergence theorem with (3.17) and (3.18) to establish (3.19). We now turn to (3.20). Since $\psi(x_n)$ and $\psi(x)$ are nondecreasing in each coordinate, the pointwise convergence established in (3.17) actually implies WM_1 convergence in (3.20); see the Corollary to Theorem 5.6.1 in Whitt (2002).

(b) First, we use the assumed convergence $x_n \to x$ in (D, SM_1) to pick $(u, r) \in \Pi_s(x)$ and $(u_n, r_n) \in \Pi_s(x_n)$, $n \geq 1$, with

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \ . \tag{7.3}$$

Since R is continuous on (D, U), we also have $\|R(u_n) - R(u)\| \to 0$. By part (a), we know that there is local uniform convergence of $R(x_n)$ to $R(x)$ at each continuity point of $R(x)$. Thus, by Theorem 5.6.1(v) of Whitt (2002), to establish $R(x_n) \to R(x)$ in (D, WM_1), it suffices to show that

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \; w_s(R^i(x_n), t, \delta) = 0 \tag{7.4}$$

for each i, $1 \leq i \leq 2k$, and $t \in Disc(R(x))$, where

$$w_s(x, t, \delta) = \sup\{\|x(t_2) - [x(t_1), x(t_3)]\| : (t_1, t_2, t_3) \in A(t, \delta)\} \tag{7.5}$$

for

$$A(t, \delta) \equiv \{(t_1, t_2, t_3) : (t - \delta) \vee 0 \leq t_1 < t_2 < t_3 \leq (t + \delta) \wedge T\} \; . \tag{7.6}$$

(Since we are considering the i^{th} coordinate function $R^i(x_n)$, the function x in (7.5) is real-valued here.) Suppose that (7.4) fails for some i and t. Then there exist $\epsilon > 0$ and subsequences $\{\delta_k\}$ and $\{n_k\}$ such that $\delta_k \downarrow 0$, $n_k \to \infty$ and

$$w_s(R^i(x_{n_k}), t, \delta_k) > \epsilon \quad \text{for all} \quad \delta_k \quad \text{and} \quad n_k \; . \tag{7.7}$$

That is, there exist time points t_{1,n_k}, t_{2,n_k} and t_{3,n_k} with

$$(t - \delta_k) \vee 0 \leq t_{1,n_k} < t_{2,n_k} < t_{3,n_k} \leq (t + \delta_k) \wedge T \tag{7.8}$$

and

$$\|R^i(x_{n_k})(t_{2,n_k}) - [R^i(x_{n_k}(t_{1,n_k}), R^i(x_{n_k}(t_{3,n_k})]\| > \epsilon \; . \tag{7.9}$$

Since the values $R^i(x_{n_k})(t)$ are contained in the values $R^i(u_{n_k})(s)$ where $(u_{n_k}, r_{n_k}) \in \Pi_s(x_{n_k})$, we can deduce that there are points s_{j,n_k} for $j = 1, 2, 3$ such that $0 \leq s_{1,n_k} < s_{2,n_k} < s_{3,n_k} \leq 1$, $r_{n_k}(s_{j,n_k}) = t_{j,n_k}$ for $j = 1, 2, 3$ and all n_k, and

$$\|R^i(u_{n_k})(s_{2,n_k}) - [R^i(u_{n_k})(s_{1,n_k}), R^i(u_{n_k})(s_{3,n_k})]\| > \epsilon \; . \tag{7.10}$$

By (7.8) and (7.10), there then exists a further subsequence $\{n'_k\}$ such that $t_{j,n'_k} \to t$ and $s_{j,n'_k} \to s_j$ as $n'_k \to \infty$ for $j = 1, 2, 3$, where $0 \leq s_1 \leq s_2 \leq s_3 \leq 1$, $r_{n'_k}(s_{j,n'_k}) \to r(s_j) = t$ and

$$\|R^i(u)(s_2) - [R^i(u)(s_1), R^i(u)(s_3)]\| \geq \epsilon > 0 \; . \tag{7.11}$$

However, by Theorem 6.2, $(R(u), r) \in \Pi_w(R(x))$ since $x \in D_s$, so that $(R^i(u), r) \in \Pi_s(R^i(x))$. Hence $(R^i(u), r) \in \Pi_s(R^i(x))$. Since $R^i(u)$ is monotone on $[s_-(t), s_+(t)]$, (7.11) cannot occur. Hence (7.4) must in fact hold and $R^i(x_n) \to R^i(x)$ in (D, M_1). Since that is true for all i, we must have $R(x_n) \to R(x)$ in (D, WM_1).

Proof of Theorem 3.4. The proof is the same as for Theorem 4.2: Given that $x \in D_+$, apply Theorem 6.2(a) to get $(R(u), r) \in \Pi_s(R(x))$ when $(u, r) \in \Pi_s(x)$. Given that $x \in D_s$, apply Theorem 6.3 to get $(R(u), r) \in \Pi_w(R(x))$ when $(u, r) \in \Pi_w(x)$.

Proof of Theorem 3.5. Suppose that $x_n \to x$ in (D, SM_1). By Theorem 3.2(a), we have $\psi(x_n) \to \psi(x)$ in (D, WM_1). Since $x \in D_+$, $\psi(x) \in C$, by Corollary 5.4. Hence the WM_1 convergence is equivalent to uniform convergence; i.e.,

$$\psi(x_n) \to \psi(x) \quad \text{in} \quad D([0,T], \mathbb{R}^k, U) \ .$$

We can then apply addition with (3.1) to get

$$R(x_n) \to R(x) \quad \text{in} \quad D([0,T], \mathbb{R}^{2k}, SM_1) \ . \quad \blacksquare$$

We now work toward the proof of Theorem 3.3. We base our proof on several elementary lemmas, which we state without proof. The first two lemmas appear in Section 5.5 of Whitt (2002). They are easy consequences of the local uniform convergence at continuity points, implied by convergence in any of the Skorohod (1956) non-uniform topologies on D.

Lemma 7.1 *If $x_n \to x$ in (D^1, M_1), $t_n \to t$ in $(0, T)$ and, for some $c > 0$, $x_n(t_n) - x_n(t_n-) \geq c$ for all n, then $x(t) - x(t-) \geq c$.*

For $x \in D$ and $t \in Disc(x)$, let $\gamma(x, t)$ be the largest magnitude (absolute value) of the jumps in x at time t of opposite sign to the sign of the largest jump in x at time t. Let $\gamma(x)$ be the maximum of $\gamma(x, t)$ over all $t \in Disc(x)$. We apply Lemma 7.1 to establish the next result.

