Unstable Asymptotics for Nonstationary Queues

by

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ABSTRACT

We relate laws of large numbers and central limit theorems for nonstationary counting processes to corresponding limits for their inverse processes. We apply these results to develop approximations for queues that are unstable in a nonstationary manner. We obtain unstable nonstationary analogs of the queueing relation $L = \lambda W$ and associated central-limit-theorem versions. For modeling and to obtain the first limits, we can construct nonstationary point processes as random time-transformations of familiar point processes, such as renewal processes and stationary point processes. We deduce the asymptotic behavior of the nonstationary point process from the asymptotic behavior of the familiar point process and the time transformation.

Keywords. nonstationary queues, nonstationary point processes, unstable queues, law of large numbers, central limit theorem, random time change, Little's law, $L = \lambda W$.

1. Introduction

In this paper we prove limit theorems to help describe the behavior of queueing systems that are unstable in a (possibly) nonstationary manner for a period of time. Let $A(t)$ and $D(t)$ count the number of arrivals and departures, respectively, in the time interval $[0,t]$. The arrival rate might be increasing while the service rate remains fixed, so that $A(t) \approx \lambda t^p$ for $p > 1$ and $D(t) \approx \mu t$. Alternatively, there might be a degradation of service with constant arrival rate, so that $A(t) \approx \lambda t$ and $D(t) \approx \mu t^q$ for $q < 1$. There could even be a combination of these factors. We want to see how the arrival and departure times, queue-length process (number in system) and the waiting times (time in system) behave in these circumstances. In particular, one purpose of this paper is to investigate the possibility of establishing unstable nonstationary analogs of the familiar stable stationary queueing relation $L = \lambda W$ in Little (1961) and Stidham (1974) and the central-limit-theorem versions in Glynn and Whitt (1986, 1988, 1989); see Bremaud (1992), Stidham and El-Taha (1989) and Whitt (1991, 1992) for background. Unstable stationary analogs of $L = \lambda W$ appear in Iglehart and Whitt (1970a), Whitt (1971), Szczotka (1986, 1992), Szczotka and Topolski (1991) and Serfozo, Szczotka and Topolski (1992).

Example 1.1. A simple motivating example is the standard $GI/GI/1$ queue with unlimited waiting space, the first-in first-out (FIFO) discipline, arrival rate λ and service rate μ. Since the arrival and service rates are constant, we regard this as a *stationary model*. Let $Q(t)$ represent the number of customers in the system at time t and W_n the time spent in the system by customer n. If $\lambda > \mu$, then $Q(t)$ and W_n fail to have proper limiting distributions as $t \to \infty$ and $n \to \infty$ for any initial conditions. Then we say that this stationary model is *unstable*. In this unstable situation, we have nondegenerate limits (for any finite initial conditions) for the *normalized* processes, i.e.,

$$t^{-1} Q(t) \to L \equiv \lambda - \mu \text{ w.p.1 as } t \to \infty \tag{1.1}$$

and

$$n^{-1} W_n \to W \equiv \mu^{-1} - \lambda^{-1} \text{ w.p.1. as } n \to \infty . \tag{1.2}$$

Note that $L = \lambda \mu W$ for L and W in (1.1) and (1.2); this can be regarded as an unstable stationary analog of the familiar stable stationary relation $L = \lambda W$.

If, in addition, the interarrival times and service times have finite variances σ_a^2 and σ_s^2, respectively, one of which is strictly positive, then also

$$[Q(t) - (\lambda - \mu) t] / \sqrt{(\lambda^3 \sigma_a^2 + \mu^3 \sigma_s^2) t} \Rightarrow N(0,1) \text{ as } t \to \infty \tag{1.3}$$

and

$$[W_n - (\mu^{-1} - \lambda^{-1}) n] / \sqrt{(\sigma_a^2 + \sigma^2) n} \Rightarrow N(0,1) \text{ as } n \to \infty , \tag{1.4}$$

where $\Rightarrow$ denotes convergence in distribution and $N(0,1)$ denotes a standard (mean 0, variance 1) normal random variable. In fact, a generalization of (1.3) and (1.4) holds with joint convergence, so that $[Q(t) - \lambda W_{\lfloor \mu t \rfloor}] / \sqrt{t}$ converges to a nondegenerate limit as $t \to \infty$; see Proposition 6.3 below.

The limits (1.1) and (1.2) are the familiar heavy-traffic *strong laws of large numbers* (SLLNs), while the limits (1.3) and (1.4) are the associated heavy-traffic *central limit theorems* (CLTs). They follow from Whitt (1971) or Iglehart and Whitt (1970a). The key to (1.1) and (1.3) is the relation

$$Q(t) = A(t) - D(t) , \ t \geq 0 , \tag{1.5}$$

while the key to (1.2) and (1.4) is the relation

$$W_n = \hat{D}_n - \hat{A}_n , \ n \geq 1 , \tag{1.6}$$

where $\hat{A}_n$ and $\hat{D}_n$ are the arrival and departure epochs of customer n. Since the arrival process is a renewal process, the SLLN and CLT for $A(t)$ and its ‘‘inverse’’ $\hat{A}_n$ are standard; see Section 17 of Billingsley (1968), Chapter 11 of Feller (1971), Section 7 of Whitt (1980) and Theorem 6 of Glynn and Whitt (1988). Since the queue is unstable, the departure process is asymptotically equivalent to the service renewal process, obtained by letting the server run continuously, which is independent of the arrival process. Hence the SLLN and CLT for $D(t)$ and its inverse hold as well. Moreover, it is known that similar limits often hold when the independence conditions are relaxed, see Theorems 1 and 2 of Iglehart and Whitt (1970b) and Theorem 6.4 of Whitt (1980). *Our primary purpose here is to establish corresponding results when the model is nonstationary as well as unstable.*