Lemma 7.2 *If $x_n \to x$ in (D, WM_1), then*

$$\overline{\lim_{n \to \infty}} \ \gamma(x_n) \leq \gamma(x) \ .$$

We only use the following consequence of Lemma 7.2.

Lemma 7.3 *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then $\gamma(x_n) \to 0$.*

We also use a generalization of Lemma 6.4, which is established in the same way.

Lemma 7.4 *For any $x \in D$, there exist $x_n \in D_{s,l}$ such that $\|x_n - x\| \to \gamma(x)$ as $n \to \infty$.*

We combine Lemmas 7.2 and 7.4 to obtain the tool we need.

Lemma 7.5 *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then there exists $x'_n \in D_{s,l}$ for $n \geq 1$ such that $\|x'_n - x_n\| \to 0$.*

Proof of Theorem 3.3. Given $x_n \to x$ in (D, WM_1), apply Lemma 7.5 to find $x_n' \in D_{s,l}$ for $n \geq 1$ such that $\|x_n' - x_n\| \to 0$ as $n \to \infty$. Then, by the triangle inequality and Theorems 3.1 and 3.4,

$$\begin{aligned} d_p(R(x_n), R(x)) &\leq d_p(R(x_n), R(x_n')) + d_p(R(x_n'), R(x)) \\ &\leq \|R(x_n) - R(x_n')\| + d_w(R(x_n'), R(x)) \\ &\leq K\|x_n - x_n'\| + K d_w(x_n', x). \end{aligned}$$

Since

$$\begin{aligned} d_p(x_n', x) &\leq d_p(x_n', x_n) + d_p(x_n, x) \\ &\leq \|x_n' - x_n\| + d_p(x_n, x) \\ &\to 0 \, , \end{aligned}$$

$d_w(x_n', x) \to 0$. Hence, $d_p(R(x_n), R(x)) \to 0$ as claimed. ∎

8. The Function Space $D([0, \infty), \mathbb{R}^k)$

It is often convenient to consider the function space $D([0, \infty), \mathbb{R}^k)$ with domain $[0, \infty)$ instead of $[0, T]$. For a sequence or net $\{x_n\}$ in $D([0, \infty), \mathbb{R}^k)$, we define convergence $x_n \to x$ as $n \to \infty$ in $D([0, \infty), \mathbb{R}^k)$ with some topology to be convergence $x_n \to x$ as $n \to \infty$ in $D([0, t], \mathbb{R}^k)$ with that same topology for the restrictions of x_n and x to $[0, t]$ for $t = t_k$ for each t_k in some sequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$, where $\{t_k\}$ can depend on x. It suffices to let t_k be continuity points of the limit function x; see Lindvall (1973), Whitt (1980) and Jacod and Shiryaev (1987).

Let $r_t : D([0, \infty), \mathbb{R}^k) \to D([0, t], \mathbb{R}^k)$ be the *restriction map* with $r_t(x)(s) = x(s)$, $0 \leq s \leq t$. Suppose that $f : D([0, \infty), \mathbb{R}^k) \to D([0, \infty), \mathbb{R}^k)$ and $f_t : D([0, t], \mathbb{R}^k) \to D([0, t], \mathbb{R}^k)$ for $t > 0$ are functions with

$$f_t(r_t(x)) = r_t(f(x)) \tag{8.1}$$

for all $x \in D([0, \infty), \mathbb{R}^k)$ and all $t > 0$. We then call the functions f_t *restrictions of the function* f. It is easy to see that the reflection maps on $D([0, t], \mathbb{R}^k)$ are restrictions of the reflection map on $D([0, \infty), \mathbb{R}^k)$. Hence we have the following result.

Theorem 8.1 *The convergence-preservation results in Theorems 3.2, 3.3 and 3.5 and Corollary 3.1 extend to* $D([0, \infty), \mathbb{R}^k)$.

Proof. Suppose that $x_n \to x$ in $D([0,\infty),\mathbb{R}^k)$ with the appropriate topology and that $\{t_j : j \ge 1\}$ is a sequence of positive numbers with $t_j \in Disc(x)^c$ and $t_j \to \infty$ as $j \to \infty$. Then, $r_{t_j}(x_n) \to r_{t_j}(x)$ in $D([0,\infty),\mathbb{R}^k)$ with the same topology as $n \to \infty$ for each j and, under the specified assumptions,

$$r_{t_j}(R(x_n)) = R_{t_j}(r_{t_j}(x_n)) \to R_{t_j}(r_{t_j}(x)) = r_{t_j}(R(x)) \tag{8.2}$$

in $D([0,t_j],\mathbb{R}^{2k})$ with the specified topology as $n \to \infty$ for each j, which implies that

$$R(x_n) \to R(x) \quad \text{in} \quad D([0,\infty),\mathbb{R}^{2k}) \tag{8.3}$$

with the same topology as in (8.2). ■

We now consider the extension of the Lipschitz properties to $D([0,\infty),\mathbb{R}^k)$. For this purpose, suppose that μ_t is one of the M_1 metrics on $D([0,t],\mathbb{R}^k)$ for $t > 0$, either d_s or d_p. As in Section 2 of Whitt (1980), an associated metric μ can be defined on $D([0,\infty),\mathbb{R}^k)$ by

$$\mu(x_1,x_2) = \int_0^\infty e^{-t}[\mu_t(r_t(x_1),r_t(x_2)) \wedge 1]dt. \tag{8.4}$$

The following result implies that the integral in (8.4) is well defined.

Theorem 8.2 *Let μ_t be one of the M_1 metrics on $D([0,t],\mathbb{R}^k)$. For all $x_1, x_2 \in D([0,\infty),\mathbb{R}^k)$, $\mu_t(x_1,x_2)$ as a function of t is right-continuous with left limits. Moreover, $\mu_t(x_1,x_2)$ is continuous at t whenever x_1 and x_2 are both continuous at t.*

We prove Theorem 8.2 by applying the following two lemmas. Let $D_c \equiv D_c([0,\infty),\mathbb{R}^k)$ be the subset of piecewise-constant functions in D with only finitely many discontinuities in any finite interval.

Lemma 8.1 *Suppose that $x_1, x_2 \in D_c([0,\infty),\mathbb{R}^k)$.*

(a) For any $t > 0$, there exists $\delta > 0$ such that

$$\mu_{t_1}(x_1,x_2) = \mu_{t_2}(x_1,x_2) \text{ for } t-\delta < t_1 < t_2 < t$$

and for $t \le t_1 < t_2 < t+\delta$.

(b) If, in addition, x_1 and x_2 are continuous at t, then there exists $\delta > 0$ such that

$$\mu_{t_1}(x_1,x_2) = \mu_{t_2}(x_1,x_2) \text{ for } t-\delta < t_1 < t_2 < t+\delta\ .$$

Proof. Since the arguments are similar for the different cases, we only do the first part of (a). The condition implies that there is an interval $(t-\delta, t)$ on which both x_1 and x_2 are constant. Let t_1 and t_2 be such that $t-\delta < t_1 < t_2 < t$. Let $\Pi_t(x)$ denote the appropriate parametric representations of x for the domain $[0,t]$. (Recall that the parametric representations themselves have domain $[0,1]$.) Let $\|\cdot\|_t$ denote the uniform norm over $[0,t]$. For $\epsilon > 0$ given, let $(u_i, r_i) \in \Pi_{t_1}(x_i)$ be such that

$$\|u_1 - u_2\|_1 \vee \|r_1 - r_2\|_1 \leq \mu_{t_1}(x_1, x_2) + \epsilon \ .$$

Now construct $(u_i', r_i') \in \Pi_{t_2}(x_i)$ by letting, for some γ with $0 < \gamma < 1$,

$$u_i'(s) = u_i(s/\gamma) \quad \text{and} \quad r_i'(s) = r_i(s/\gamma)$$

for $0 \leq s \leq \gamma$. Then let $u_i'(1) = x_i(t_2)$, $r_i'(1) = t_2$ and let u_i' and r_i' be defined by linear interpolation elsewhere. This construction yields

$$\|u_1' - u_2'\|_1 \vee \|r_1' - r_2'\|_1 = \|u_1 - u_2\|_1 \vee \|r_1 - r_2\|_1$$

so that

$$\mu_{t_2}(x_1, x_2) \leq \|u_1' - u_2'\|_1 \vee \|r_1' - r_2'\|_1 \leq \mu_{t_1}(x_1, x_2) + \epsilon \ .$$

Since ϵ was arbitrary, $\mu_t(x_1, x_2) \leq \mu_{t_1}(x_1, x_2)$. We now establish the inequality in the other direction. For $\epsilon > 0$ given let $(u_i, r_i) \in \Pi_{t_2}(x_i)$ be such that

$$\|u_1 - u_2\|_1 \vee \|r_1 - r_2\|_1 \leq \mu_{t_2}(x_1, x_2) + \epsilon \ .$$

Let t^* be a point in the interval $(t-\delta, t_1)$. Since r_i is continuous with $r_i(1) = t_2$, we can find s_i such that $r_i(s_i) = t^*$ for $i = 1, 2$. Let $s^* = (s_1 \vee s_2 + 1)/2$ and let $r_i'(s^*) \equiv t_1$ for $i = 1, 2$. Let $r_i'(s) = r_i(s)$ for $0 \leq s \leq s_i$ and let r_i' be defined on (s_i, s^*) and $(s^*, 1)$ by linear interpolation. Then $\|r_1' - r_2'\|_1 = \|r_1 - r_2\|_1$ and (u_i, r_i') is a legitimate parametric representation of x_i on both $[0, t_1]$ and $[0, t_2]$. Then

$$\|u_1 - u_2\|_1 \vee \|r_1' - r_2'\|_1 = \|u_1 - u_2\|_1 \vee \|r_1 - r_2\|_1 \ ,$$

so that

$$\mu_{t_1}(x_1, x_2) \leq \|u_1 - u_2\|_1 \vee \|r_1' - r_2'\|_1 \leq \mu_{t_2}(x_1, x_2) + \epsilon \ .$$

Since ϵ was arbitrary, $\mu_{t_1}(x_1, x_2) \leq \mu_{t_2}(x_1, x_2)$, and the proof is complete. ∎

Lemma 8.2 *For any $x_1, x_2 \in D \equiv D([0,\infty), \mathbb{R}^k)$, $t > 0$ and $\epsilon > 0$, there exist $x_{1,c}$, $x_{2,c} \in D_c$ such that*

$$|\mu_s(x_1, x_2) - \mu_s(x_{1,c}, x_{2,c})| \leq \epsilon \quad \textit{for} \quad 0 < s \leq t \ .$$

Proof. By the triangle inequality,

$$\begin{aligned}\mu_s(x_1,x_2) &\le \mu_s(x_{1,c},x_{2,c}) + \mu_s(x_1,x_{1,c}) + \mu_s(x_2,x_{2,c})\\ &\le \mu_s(x_{1,c},x_{2,c}) + \|x_1 - x_{1,c}\|_t + \|x_2 - x_{2,c}\|_t\end{aligned}$$

for $0 < s \le t$. By the basic approximation lemma, Lemma 5.4, we can choose $x_{1,c}$ and $x_{2,c}$ for any given t and ϵ so that $\|x_1 - x_{1,c}\|_t + \|x_2 - x_{2,c}\|_t < \epsilon$. A similar inequality holds in the other direction. ∎

We then can establish the following result, paralleling Lemma 2.2 and Theorem 2.5 of Whitt (1980). For (iii), see Theorem 5.6.1(ii) of Whitt (2002).

Theorem 8.3 *Suppose that μ and μ_t, $t > 0$ are the SM_1 metrics on $D([0,\infty),\mathbb{R}^k)$ and $D([0,t],\mathbb{R}^k)$. Then the following are equivalent for x and x_n, $n \ge 1$, in $D([0,\infty),\mathbb{R}^k)$.*

(i) $\mu(x_n,x) \to 0$ *as* $n \to \infty$;

(ii) $\mu_t(r_t(x_n), r_t(x)) \to 0$ *as* $n \to \infty$ *for all* $t \notin Disc(x)$;

(iii) there exist parametric representations (u,r) and (u_n,r_n) of x and x_n mapping $[0,\infty)$ into the graphs such that

$$\|u_n - u\|_t \vee \|r_n - r\|_t \to 0 \quad as \quad n \to \infty$$

for each $t > 0$.

We now show that the Lipschitz property extends from $D([0,t],\mathbb{R}^k)$ to $D([0,\infty),\mathbb{R}^k)$.

Theorem 8.4 *If a function*

$$f : D([0,\infty),\mathbb{R}^k) \to D([0,\infty),\mathbb{R}^k) \tag{8.5}$$

has restrictions

$$f_t : D([0,T],\mathbb{R}^k) \to D([0,T],\mathbb{R}^k) \tag{8.6}$$

satisfying

$$\mu_{2,t}(f_t(r_t(x_1)), f_t(r_t(x_2))) \le K\mu_{1,t}(r_t(x_1), r_t(x_2)) \quad for\ all \quad t > 0\ , \tag{8.7}$$

where $\mu_{1,t}$ and $\mu_{2,t}$ are two metrics on $D([0,t],\mathbb{R}^k)$ (defined consistently for all $t > 0$) and K is independent of t, then

$$\mu_2(f(x_1), f(x_2)) \le (K \vee 1)\mu_1(x_1,x_2)\ , \tag{8.8}$$

where μ_1 and μ_2 are the associated metrics on $D([0,\infty),\mathbb{R}^k)$ in (8.4).