Example 1.2. We were primarily motivated by studying queues with *nonstationary arrival processes;* e.g., see Massey and Whitt (1992). For a concrete example of this type, consider an $M_t/GI/1$ queue with one server, unlimited waiting space, the FIFO discipline, a nonhomogeneous Poisson arrival process with deterministic nonstationary arrival-rate function $\lambda(t)$, and i.i.d. service times that are independent of the arrival process. As in Example 1.1, let the service-time distribution have mean μ^{-1} and finite variance σ_s^2. Suppose that $\lambda(t) = at^b$ for positive constants a and b. It is well known that $A(t)$ has a Poisson distribution for each t with time-dependent mean

$$\phi(t) = \int_0^t \lambda(s)\, ds = \frac{at^{b+1}}{b+1} . \tag{1.7}$$

We approximate the behavior at time t by considering limits as $t \to \infty$. By the SLLN, $A(t)/\phi(t) \to 1$ w.p.1 as $t \to \infty$ and, by the CLT,

$$[A(t) - \phi(t)]/\sqrt{\phi(t)} \Rightarrow N(0,1) \text{ as } t \to \infty . \tag{1.8}$$

(The SLLN and CLT apply because any Poisson random variable with mean m has the distribution of the sum of n independent Poisson random variables each with mean m/n.)

Since the arrival rate is increasing without bound, after some initial period the server will be continuously busy. Thus, we can conclude that $D(t)/t \to \mu$ w.p.1 as $t \to \infty$ and

$$\left[\frac{A(t) - \phi(t)}{\sqrt{\phi(t)}} \; , \; \frac{D(t) - \mu t}{\sqrt{\sigma_s^2 \mu^3 t}} \right] \Rightarrow (N_1(0,1), N_2(0,1)) \text{ as } t \to \infty \, , \tag{1.9}$$

where $N_1(0,1)$ and $N_2(0,1)$ are independent standard normal random variables. By (1.5) and the continuous mapping theorem (Theorem 5.1 of Billingsley (1968)), we can deduce that $Q(t)/\phi(t) \to 1$ w.p.1 as $t \to \infty$ and

$$[Q(t) - \phi(t) + \mu t]/\sqrt{\phi(t)} \Rightarrow N_1(0,1) \text{ as } t \to \infty \tag{1.10}$$

jointly with (1.9) for $\phi(t)$ in (1.7). We obtain (1.10) from (1.5) and (1.9) because $t/\phi(t) \to 0$ as $t \to \infty$.

In this paper, we establish generalizations of (1.9) and (1.10), and investigate the associated asymptotic behavior of the arrival times, departure times and waiting times. ■

Here is how the rest of this paper is organized. In §2 we discuss SLLNs for nonstationary counting processes and their inverses. In §3 we establish SLLNs for unstable nonstationary queues. In §4 we discuss CLTs for nonstationary counting processes and their inverses. In §5 we discuss associated functional central limit theorems (FCLTs). In §6 we show how the CLTs for point processes imply CLTs for the queueing processes. In §7 we consider how to construct nonstationary counting processes as random time transformations of familiar counting processes. e.g., renewal processes. We obtain LLNs and CLTs for the nonstationary counting process from corresponding limits for the familiar counting process and the random time transformation. In Section 8 we discuss how to obtain CLTs for nonstationary point processes by exploiting their compensators. Sections 4-8 combine to yield general conditions for CLTs for nonstationary queueing processes.

In contrast to the unstable asymptotics for nonstationary queues considered here, *stable asymptotics for possibly nonstationary queues* are discussed in Gelenbe (1983), Heyman and Whitt (1984), Gelenbe and Finkel (1987), Lemoine (1989), Rolski (1989) and El-Taha and

Stidham (1991) and references cited there, and *uniform acceleration asymptotics for nonstationary queues* are discussed in Massey (1985), Whitt (1991), Mandelbaum and Massey (1992), Massey and Whitt (1992) and Grier, Massey and Whitt (1992).

2. SLLNs for Nonstationary Counting Processes and Their Inverses

Let $\hat{A} \equiv \{\hat{A}_n : n \geq 1\}$ be a nondecreasing sequence of nonnegative numbers and let $A \equiv \{A(t) : t \geq 0\}$ be the associated counting function, defined by

$$A(t) = \max\{n \geq 1 : \hat{A}_n \leq t\} , \ t \geq 0 , \tag{2.1}$$

with $A(t) = 0$ if $\hat{A}_1 > t$. The pair $\{\hat{A}_n\}$ and $\{A(t)\}$ provide two different representations of a sample path of a stochastic point process. In this section we relate limits for normalized versions of $\{\hat{A}_n\}$ and $\{A(t)\}$. Since we are thinking of nonstationary point processes, we allow general nonlinear normalization functions. In the stochastic setting, the limits will hold w.p.1 and can be regarded as SLLNs.

Our normalization functions will be *homeomorphisms* of the positive real line $\mathbb{R}_+$. Such a function ϕ is strictly increasing and continuous on $\mathbb{R}_+$ with $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. Let $\circ$ denote the *composition map,* i.e., $(\phi_1 \circ \phi_2)(t) = \phi_1(\phi_2(t))$, and let e denote the *identity map,* i.e., $e(t) = t$. Of great importance is the fact that ϕ has an *inverse* ϕ^{-1} with $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = e$. Also note that $(\phi_1 \circ \phi_2)^{-1} = \phi_2^{-1} \circ \phi_1^{-1}$ for two homeomorphisms of $\mathbb{R}_+$. Thus, if ϕ is a homeomorphism of $\mathbb{R}_+$, then so is $\lambda\, \phi(t)$ for any positive real number λ, and its inverse is $\phi^{-1}(t/\lambda)$.