Proof. By (8.4) and (8.7),

$$\begin{aligned}
\mu_2(f(x_1), f(x_2)) &= \int_0^\infty e^{-t}[\mu_{2,t}(r_t(f(x_1)), r_t(f(x_2))) \wedge 1]dt \\
&= \int_0^\infty e^{-t}[\mu_{2,t}(f_t(r_t(x_1)), f_t(r_t(x_2))) \wedge 1]dt \\
&\leq \int_0^\infty e^{-t}[K\mu_{1,t}(r_t(x_1), r_t(x_2)) \wedge 1]dt \\
&\leq (K \vee 1) \int_0^\infty e^{-t}[\mu_{1,t}(r_t(x_1), r_t(x_2)) \wedge 1]dt \\
&\leq (K \vee 1)\mu_1(x_1, x_2) \ . \quad \blacksquare
\end{aligned}$$

Corollary 8.1 *Let $R : D([0,\infty), \mathbb{R}^k) \to D([0,\infty), \mathbb{R}^{2k})$ be the reflection map with function domain $[0,\infty)$ defined by (3.1)–(3.7). Let metrics associated with domain $[0,\infty)$ be defined in terms of restrictions by (8.4). Then the conclusions of Theorems 3.4, 4.1 and 4.2 also hold for domain $[0,\infty)$.*

9. The Reflection Map as a Function of the Reflection Matrix

In this section we discuss the behavior of the reflection map as a function of the reflection matrix Q as well as the net-input function x. We will apply the results here in our limits for stochastic fluid networks in the next section.

Let $\mathcal{Q}$ be the set of all $k \times k$ reflection matrices (substochastic matrices Q such that $Q^n \to 0$ as $n \to \infty$). Let $\pi_{x,Q} : D \to D$ be the map defined by

$$\pi_{x,Q}(y) = (Q^t y - x)^{\uparrow} \vee 0 \ , \tag{9.1}$$

where $x^{\uparrow}(t) \equiv \sup_{0 \leq s \leq t} x(s), \quad 0 \leq t \leq T$. The map $\pi_{x,Q}$ is used to establish Theorem 3.1; see Harrison and Reiman (1981) and Chen and Whitt (1993). Let $R_Q \equiv (\psi_Q, \phi_Q)$ be the reflection map in (3.1)–(3.3) as a function of Q as well as x.

Theorem 9.1 *Let $Q_1, Q_2 \in \mathcal{Q}$ with $\|Q_1^t\| = \gamma_1 < 1$ and $\|Q_2^t\| = \gamma_2 < 1$. For all $n \geq 1$,*

$$\|\pi^n_{x,Q_j}(0)\| \leq (1 + \gamma_j + \cdots + \gamma_j^{n-1})\|x\| \tag{9.2}$$

and

$$\|\pi^n_{x,Q_1}(0) - \pi^n_{x,Q_2}(0)\| \leq (1 + \gamma_2 + \cdots + \gamma_2^{n-1}) \frac{\|x\| \cdot \|Q_1^t - Q_2^t\|}{1 - \gamma_1} \ , \tag{9.3}$$

so that

$$\|\psi_{Q_j}(x)\| \leq \frac{\|x\|}{1 - \gamma_j} \tag{9.4}$$

and

$$\|\psi_{Q_1}(x) - \psi_{Q_2}(x)\| \le \frac{\|x\| \cdot \|Q_1^t - Q_2^t\|}{(1-\gamma_1)(1-\gamma_2)} \ . \tag{9.5}$$

Proof. First

$$\|\pi^1_{x,Q_j}(0)\| = \|(-x)^{\uparrow} \vee 0\| \le \|x\| \ . \tag{9.6}$$

Next, by induction,

$$\begin{aligned} \|\pi^{n+1}_{x,Q_j}(0)\| &= \|(Q_j^t \pi^n_{x,Q_j}(0) - x)^{\uparrow} \vee 0| \\ &\le \|Q_j^t\| \cdot \|\pi^n_{x,Q_j}(0)\| + \|x\| \\ &\le \gamma_j(1 + \gamma_j + \cdots + \gamma_j^{n-1})\|x\| + \|x\| = (1 + \gamma_j + \cdots + \gamma_j^n)\|x\| . \end{aligned} \tag{9.7}$$

Similarly, by induction (assuming (9.3) for n)

$$\begin{aligned} \|\pi^{n+1}_{x,Q_1}(0) - \pi^{n+1}_{x,Q_2}(0)\| &\le \|Q_1^t \pi^n_{x,Q_1}(0) - Q_2^t \pi_{x,Q_2}(0)\| \\ &\le \|Q_1^t \pi^n_{x,Q_1}(0) - Q_2^t \pi^n_{x,Q_1}(0)\| + \|Q_2^t \pi^n_{x,Q_1}(0) - Q_2^t \pi^n_{x,Q_2}(0)\| \\ &\le \|Q_1^t - Q_2^t\| \cdot \|x\|/(1-\gamma_1) + \|Q_2^t\| \cdot \|\pi^n_{x,Q_1}(0) - \pi^n_{x,Q_2}(0)\| \\ &\le (1 + \gamma_2 + \cdots + \gamma_2^n)\|Q_1^t - Q_2^t\| \cdot \|x\|/(1-\gamma_1) \ . \end{aligned} \tag{9.8}$$

Finally, since $\|\pi^n_{x,Q}(0) - \psi_Q(x)\| \to 0$ as $n \to \infty$, the final two bounds (9.4) and (9.5) follows. ■

Theorem 9.2 *If* $\|x_n - x\| \to 0$ *in* D^k *and* $Q_n \to Q$ *in* $\mathcal{Q}$, *then*

$$\|R_{Q_n}(x_n) - R_Q(x)\| \to 0 \quad in \quad D^{2k} \ . \tag{9.9}$$

Proof. As in Harrison and Reiman (1981), we can find a positive diagonal matrix Λ so that $Q_*^t = \Lambda^{-1}Q^t\Lambda$ and $\|Q_*^t\| = \gamma < 1$. Since $Q_n \to Q$ as $n \to \infty$, $\|Q_{n*}^t\| \equiv \gamma_n \to \gamma$, where $Q_{n*}^t \equiv \Lambda^{-1}Q_n^t\Lambda$ with the same diagonal matrix used above. Consider n sufficiently large that $\gamma_n < 1$. Since $\psi_{Q^*}(\Lambda x) = \Lambda\psi_Q(x)$, for such n we have

$$\begin{aligned} \|\psi_{Q_n}(x_n) - \psi_Q(x)\| &= \|\Lambda^{-1}\Lambda\psi_{Q_n}(x_n) - \Lambda^{-1}\Lambda\psi_Q(x)\| \\ &\le \|\Lambda^{-1}\| \cdot \|\psi_{Q_{n*}}(\Lambda x_n) - \psi_{Q_*}(\Lambda x)\| \\ &\le \|\Lambda^{-1}\|(\|\psi_{Q_{n*}}(\Lambda x_n) - \psi_{Q_{n*}}(\Lambda x)\| \\ &\qquad + \|\psi_{Q_{n*}}(\Lambda x) - \psi_{Q_*}(\Lambda x)\|) \ . \end{aligned} \tag{9.10}$$

Thus, by (9.4) and (9.5),

$$\begin{aligned}
\|\psi_{Q_n}(x_n) - \psi_Q(x)\| &\leq \|\Lambda^{-1}\| \left(\frac{\|\Lambda x_n - \Lambda x\|}{1-\gamma_n} + \frac{\|\Lambda x\| \cdot \|Q_{n*} - Q_*\|}{(1-\gamma_n)(1-\gamma)} \right) \\
&\leq M_n \left(\|x_n - x\| + \frac{\|x\| \cdot M_n \cdot (1-\gamma_n) \cdot \|Q_n - Q\|}{1-\gamma} \right) \qquad (9.11)
\end{aligned}$$

for

$$M_n \equiv \frac{\|\Lambda^{-1}\| \cdot \|\Lambda\|}{1-\gamma_n} \ .$$

Hence,

$$\|\psi_{Q_n}(x_n) - \psi_Q(x)\| \to 0 \quad \text{as} \quad n \to \infty \ . \quad \blacksquare$$

Now we obtain corresponding results with the M_1 topologies.

Theorem 9.3 *Suppose that $Q_n \to Q$ in $\mathcal{Q}$.*

(a) If $x_n \to x$ in (D^k, WM_1) and $x \in D_s$, then

$$R_{Q_n}(x_n) \to R_Q(x) \quad \mathit{in} \quad (D^{2k}, WM_1) \ . \qquad (9.12)$$

(b) If $x_n \to x$ in (D^k, SM_1) and $x \in D_+$, then

$$R_{Q_n}(x_n) \to R_Q(x) \quad \mathit{in} \quad (D^{2k}, SM_1) \ . \qquad (9.13)$$

Proof. We only prove the first of the two results, since the two proofs are essentially the same. If $x_n \to x$ in (D, WM_1) with $x \in D_s$, then we can find $x_n' \in D_{s,l}$ for $n \geq 1$ such that $\|x_n - x_n'\| \to 0$ by Lemma 7.5. By Theorem 3.1

$$\|R_{Q_n}(x_n) - R_{Q_n}(x_n')\| \leq K_n \|x_n - x_n'\| \to 0 \qquad (9.14)$$

because $K_n \to K < \infty$. By Theorem 6.3, $(R_Q(u), r) \in \Pi_w(R(x))$ when $x \in D_s$. So, for any $\epsilon > 0$ given, let $(u, r) \in \Pi_w(x)$ and $(u_n, r_n) \in \Pi_w(x_n')$ such that $\|u_n - u\| \vee \|r_n - r\| \leq \epsilon$. Then $(R_Q(u), r) \in \Pi_w(R_Q(x))$, $(R_{Q_n}(u_n), r_n) \in \Pi_w(R_{Q_n}(x_n'))$ for $n \geq 1$ and

$$\|R_{Q_n}(u_n) - R_Q(u)\| < K(\epsilon + \|Q_n - Q\|) \qquad (9.15)$$

by Theorem 9.2 and (9.11), so that

$$R_{Q_n}(x_n') \to R_Q(x) \quad \text{in} \quad (D^{2k}, WM_1) \ . \qquad (9.16)$$

Combining (9.14), (9.16) and the triangle inequality with the metric d_p, we obtain (9.12). $\blacksquare$

10. Limits for Stochastic Fluid Networks

In this section we provide concrete stochastic applications of the convergence-preservation results. Following Kella and Whitt (1996) and references therein, we characterize a single-class open stochastic fluid network with Markovian routing by a four-tuple $\{A, r, Q, X(0)\}$, where $A \equiv (A^1, \ldots, A^k)$ is the vector of exogenous input stochastic processes at the k stations, $r = (r^1, \ldots, r^k)$ is the vector of potential output rates at the stations, $Q \equiv (Q_{i,j})$ is the routing matrix and $X(0) \equiv (X^1(0), \ldots, X^k(0))$ is the nonnegative random vector of initial buffer contents. The stochastic processes $A^j \equiv \{A^j(t) : t \geq 0\}$ have nondecreasing nonnegative sample paths; $A^j(t)$ represents the cumulative input at station j during the time interval $[0, t]$. When the buffer at station j is nonempty, there is fluid output from station j at constant rate r_j. When the buffer is empty, the output rate is the minimum of the net (exogenous plus internal) input rate and r_j. A proportion $Q_{i,j}$ of all output from station i is routed to station j, while a proportion $q_i \equiv 1 - \sum_{j=1}^{k} Q_{i,j}$ is routed out of the network. We assume that Q is substochastic so that $Q_{i,j} \geq 0$, $1 \leq j \leq k$, and $q_i \geq 0$, $1 \leq i \leq k$. Moreover, we assume that $Q^n \to 0$ as $n \to \infty$, where Q^n is the n^{th} power of Q, as in the definition of the reflection map.

As a more concrete example, suppose that the exogenous input to station j is the sum of the inputs from m_j separate on-off sources. Let (j, i) index the i^{th} on-off source at station j. When the (j, i) source is on, it sends fluid input at rate $\lambda_{j,i}$; when it is off, it sends no input. Let $B_{j,i}(t)$ be the cumulative on time for source (j, i) during the time interval $[0, t]$. Then the exogenous input process at station j is

$$A^j(t) = \sum_{i=1}^{m_j} \lambda_{j,i} B_{j,i}(t), \quad t \geq 0 . \tag{10.1}$$

Since $B_{j,i}$ necessarily has continuous sample paths, the exogenous input processes A^j and A also have continuous sample paths in this special case.

Given the defining four-tuple $(A, r, Q, X(0))$, the associated $\mathbb{R}^k$-valued potential net-input process is

$$X(t) = X(0) + A(t) - (I - Q^t) rt, \quad t \geq 0 . \tag{10.2}$$

where Q' is the transpose of Q. Indeed, proceeding formally, we regard (10.2) as a definition. Since A^j has nondecreasing sample paths for each j, the sample paths of X are of bounded variation. In many special cases, the sample paths of X will be continuous as well.