The basis for our results is the fundamental inverse relation, which we state without proof.

Lemma 2.1. *For all* $n \geq 1$ *and* $t > 0$, $\hat{A}_n \leq t$ *if and only if* $A(t) \geq n$.

The relation between the limits for $\hat{A}$ and A follows easily from the following bounds, which are of independent interest. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\lceil x \rceil$ be the least integer greater than or equal to *x*.

One-sided bounds are obtained below by either setting $\varepsilon = 1$ or setting $\delta = \infty$. Let $1/0 = \infty$ and $1/\infty = 0$.

Lemma 2.2. *Suppose that* ϕ_1 *and* ϕ_2 *are homeomorphisms of* $\mathbb{R}_+$, $0 < \varepsilon \le 1$ *and* $0 < \delta \le \infty$.

(a) If

$$1 - \varepsilon \le \frac{\phi_2(A(t))}{\phi_1(t)} < 1 + \delta \text{ for } all\ t \ge t_0\ , \tag{2.2}$$

then

$$\frac{1}{1+\delta} < \frac{\phi_1(\hat{A}_n)}{\phi_2(n)} \le \frac{1}{1-\varepsilon} \quad \text{for } all\ n \ge n_0 \equiv \left\lceil \phi_2^{-1}(\phi_1(t_0)(\lambda + \delta)) \right\rceil . \tag{2.3}$$

(b) If

$$1 - \varepsilon < \frac{\phi_1(\hat{A}_n)}{\phi_2(n)} \le 1 + \delta \quad \text{for } all\ n \ge n_0\ , \tag{2.4}$$

then

$$\frac{\phi_2(A(t))}{\phi_1(t)} \le \frac{1}{1-\varepsilon} \tag{2.5}$$

and

$$\frac{\phi_2(A(t)+1)}{\phi_1(t)} \ge \frac{1}{1+\delta}\ . \tag{2.6}$$

for all $t \ge t_0 \equiv \left\lceil \phi_1^{-1}(\phi_2(t_0)(1+\delta)) \right\rceil$. *Moreover, there is a sequence of times* $\{t_k\}$ *such that* $t_k \to \infty$ *as* $k \to \infty$ *and*

$$\frac{\phi_2(A(t_k))}{\phi_1(t_k)} \ge \frac{1}{1+\delta} \tag{2.7}$$

for all $t_k \ge t_0$.

Proof. (a) If (2.2) holds, then

$$n_1(t) \equiv \lfloor \phi_2^{-1}(\phi_1(t)(1-\varepsilon)) \rfloor \le A(t) < \lceil \phi_2^{-1}(\phi_1(t)(1+\delta)) \rceil \equiv n_2(t)$$

for all $t \ge t_0$ and, by Lemma 2.1,

$$\hat{A}_{n_1(t)} \le t < \hat{A}_{n_2(t)} \text{ for all } t \ge t_0 \ . \tag{2.8}$$

Let t_1 and t_2 be functions of n defined by

$$t_1(n) = \phi_1^{-1}(\phi_2(n)/(1-\varepsilon)) \text{ and } t_2(n) = \phi_1^{-1}(\phi_2(n)/(1+\delta)) \ ,$$

and note that $n_1(t_1(n)) = n_2(t_2(n)) = n$ for all n. Hence, for all $n \ge n_0 = \lceil \phi_2^{-1}(\phi_1(t_0)(1+\delta)) \rceil$, we have $t_1(n_0) \ge t_2(n_0) \ge t_0$ and, by (2.8),

$$t_2(n) < \hat{A}_{n_2(t_2(n))} = \hat{A}_n = \hat{A}_{n_1(t_1(n))} \le t_1(n)$$

or, equivalently,

$$\phi_2(n)\left[\frac{1}{1+\delta} - 1\right] < \phi_1(\hat{A}_n) - \phi_2(n) \le \phi_2(n)\left[\frac{1}{1-\varepsilon} - 1\right]$$

which implies (2.3). (b) If (2.4) holds, then

$$\tilde{t}_1(n) \equiv \phi_1^{-1}(\phi_2(n)(1-\varepsilon)) < \hat{A}_n \le \phi_1^{-1}(\phi_2(n)(1+\delta)) \equiv \tilde{t}_2(n)$$

for all $n \ge n_0$ and, by Lemma 2.1,

$$A(\tilde{t}_1(n)) < n \le A(\tilde{t}_2(n)) \text{ for all } n \ge n_0 \ . \tag{2.9}$$

Let $\tilde{n}_1$ and $\tilde{n}_2$ be functions of t defined by

$$\tilde{n}_1(t) = \lceil \phi_2^{-1}(\phi_1(t)/(1-\varepsilon)) \rceil \text{ and } \tilde{n}_2(t) = \lfloor \phi_2^{-1}(\phi_1(t)/(1+\delta)) \rfloor$$

and note that

$$\tilde{t}_2(\tilde{n}_2(t)) \le t \le \tilde{t}_1(\tilde{n}_1(t)) ,$$

so that, by (2.9),

$$\tilde{n}_2(t) \le A(\tilde{t}_2(\tilde{n}_2(t))) \le A(t) \le A(\tilde{t}_1(\tilde{n}_1(t))) < \tilde{n}_1(t)$$

and

$$\phi_2^{-1}(\phi_1(t)/(1+\delta)) - 1 \le A(t) \le \phi_2^{-1}(\phi_1(t)/(1-\varepsilon))$$

for all $t \ge t_0 \equiv \phi_1^{-1}(\phi_2(n_0)(1+\delta))$, because $\tilde{n}_1(t_0) \ge \tilde{n}_2(t_0) = n_0$, which implies (2.5) and (2.6) by the reasoning for part (a). For (2.7), choose the sequence $\{t_k\}$ so that $\phi_2^{-1}(\phi_1(t_k)/(1+\delta))$ is an integer. Then we have the lower bound $A(t)_k \ge \phi_2^{-1}(\phi_1(t_k)/(1+\delta))$ for all k, which implies (2.7). ■

We now apply Lemma 2.2 to characterize the asymptotic behavior.

Theorem 2.1. *Suppose that* ϕ_1 *and* ϕ_2 *are homeomorphisms of* $\mathbb{R}_+$ *and* $0 \le \lambda \le \infty$.

(a) If $\phi_2(A(t))/\phi_1(t) \to \lambda$ *as* $t \to \infty$, *then* $\phi_1(\hat{A}_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$.

(b) If $\phi_1(\hat{A}_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$, *then*

$$\overline{\lim_{t \to \infty}} \; \phi_2(A(t))/\phi_1(t) = \lambda . \tag{2.10}$$

(c) If, in addition to the condition for (b), either

$$\frac{\phi_2(A(t)+1) - \phi_2(A(t))}{\phi_1(t)} \to 0 \text{ as } t \to \infty \tag{2.11}$$

or

$$\frac{\phi_2(n+1)}{\phi_2(n)} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty \ , \tag{2.12}$$

then $\phi_2(A(t))/\phi_1(t) \rightarrow \lambda$ *as* $t \rightarrow \infty$.

(d) If $\phi_1(\hat{A}_n)/\phi_2(n) \rightarrow 0$ *as* $n \rightarrow \infty$ *and either*

$$\overline{\lim_{t\rightarrow\infty}} \ \frac{\phi_2(A(t)+1) - \phi_2(A(t))}{\phi_1(t)} < \infty \tag{2.13}$$

or

$$\underline{\lim_{n\rightarrow\infty}} \ \frac{\phi_2(n)}{\phi_2(n+1)} > 0 \ , \tag{2.14}$$

then $\phi_2(A(t))/\phi_1(t) \rightarrow \infty$ *as* $t \rightarrow \infty$.

Proof. (a) First suppose that $0 < \lambda < \infty$. Then incorporate λ into $\phi_1(t)$ by dividing by λ. The condition implies that for all appropriate ε and δ there exists t_0 such that (2.2) holds. By Lemma 2.2(a), (2.3) holds. Since ε and δ are arbitrary in (2.3), it implies the desired conclusion. To treat the cases $\lambda = 0$ and $\lambda = \infty$, use the one-sided bounds in Lemma 2.2. For example, if $\phi_2(A(t))/\phi_1(t) \rightarrow 0$ as $t \rightarrow \infty$, then for all positive ε and δ there exists t_0 such that $\phi_2(A(t))/\varepsilon\,\phi_1(t)) < 1 + \delta$ for all $t \geq t_0$. By Lemma 2.2(a), $\varepsilon\,\phi_1(\hat{A}_n)/\phi_2(n) > 1/(1+\delta)$ for all $n \geq n_0$. Since ε can be arbitrarily small, $\phi_1(\hat{A}_n)/\phi_2(n) \rightarrow \infty$ as $n \rightarrow \infty$. (b) Reason as in (a) using (2.4), (2.5) and (2.7). (c) Use (2.6), (2.11) and (2.12), noting that

$$\frac{1}{1-\varepsilon} - \frac{\phi_2(A(t)+1) - \phi_2(A(t))}{\phi_1(t)} \leq \frac{\phi_2(A(t))}{\phi_1(t)} \leq \frac{1}{1-\varepsilon} \tag{2.15}$$

and

$$\frac{\phi_2(A(t))}{\phi_2(A(t)+1)(1+\varepsilon)} \leq \frac{\phi_2(A(t))}{\phi_1(t)} \leq \frac{1}{1-\varepsilon} \ . \tag{2.16}$$

(d) Reason as in (c), using (2.13) and (2.14) with (2.15) and (2.16). ■

Remark 2.1. Note that $\phi_2(A(t))/\phi_1(t) \rightarrow \lambda$ as $t \rightarrow \infty$ if and only if $\phi_2(A(\phi_1^{-1}(t)))/t \rightarrow \lambda$

as $t \to \infty$; i.e., the spatial normalization $\phi_1(t)$ is equivalent to the standard normalizing function e after making a time transformation by ϕ^{-1} ■

Example 2.1. To see that an extra condition is needed in Theorem 2.1(c), let $\hat{A}_n = n$ for all n, so that $A(t) = \lfloor t \rfloor$ for all t. Also let $\phi_1(t) = \phi_2(t) = e^t$ for all t. Then $\phi_1(\hat{A}_n)/\phi_2(n) = 1$ for all n, while

$$\phi_2(A(t))/\phi_1(t) = e^{\lfloor t \rfloor - t} ,$$

which has limit supremum 1 and limit infimum e^{-1}. Also note that neither (2.11) nor (2.12) is satisfied.

Example 2.2. To see that the extra conditions in Theorem 2.1(c) are not necessary, let $\hat{A}_n = e^n$ for all n, so that $A(t) = \lfloor \log t \rfloor$. Let $\phi_2(t) = e^t$ and $\phi_1(t) = t$ for all t. Then $\phi_1(\hat{A}_n)/\phi_2(n) = 1$ for all n and

$$\frac{\phi_2(A(t))}{\phi_1(t)} = \frac{e^{\lfloor \log t \rfloor}}{t} \to 1 \text{ as } t \to \infty ,$$

but $\phi_2(n+1)/\phi_2(n) = e$ for all n and

$$\frac{\phi_2(A(t)+1) - \phi_2(A(t))}{\phi_1(t)} = \frac{(e-1)e^{\lfloor \log t \rfloor}}{t} \to e - 1 \text{ as } t \to \infty . \quad ■$$

A special case of interest is when the homeomorphisms are of the form $\phi(t) = t^p$ for $p > 0$.