The buffer-content stochastic process $Z \equiv (Z^1, \ldots, Z^k)$ is simply obtained by applying the reflection map to the net-input process X in (10.2), in particular,

$$Z = \phi(X) \ , \tag{10.3}$$

where $R = (\psi, \phi)$ in (3.1)–(3.7). Since the potential output vector is $(I - Q^t)rt$, the definition (3.1)–(3.7) is very natural here, just as for the instantaneous net input in (5.3). Again proceeding formally, as in Harrison and Reiman (1981), we regard (10.3) as a definition. This stochastic fluid network model is more elementary than the queue-length processes in the queueing network in Chen and Whitt, because the content process of interest Z is defined directly in terms of the reflection map, requiring only (10.2) and (10.3).

We now want to establish some limit theorems for the stochastic processes. First, we obtain a model continuity or stability result. For this purpose, we define a sequence of fluid network models indexed by n characterized by four-tuples $(A_n, r_n, Q_n, X_n(0))$. Let $\Rightarrow$ denote convergence in distribution.

Theorem 10.1 *If $(A_n, X_n(0)) \Rightarrow (A, X(0))$ in $D([0,\infty), \mathbb{R}^k) \times \mathbb{R}^k$, where the topology is either SM_1 or WM_1, $r_n \to r$ and $Q_n \to Q$ in $\mathcal{Q}$ as $n \to \infty$, then*

$$(X_n, Y_n, Z_n) \Rightarrow (X, Y, Z) \quad as \quad n \to \infty \quad in \quad D([0,\infty), \mathbb{R}^{3k}) \ ,$$

with the same topology on D, where X_n and X are the associated net-input processes defined by (10.2), Y_n and Y are the associated regulator processes, and Z_n and Z as the associated buffer-content processes, with

$$R(X_n) \equiv (\psi(X_n), \phi(X_n)) \equiv (Y_n, Z_n) \ .$$

Proof. Apply the continuous mapping theorem with the continuous functions in (10.2) and (3.1)–(3.7), invoking Theorem 9.3. Note that A_n, A, X_n and X have sample paths in D_+. First apply the linear function in (10.2) mapping $(A_n, r_n, Q_n, X_n(0))$ into X_n; then apply R mapping X_n into (Y_n, Z_n). For the special case of common Q, we can invoke Theorem 3.4 instead of Theorem 9.3. ∎

Remark 10.1 If $P(A \in D_1) = 1$, i.e., if

$$P(Disc(A^i) \cap Disc(A^j) = \phi) = 1 \tag{10.4}$$

for all i, j with $1 \leq i, j \leq k$ and $i \neq j$, then the assumed SM_1 convergence $A_n \Rightarrow A$ is implied by WM_1 convergence. Since A_n and A have nondecreasing sample paths, the condition $A_n^i \Rightarrow A^i$ $D([0,\infty), \mathbb{R}, M_1)$ is equivalent to convergence of the finite-dimensional distributions at all time points t for which $P(t \in Disc(A^i)) = 0$, where $Disc(A^i)$ is the set of discontinuity points of A^i. ∎

Remark 10.2 As we have indicated, it is natural for A_n to have continuous sample paths, but that does not imply that the limit A necessarily must have continuous sample paths. If A does in fact have continuous sample paths, then so do X, Y and Z. Then, the SM_1 topology reduces to the topology of uniform convergence on compact subsets. ■

Remark 10.3 We can also obtain a bound on the distance between (X_n, Y_n, Z_n) and (X, Y, X) using the Prohorov metric π on the probability measures on (D_+, SM_1); see p. 237 of Billingsley (1968) and Whitt (1974). For random elements X_1 and X_2, let $\pi(X_1, X_2)$ denote the Prohorov metric applied to the probability laws of X_1 and X_2. The statement then is: For common Q, there exists a constant K such that

$$\pi((X_n, Y_n, Z_n), (X, Y, Z)) \leq K\pi((A_n, X_n(0)), (A, X(0))) \ . \tag{10.5}$$

For common Q, we apply Theorem 3.4. ■

We also can obtain heavy-traffic FCLTs for stochastic fluid networks by considering a sequence of models with appropriate scaling.

Theorem 10.2 *Consider a sequence of stochastic fluid networks* $\{(A_n, r_n, Q_n, X_n(0)) : n \geq 1\}$. *If there exist a constant* $q > 0$*, an* $\mathbb{R}^k$*-valued random vector* $X(0)$*, vectors* $\alpha_n \in \mathbb{R}^k$*,* $n \geq 1$*, and a stochastic process* A *such that*

$$n^{-q}[A_n(nt) - \alpha_n nt, X_n(0)] \Rightarrow [A(t), X(0)] \quad as \quad n \to \infty \tag{10.6}$$

in $D([0,\infty), \mathbb{R}^k, WM_1) \times \mathbb{R}^k$*, where*

$$P(A \in D_s) = 1 \ , \tag{10.7}$$

and

$$n^{1-q}[\alpha_n - (I - Q_n^t) r_n] \to \alpha \quad in \quad \mathbb{R}^k \ , \tag{10.8}$$

then

$$n^{-q}(X_n(nt), Y_n(nt), Z_n(nt)) \Rightarrow (X(t), Y(t), Z(t)) \quad as \quad n \to \infty \tag{10.9}$$

in $D([0,\infty), \mathbb{R}^k, WM_1) \times D([0,\infty), \mathbb{R}^{2k}, WM_1)$*, where*

$$X(t) = X(0) + A(t) + \alpha(t), \quad t \geq 0 \ , \tag{10.10}$$

and $(Y, Z) = R(X)$ *for* R *in* (3.1)–(3.3).

Proof. Since

$$n^{-q}X_n(nt) = n^{-q}[X_n(0) + [A_n(nt) - \alpha_n nt] + [\alpha_n nt - (I - Q_n^t)r_n nt], \quad t \geq 0 \ , \tag{10.11}$$

$$n^{-q}X_n(nt) \Rightarrow X(t) \quad \text{in} \quad (D^k, WM_1) \ . \tag{10.12}$$

The proof is completed by applying the continuous mapping theorem, using Theorem 9.3. For common Q, we could use Theorem 3.3. ∎

Remark 10.4 In order for condition (10.6) to hold, it suffices to have $X_n(0)$ be independent of $\{A_n(t) : t \geq 0\}$ for each n,

$$X_n(0) \Rightarrow X(0) \quad \text{in} \quad \mathbb{R}^k \ ,$$

and $\{A_n^i(t) : t \geq 0\}$, $1 \leq i \leq k$, be k mutually independent processes for each n, with

$$n^{-q}[A_n^i(nt) - \alpha_n^i nt] \Rightarrow A^i \quad \text{as} \quad n \to \infty \quad \text{in} \quad D([0,\infty), \mathbb{R}^1, M_1) \tag{10.13}$$

for $1 \leq i \leq k$. If also $P(t \in Disc(A^i)) = 0$ for all i and t (so that A^i has no fixed discontinuities). then, almost surely, the limit process A has discontinuities in only one coordinate at a time, so that the assumed convergence in the WM_1 topology is actually equivalent to convergence in the SM_1 topology.