Corollary. *Suppose that* $0 < p < \infty$ *and* $0 \le \lambda \le \infty$. *The following are equivalent:*

(i) $$\frac{A(t)}{t^p} \to \lambda \text{ as } t \to \infty , \qquad (ii) \ \frac{(A(t))^{1/p}}{t} \to \lambda^{1/p} \text{ as } t \to \infty ,$$

(iii) $$\frac{\hat{A}_n}{n^{1/p}} \to \lambda^{-1/p} \text{ as } n \to \infty , \qquad (iv) \ \frac{(\hat{A}_n)^p}{n} \to \lambda^{-1} \text{ as } n \to \infty .$$

Proof. Apply Theorem 2.1 with $\phi_2(t) = t$ and $\phi_1(t) = t^p$ to relate (i) and (iv). Note that (2.12) holds. To relate (i) and (ii), note that $(A(t))^{1/p}/t = (A(t)/t^p)^{1/p}$, and similarly for (iii) and (iv). ■

We used the property that $\phi(x/y) = \phi(x)/\phi(y)$ for $\phi(x) = x^p$ in the Corollary. The following classic lemma shows that this does not hold more generally.

Lemma 2.3. *A homeomorphism ϕ of $\mathbb{R}_+$ satisfies $\phi(xy) = \phi(x)\phi(y)$ for all nonnegative x and y if and only if $\phi(t) = t^p$ for some $p > 0$.*

Proof. The sufficiency is immediate. For the necessity, suppose that $\phi(xy) = \phi(x)\phi(y)$ for all nonnegative x and y. If we let $\psi(x) = \log \phi(e^x)$, then $\psi(x + y) = \psi(x) + \psi(y)$ for all real x and y. It is well known and easy to see that $\psi(x) = px$ for some real number p, which implies that $\phi(x) = e^{\psi(\log x)} = e^{p \log x} = x^p$. Since ϕ is strictly increasing, we must have $p > 0$. ■

The Corollary to Theorem 2.1 is very useful in queueing applications involving (1.5) and (1.6) because it enables us to replace $\phi_2(A(t))/\phi_1(t)$ and $\phi_1(\hat{A}_n)/\phi_2(n)$ by $A(t)/\phi_2^{-1}(\phi_1(t))$ and $\hat{A}_n/\phi_1^{-1}(\phi_2(n))$ respectively. The following lemma shows that we can do this more generally.

Lemma 2.4. *Suppose that $a_n \to \infty$ and $a_n/b_n \to 1$ as $n \to \infty$. If there is a t_0 such that $\log\phi(e^t)$ is uniformly continuous in (t_0, ∞), then $\phi(a_n)/\phi(b_n) \to 1$ as $n \to \infty$.*

Proof. Since $a_n \to \infty$ and $a_n/b_n \to 1$ as $n \to \infty$, $\log a_n - \log b_n \to 0$, $\log a_n \to \infty$ and $\log b_n \to \infty$ as $n \to \infty$. If $\log(\phi(e^t))$ is uniformly continuous in (t_0, ∞), then

$$\log \phi(e^{\log a_n}) - \log \phi(e^{\log b_n}) = \log \phi(a_n) - \log \phi(b_n)$$

$$= \log(\phi(a_n)/\phi(b_n)) \to 0 \quad \text{as } n \to \infty ,$$

so that $\phi(a_n)/\phi(b_n) \to 1$ as $n \to \infty$. ■

The following Corollary to Lemma 2.4 indicates how Lemma 2.4 can be applied in our context.

Corollary. *If $\phi_2(A(t))/\phi_1(t) \to \lambda$ as $t \to \infty$ and $\log \phi_2^{-1}(e^t)$ is uniformly continuous in (t_0, ∞) for some t_0, then $A(t)/\phi_2^{-1}(\lambda\phi_1(t)) \to 1$ as $t \to \infty$.*

Remark 2.2. Lemma 2.4 implies the Corollary to Theorem 2.1 because $\log \phi(e^t) = \log \lambda + pt$ when $\phi(t) = \lambda t^p$. Another function covered by Lemma 2.4 is $\phi(t) = a \log bt$; then $\log \phi(e^t)) = \log a + \log (\log b + t)$. However, $\log \phi(e^t) = \log a + be^t$ when $\phi(t) = ae^{bt}$, so that the uniform continuity does not hold when $\phi(t) = ae^{bt}$. ■

The following result is also useful to characterize the normalizing functions.

Lemma 2.5. *Suppose that* $0 < \lambda < \infty$ *and* $a_n \to \infty$ *as* $n \to \infty$. *If there is a* t_0 *such that* $\log \phi(e^t)$ *is uniformly continuous in* (t_0, ∞), *then*

$$\overline{\lim_{n \to \infty}} \left| \frac{\phi(a_n)}{\phi(\lambda a_n)} \right| < \infty .$$

Proof. Recall that if a function ψ is uniformly continuous in (t_0, ∞), then

$$\sup \{ |\psi(t + x) - \psi(t)| : t \geq t_0 \} < \infty$$

for any positive *x*. Since

$$\log \lambda a_n - \log a_n = \log \lambda ,$$

$$\overline{\lim_{n \to \infty}} \{ |\log \phi(e^{\log \lambda a_n}) - \log \phi(e^{a_n})| \}$$

$$= \overline{\lim_{n \to \infty}} \{ |\log \phi(\lambda a_n) - \log \phi(a_n)| \}$$

$$= \overline{\lim_{n \to \infty}} \{ |\log(\phi(\lambda a_n)/\phi(a_n))| < \infty ,$$

which implies the desired conclusion. ■

We are thinking of $\{\hat{A}_n : n \geq 1\}$ being the points in a point process sample path, so it is natural to assume that $\{\hat{A}_n\}$ is nondecreasing. However, we could start with a general sequence of real numbers $\{X_n : n \geq 1\}$ and obtain $\{\hat{A}_n\}$ as the successive maxima, i.e.,

$$\hat{A}_n = \max\{X_k : 0 \le k \le n\}\ ,\ n \ge 1\ , \tag{2.17}$$

where $X_0 = 0$. A similar result holds for $A(t)$ defined is

$$A(t) = \sup\{0, X(s), 0 \le s \le t\}\ , \tag{2.18}$$

$t \ge 0$, where $\{X(t) : t \ge 0\}$ is a right-continuous integer-valued process with left limits.

Proposition 2.2. *Suppose that* ϕ_1 *and* ϕ_2 *are two homeomorphisms of* $\mathbb{R}_+$ *and* $0 \le \lambda \le \infty$. *If* $\phi_1(X_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$, *then* $\phi_1(\hat{A}_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$ *for* $\hat{A}_n$ *in (2.17).*

Proof. First assume that $0 < \lambda < \infty$. Given the assumed convergence, for all $\varepsilon > 0$, there is an n_0 such that

$$\phi_1^{-1}(\phi_2(n)/\lambda(1+\varepsilon)) \le X_n \le \phi_1^{-1}(\phi_2(n)/\lambda(1-\varepsilon)) \ \text{for all}\ \ n \ge n_0\ ,$$

which implies

$$\phi_1^{-1}(\phi_2(n)/\lambda(1+\varepsilon)) \le \hat{A}_n \le \max\{\hat{A}_{n_0}\ ,\ \phi_1^{-1}(\phi_2(n)/\lambda(1-\varepsilon))\}\ \text{for all}\ n \ge n_0\ .$$

Let n_1 be such that

$$\phi_1^{-1}(\phi_2(n)/\lambda(1-\varepsilon)) \ge \hat{A}_{n_0}\ .$$

Then, for all $n \ge n_1$,

$$\frac{1}{\lambda(1+\varepsilon)} \le \frac{\phi_1(\hat{A}_n)}{\phi_2(n)} \le \frac{1}{\lambda(1-\varepsilon)}\ ,$$

which implies the conclusion. For $\lambda = 0$ and $\lambda = \infty$ use associated one-sided inequalities. ■

3. SLLNs for Queueing Processes

In this section we investigate nonstationary analogs of the queueing relation $L = \lambda W$. Consider the deterministic framework based on a sequence of ordered pairs of real numbers $\{(\hat{A}_k, \hat{D}_k) : k \ge 1\}$, where $0 \le \hat{A}_k \le \hat{A}_{k+1}$ and $\hat{A}_k \le \hat{D}_k$ for all k. We interpret $\hat{A}_k$ as the k^{th} arrival epoch and $\hat{D}_k$ as the departure epoch of the k^{th} arrival, so that $W_k \equiv \hat{D}_k - \hat{A}_k$ is the time spent by customer k in the system. It is important to note that *the service discipline makes a big difference with unstable queues.* For example, with the last-come first-served discipline, many early customers might never depart. Hence, we consider only the FIFO discipline, under which, $D_k \le D_{k+1}$ for all k. However, the results apply to other disciplines if we just interpret $\hat{D}_k$ as the k^{th} departure epoch.

As in Section 1, we state our conditions in terms of the departure epochs $\hat{D}_k$, which are not primitive model elements. However, in unstable situations, limits for the service times usually translate easily into limits for the departure times.

Limits for the sequences $\{\hat{A}_n\}$ and $\{\hat{D}_n\}$ can be related to limits for the associated arrival and departure counting processes $\{A(t)\}$ and $\{D(t)\}$ defined by (2.1) as indicated in Section 2. These limits in turn easily imply limits for $Q(t)$ and W_n via (1.5) and (1.6). We state two sample results.

Proposition 3.1. *Suppose that the discipline is FIFO,* $0 \le \lambda \le \infty$, $0 \le \mu \le \infty$ *and* $0 < q \le p < \infty$.

(a) The limits

$$\frac{A(t)}{t^p} \to \lambda \ \ \textit{and} \ \ \frac{D(t)}{t^q} \to \mu \ \textit{as} \ t \to \infty \, , \tag{3.1}$$

hold if and only if the limits

$$\frac{\hat{A}_n}{n^{1/p}} \to \lambda^{-1/p} \ \ \textit{and} \ \ \frac{\hat{D}_n}{n^{1/q}} \to \mu^{-1/q} \ \ \textit{as} \ \ n \to \infty \tag{3.2}$$

hold.

(b) If the limits in (a) *hold with* $\mu < \infty$ *and* $\lambda > 0$ *when* $p = q$, *then*

$$\frac{Q(t)}{t^p} \to L \quad as\ t \to \infty\ , \tag{3.3}$$

$$t^{-(1+p)} \int_0^t Q(s)\, ds \to L/(1+p) \quad as\ t \to \infty\ , \tag{3.4}$$

$$\frac{W_n}{n^{1/q}} \to W \quad as\ n \to \infty\ , \tag{3.5}$$

$$n^{-(1+q^{-1})} \sum_{k=1}^{n} W_k \to W/(1+q^{-1}) \quad as\ n \to \infty\ . \tag{3.6}$$

If $p > q$, *then* $L = \lambda$ *and* $W = \mu^{-1/q}$. *If* $p = q$, *then* $L = \lambda - \mu$ *and* $W = \mu^{-1/p} - \lambda^{-1/p}$.