Remark 10.5 If condition (10.7) does not hold, then we obtain (10.9) with $(Y_n, Z_n) \Rightarrow (Y, Z)$ in $D([0,\infty), \mathbb{R}^{2k})$ with the L_1 topology instead of the WM_1 topology, by Theorem 3.2(a).

Remark 10.6 In many applications the limiting form of the initial conditions can be considered deterministic; i.e. $P(X(0) = c) = 1$ for some $c \in \mathbb{R}^k$. Then (Y, Z) is simply a reflection of A, modified by the deterministic initial condition c and the deterministic drift $\alpha(t)$. In Whitt (2000a) conditions are determined to have the convergence $A_n^i \Rightarrow A^i$. Then A^i is often a Lévy process. When A is a Lévy process, Z and (Y, Z) are reflected Lévy processes. See the references in Whitt (2000a) and Kella and Whitt (1996) for more on these processes. In some cases explicit expressions for non-product-form steady-state distributions have been derived; see Kella and Whitt (1992) and Kella (1993, 1996). ∎

Remark 10.7 Clearly, we can obtain similar results for more general models by similar methods. For example, the prevailing rates might be stochastic processes. The potential output rate from station j at time t can be the random variable $R_j(t)$. Then the net-input process in (10.2) should be changed to

$$X(t) = X(0) + A(t) - (I - Q^t)S(t), \quad t \geq 0 \ , \tag{10.14}$$

where $S \equiv (S^1, \ldots, S^k)$ is the $\mathbb{R}^k$-valued potential output process, having

$$S^j(t) = \int_0^t R^j(u)du, \quad t \geq 0 \ . \tag{10.15}$$

Similarly, with the on-off sources, the input rates during the on periods might be stochastic processes instead of the constant rates $\lambda_{j,i}$ in (10.1). Extensions of Theorems 10.1 and 10.2 are straightforward with such generalizations, but we must be careful that the assumptions of Theorems 3.2–3.5 are satisfied. ∎

11. Other Reflection Maps

In this section we show that the continuity and Lipschitz properties of the reflection map extend to more general reflection maps, such as those considered by Dupuis and Ishii (1991), Williams (1987, 1995) and Dupuis and Ramanan (1999a,b). We assume that the reflected process has values in a closed subset S of $\mathbb{R}^k$. We assume that we are given an instantaneous reflection map $\phi_0 : S \times \mathbb{R}^k \to S$. The idea is that an initial position s_0 in S and an instantaneous net input u are mapped by ϕ_0 into the new position $s_1 \equiv \phi_0(s_0, u_0)$ in S. In many cases $\phi_0(s_0, u_0)$ will depend upon (s_0, u_0) only through their sum $s_0 + u_0$, but we allow more general possibilities. It is also standard to have S be convex and $\phi_0(s, u) = s + u$ if $s + u \in S$, while $\phi_0(s, u) \in \partial S$ if $s + u \notin S$, where ∂S is the boundary of S, but again we do not directly require it. Under extra regularity conditions, ϕ_0 becomes the projection in Dupius and Ramanan (1999a).

As in Section 5, we use ϕ_0 to define a reflection map on $D_c \equiv D_c([0, T], \mathbb{R}^k)$. However, we also allow dependence upon the initial position in S. Thus, we define $\phi : S \times D_c \to D_c$ by letting

$$\phi(z(0-), x)(t_i) \equiv z(t_i) \equiv \phi_0(z(t_{i-1}), x(t_i) - x(t_{i-1})), \quad 0 \leq i \leq m \ , \tag{11.1}$$

where $t_1, \ldots, t_m$ are the discontinuity points of x, with $t_0 = 0 < t_1 < \cdots < t_m < T$, $x^i(t_{-1}) = 0$ for all i and $z(t_{-1}) \equiv z(0-) \in S$ is the initial position. A standard case is $x^i(0) = 0$ for all i and $z(0) = z(0-)$. We let z be constant in between these discontinuity points.

We then make two general assumptions about the instantaneous reflection map ϕ_0 and the associated reflection map ϕ on $S \times D_c$ in (11.1). One is a Lipschitz assumption and the other is a monotonicity assumption.

Lipschitz Assumption. There is a constant K such that

$$\|\phi(s_1, x_1) - \phi(s_2, x_2)\| \le K(\|x_1 - x_2\| \vee \|s_1 - s_2\|)$$

for all $s_1, s_2 \in S$ and $x_1, x_2 \in D_c$, where ϕ is the reflection map in (11.1).

We now turn to the monotonicity. Let e_i be the vector in $\mathbb{R}^k$ with a 1 in the i^{th} coordinate and 0's elsewhere. Let $\phi_0^j(s, u)$ be the j^{th} coordinate of the reflection. We require monotonicity af all these coordinate maps, but we allow the monotonicity to be in different directions in different coordinates.

Monotonicity Assumption. For all $s_0 \in \mathbb{R}^k$, i, $1 \le i \le k$ and j, $1 \le j \le k$, $\phi_0^j(s_0, \alpha e_i)$ is monotone in the real variable α for $\alpha > 0$ and for $\alpha < 0$.

Just as in Theorem 5.2, we can use the Lipschitz assumption to extend the reflection map from D_c to D. The proof is essentially the same as before.

Theorem 11.1 *If the reflection map $\phi : S \times D_c \to D_c$ in (11.1) satisfies the Lipschitz assumption, then there exists a unique extension $\phi : S \times D \to D$ of the reflection map in (11.1) satisfying $\|\phi(s, x_n) - \phi(s, x)\| \to 0$ if $s \in S$, $x_n \in D_c$ and $\|x_n - x\| \to 0$. Moreover, $\phi : S \times D \to D$ inherits the Lipschitz property.*

We now want to establish sufficient conditions for the reflection map to inherit the Lipschitz property when we use appropriate M_1 topologies on D. From our previous analysis, we know that we need to impose regularity conditions. With the monotonicity assumption above, it is no longer sufficient to work in D_s. We assume that the sample paths have discontinuities in only one coordinate at a time, i.e., we work in the space D_1. We exploit another approximation lemma. Let $D_{c,1}$ be the subset of D_c in which all discontinuities occur in only one coordinate at a time, i.e., $D_{c,1} \equiv D_c \cap D_1$. The following is another variant of Lemma 5.4, which can be established using it.