Proof. (a) Apply the Corollary to Theorem 2.1. (b) The limits (3.3) and (3.5) follow from (1.5), (1.6), (3.1) and (3.2). To see that (3.3) implies (3.4), first suppose that $0 < L < \infty$ and then note that (3.1) implies that, for all $\varepsilon > 0$, there exists t_0 such that

$$(1-\varepsilon) L t^p \leq Q(t) \leq (1+\varepsilon) L t^p \text{ for } t \geq t_0\ . \tag{3.7}$$

Hence,

$$(1-\varepsilon) \frac{L(t^{p+1} - t_0^{p+1})}{p+1} \leq \int_0^t Q(s)\, ds \leq \int_0^{t_0} Q(s)\, ds + \frac{(1+\varepsilon) L(t^{p+1} - t_0^{p+1})}{p+1}$$

$$\leq t_0 A(t_0) + \frac{(1+\varepsilon) L(t^{p+1} - t_0^{p+1})}{p+1}\ ,$$

so that

$$\frac{(1-\varepsilon)L}{p+1} \leq \underline{\lim}_{t \to \infty} t^{-(p+1)} \int_0^t Q(s)\, ds \leq \overline{\lim}_{t \to \infty} t^{-(p+1)} \int_0^t Q(s)\, ds \leq \frac{(1+\varepsilon)L}{p+1}\ . \tag{3.8}$$

Since ε was arbitrary, we can let $\varepsilon \to 0$ in (3.8) to complete the proof. When $L = 0$ or $L = \infty$, the argument is similar, using one-sided relations; e.g., when $L = 0$ instead of (3.7) we have $Q(t) \leq \varepsilon t^p$ for $t \geq t_0$. Similarly, the limit (3.5) implies (3.6). The argument is slightly more complicated, involving Riemann sum approximations of integrals. It is Lemma 4 of Glynn and Whitt (1992). ■

Remark 3.1. In the setting of Proposition 3.1, we can also consider the queue length seen by the n^{th} arrival, say Q_n. Given (3.2) and (3.3),

$$\frac{Q_n}{n} = \frac{Q(\hat{A}_n -)}{(\hat{A}_n -)^p} \; \frac{(\hat{A}_n -)^p}{n} \to \lambda^{-1} L \quad \text{as } n \to \infty \; .$$

If $p > q$, then $\lambda^{-1} L = 1$ and, if $p = q$, then $\lambda^{-1} L = 1 - (\mu/\lambda)$.

Proposition 3.2. *Suppose that the discipline is* FIFO and $0 < \mu \le \lambda < \infty$. *Let* ϕ *be a homeomorphism of* $\mathbb{R}_+$ *such that* $\log \phi^{-1}(e^t)$ *is uniformly continuous in* (t_0, ∞) *for some* t_0. *If*

$$\frac{A(t)}{\phi(t)} \to \lambda \text{ and } \frac{D(t)}{\phi(t)} \to \mu \;\; as \;\; t \to \infty \; , \tag{3.9}$$

then

$$\frac{\hat{A}_n}{\phi^{-1}(n/\lambda)} \to 1 \text{ and } \frac{\hat{D}_n}{\phi^{-1}(n/\mu)} \to 1 \;\; as \;\; n \to \infty \; , \tag{3.10}$$

and

$$W_n = \phi^{-1}(n/\mu) - \phi^{-1}(n/\lambda) + o(\phi^{-1}(n)) \;\; as \;\; n \to \infty \; . \tag{3.11}$$

Proof. By Theorem 2.1(a), (3.9) implies that

$$\frac{\phi(\hat{A}_n)}{n} \to \lambda^{-1} \text{ and } \frac{\phi(\hat{D}_n)}{n} \to \mu^{-1} \;\; as \;\; n \to \infty \; ,$$

which in turn implies (3.10) by Lemma 2.4. By (1.6) and Lemma 2.5, (3.10) implies (3.11). ■

Remark 3.2. The results above and the CLTs below also hold for many non-FIFO systems too. If $\hat{D}_n'$ denotes the n^{th} departure time while $\hat{D}_n$ denotes the departure time of the n^{th} arrival, then in addition to (3.2) for $\hat{D}_n'$ instead of $\hat{D}_n$, it suffices to show that $\hat{D}_n - \hat{D}_n'$ is asymptotically negligible after normalization. For example, consider the standard s-server model with the first-come first-served discipline. We can write $\hat{D}_n = \hat{A}_n + T_n + S_n$ where T_n is the delay before beginning service and S_n is the service time. The key is the delay before beginning service. As in Sections 5-7 of Iglehart and Whitt (1970a), the delay can often be related first to the virtual waiting time, then the total workload process and finally the queue length process. ■

4. CLTs for Nonstationary Counting Processes and Their Inverses

In this section we relate CLTs for normalized versions of the processes $\{A(t)\}$ and $\{\hat{A}_n\}$. For this purpose, we use the following elementary lemma, which we state without proof.

Lemma 4.1. *Let $\{F_n : n \geq 1\}$ be a sequence of cdf's and let x be a continuity point of a cdf F. Then $F_n(x_n) \to F(x)$ as $n \to \infty$ for all sequence of real numbers $\{x_n : n \geq 1\}$ with $x_n \to x$ as $n \to \infty$ if the convergence holds for one such sequence.*

The following result is a CLT analog of the SLLN in Theorem 2.1. This result extends Theorem 6 of Glynn and Whitt (1988), which covers the linear case. We note that the bounding argument used to treat discontinuous limiting cdf's in the proof there is actually not needed.

Theorem 4.1. *Let ϕ_1 and ϕ_2 be homeomorphisms of $\mathbb{R}_+$ and let ψ be a nondecreasing continuous positive real-valued function for which*

$$\psi(t)/\psi(t + x\psi(t)) \to 1 \text{ as } t \to \infty \qquad (4.1)$$

for all x.