Lemma 11.1 *For all $x \in D_1$, there exist $x_n \in D_{c,1}$, $n \ge 1$, such that $\|x_n - x\| \to 0$.*

We are now ready to state our M_1 result.

Theorem 11.2 *Suppose that the Lipschitz and monotonicity assumptions above are satisfied. Let $\phi : S \times D \to D$ be the reflection mapping obtained by extending (11.1) by applying Theorem 11.1.*

For any $s \in S$, $x \in D_1$ and $(u,r) \in \Pi_w(x)$, $(\phi(s,u),r) \in \Pi_w(\phi(s,x))$. Thus there exists a constant K such that

$$d_p(\phi(s_1,x_1),\phi(s_2,x_2)) \leq d_w(\phi(s_1,x_1),\phi(s_2,x_2)) \leq K(d_s(x_1,x_2) \vee \|s_1-s_2\|) \tag{11.2}$$

for all $s_1,s_2 \in S$ and $x_1,x_2 \in D_1$. Moreover, if $s_n \to s$ in $\mathbb{R}^k$ and $x_n \to x$ in (D,WM_1) where $x \in D_1$, then

$$\phi(s_n,x_n) \to \phi(s,x) \quad in \quad (D,WM_1)\ . \tag{11.3}$$

Proof. By Theorem 11.1, the extended reflection map $\phi : S \times D \to D$ is well defined and Lipschitz in the uniform norm. For any $x \in D_1$, apply Lemma 11.1 to obtain $x_n \in D_{c,1}$ with $\|x_n - x\| \to 0$. Since $x \in D_1$, the strong and weak parametric representations coincide. Choose $(u,r) \in \Pi_s(x) = \Pi_w(x)$. Since $\|x_n - x\| \to 0$ and $x_n \in D_{c,1}$, we can find $(u_n,r_n) \in \Pi_s(x_n) = \Pi_w(x_n)$ such that $\|u_n - u\| \vee \|r_n - r\| \to 0$. Now, paralleling Theorem 6.2, we can apply the monotonicity condition on $D_{c,1}$ to deduce that $(\phi(s,u_n),r_n) \in \Pi_w(\phi(s,x_n))$ for all n. (Note that we need not have either $\phi(s,x) \in D_1$ or $\phi(s,x_n) \in D_{c,1}$, but we do have $\phi(s,x_n) \in D_c$. Note that the componentwise monotonicity implies that $(\phi(s,u_n),r_n)$ belongs to $\Pi_w(\phi(s,x_n))$, but not necessarily to $\Pi_s(\phi(s,x_n))$.) By the Lipschitz property of ϕ,

$$\|\phi(s,u_n) - \phi(s,u)\| \vee \|r_n - r\| \to 0\ . \tag{11.4}$$

Hence, we can apply Lemma 6.5 to deduce that $(\phi(s,u),r) \in \Pi_w(\phi(s,x))$. We thus obtain the Lipschitz property (11.2), just as in Theorem 3.4, by applying the argument in Theorem 4.2. Finally, to obtain (11.3), suppose that $s_n \to s$ in S and $x_n \to x$ in (D,SM_1) with $x \in D_1$. Under that condition, by the analog of Lemma 7.5 (for D_1 instead of D_s), we can find $x'_n \in D_{1,l} \subseteq D_1$ such that $\|x_n - x'_n\| \to 0$. Since ϕ is Lipschitz on $S \times (D,U)$, there exists a constant K such that

$$\|\phi(s_n,x_n) - \phi(s_n,x'_n)\| \leq K\|x_n - x'_n\| \to 0\ . \tag{11.5}$$

By part (a), there exists a constant K such that

$$d_w(\phi(s_n,x'_n),\phi(s,x)) \leq K(d_s(x'_n,x) \vee \|s_n - s\|) \to 0\ . \tag{11.6}$$

By (11.5), (11.6) and the triangle inequality with d_p, we obtain (11.3). ∎

To give one concrete application of Theorem 11.2, we consider the two-sided regulator, $R : D([0,T],\mathbb{R}) \to D([0,T],\mathbb{R}^{3k})$ defined by $R(x) \equiv (\phi(x),\psi_1(x),\psi_2(x)) \equiv (z,y_1,y_2)$, where

$$z = x + y_1 - y_2\ ,$$

$$
\begin{aligned}
&0 \le z(t) \le c, \quad 0 \le t \le T \ , \\
&y_1(0) = -(x(0) \wedge 0)^- \quad \text{and} \quad y_2(0) = [c - x(0)]^+ \ , \\
&y_1 \quad \text{and} \quad y_2 \quad \text{are nondecreasing} \ , \\
&\int_0^T z(t) dy_1(t) = 0 \quad \text{and} \quad \int_0^T [c - z(t)] dy_2(t) = 0 \ ,
\end{aligned} \tag{11.7}
$$

as in Chapter 2 of Harrison (1985) and in Section 4.5 of Berger and Whitt (1992). Since the domain is one-dimensional, here we have $D = D^1 = D_1$ and $D_c = D_{c,1}$. The two-sided regulator can also be defined using (11.1) with the elementary instantaneous reflection map $\phi_0 : [0, c] \times \mathbb{R} \to \mathbb{R}$ defined by

$$
\phi_0(s, u) = (s + u) \vee 0 \wedge c \ . \tag{11.8}
$$

From (11.8), it is immediate that the monotonicity assumption is satisfied. Berger and Whitt proved that the reflection map $\phi : [0, c] \times D^1 \to D^1$ in this case is Lipschitz in the uniform norm with Lipschitz constant 2. (They showed that the maps ψ_1 and ψ_2 are continuous but not Lipschitz on (D, U).) Thus we have the following result.

Theorem 11.3 *Let $\phi : [0, c] \times D^1 \to D^1$ be the two-sided regulator map defined by either (11.7) or (11.8) and Theorem 11.1. Then*

$$
d(\phi(s_1, x_1), \phi(s_2, x_2)) \le 2(d(x_1, x_2) \vee |s_1 - s_2|) \tag{11.9}
$$

for all $x_1, x_2 \in D^1$, where d is the M_1 metric.

For other reflection maps, we need to verify the Lipschitz and monotonicity assumptions above. Evidently the Lipschitz assumption is the more difficult condition to verify. However, Dupuis and Ishii (1991) and Dupuis and Ramanan (1999a,b) have established general conditions under which the Lipschitz assumption is satisfied.

Acknowledgments. I am grateful to Nimrod Bayer, Jim Dai, Takis Konstantopoulos, Avi Mandelbaum, Tolya Puhalskii, Kavita Ramanan and Marty Reiman for assistance.

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