(a) If

$$X(t) \equiv [\phi_2(A(t)) - \phi_1(t)]/\psi(\phi_1(t)) \Rightarrow X \text{ as } t \to \infty\ , \qquad (4.2)$$

then

$$Y(n) \equiv [\phi_1(\hat{A}_n) - \phi_2(n)]/\psi(\phi_2(n)) \Rightarrow -X \text{ as } n \to \infty\ , \qquad (4.3)$$

(b) If (4.3) holds, then there exists an increasing sequence of positive real numbers $\{t_n : n \geq 0\}$ with $t_n \to \infty$ as $n \to \infty$ such that $X(t_n) \Rightarrow X$ as $n \to \infty$ for $X(t)$ in (4.2).

(c) If (4.3) holds with

$$\frac{\phi_2(n+1) - \phi_2(n)}{\psi(\phi_2(n))} \to 0 \text{ as } n \to \infty\ , \qquad (4.4)$$

and

$$\psi(\phi_2(n+1))/\psi(\phi_2(n)) \to 1 \text{ as } n \to \infty , \tag{4.5}$$

then (4.2) holds.

Proof. (a) If (4.2) holds, then

$$F_t(x-) \equiv P(X(t) < x) \to P(X < x) \equiv F(x) \tag{4.6}$$

as $t \to \infty$ for all continuity points x of F. Note that

$$\begin{aligned} F_t(x-) &= P([\phi_2(A(t)) - \phi_1(t)]/\psi(\phi_1(t)) < x) \\ &= P(\phi_2(A(t)) < \phi_1(t) + x\psi(\phi_1(t))) \\ &= P(A(t) < \phi_2^{-1}(\phi_1(t) + x\psi(\phi_1(t))) \end{aligned}$$

so that, by Lemma 2.1, $F_t(x) = P(\hat{A}_{n(t)} > t)$ for any t such that

$$n(t) \equiv \phi_2^{-1}(\phi_1(t) + x\psi(\phi_1(t)))$$

is an integer. For such t,

$$F_t(x) = P([\phi_1(\hat{A}_{n(t)}) - \phi_2(n(t))]/\psi(\phi_2(n(t)) > -x(t)) ,$$

where

$$\begin{aligned} x(t) &= -[\phi_1(t) - \phi_2(n(t))]/\psi(\phi_2(n(t)) \\ &= x\psi(\phi_1(t))/\psi((\phi_1(t) + x\psi(\phi_1(t)) \to x \text{ as } t \to \infty \end{aligned}$$

by (4.1). Note that, for each positive integer n sufficiently large, we can find t_n such that $n(t_n) = n$, because ϕ_1, ϕ_2 and ψ are nondecreasing and continuous, and $n(t) \to \infty$ as $t \to \infty$. Hence,

$$G_n(x_n) \equiv P(-[\phi_1(\hat{A}_n) - \phi_2(n)]/\psi(\phi_2(n)) < x_n) = F_{t_n}(x_n) \tag{4.7}$$

where $x_n = x(t_n) \to x$ as $n \to \infty$. Since x is a continuity point of F, $G_n(x) \to F(x)$ as $n \to \infty$ as well by Lemma 4.1. Hence, $Y(n) \Rightarrow -X$.

(b) The argument is similar. Note that for G_n defined in (4.7).

$$G_n(x) = P(\hat{A}_n > \phi_1^{-1}(\phi_2(n) - x\psi(\phi_2(n))) = P(A(t_n) < n)$$

for

$$t_n = \phi_1^{-1}(\phi_2(n) - x\psi(\phi_2(n))) \qquad (4.8)$$

by Lemma 2.1. Thus, for F_t in (4.6) and t_n in (4.8), $F_{t_n}(x_n-) = G_n(x)$ for

$$\begin{aligned} x_n &= [\phi_2(n) - \phi_1(t_n)]/\psi(\phi_1(t_n)) \\ &= x\psi(\phi_2(n))/\psi(\phi_2(n) - x\psi(\phi_2(n))) \to x \text{ as } n \to \infty \end{aligned}$$

by (4.1). If $G_n(x) \to F(x)$, where x is a continuity point of F, then $F_{t_n}(x_n) \to F(x)$ and $F_{t_n}(x) \to F(x)$ as $n \to \infty$, by Lemma 4.1. Hence $X(t_n) \Rightarrow X$, as claimed.

(c) For any t, let n be such that $t_n \le t < t_{n+1}$ for t_n in (4.8). Since $A(t_n) \le A(t) \le A(t_{n+1})$, it suffices to show that $A(t_n)$ and $A(t_{n+1})$ have the same limits with the same normalization; i.e., it suffices to show that

$$\frac{\phi_2(A(t_n)) - \phi_1(t_{n+1})}{\psi(\phi_1(t_{n+1}))} \Rightarrow X ,$$

which in turn holds if

$$\frac{\psi(\phi_1(t_{n+1}))}{\psi(\phi_1(t_n)} \to 1 \quad \text{as} \quad n \to \infty \qquad (4.9)$$

and

$$\frac{\phi_1(t_{n+1}) - \phi_1(t_n)}{\psi(\phi_1(t_n))} \to 0 \quad \text{as} \quad n \to \infty \qquad (4.10)$$

By (4.1) and (4.8), (4.9) is equivalent to (4.5). By (4.8) and (4.9), (4.10) is equivalent to (4.4). ■

Remark 4.1. The standard stationary case is covered as the special case of Theorem 4.1 in which

$X = N(0,1)$, $\psi(t) = \lambda\sigma\sqrt{t}$, $\phi_1(t) = \lambda t$ and $\phi_2(t) = t$ for all t, where λ and σ are positive constants. In the nonstationary case, we typically will have $X = N(0,1)$ and $\psi(t) = c\sqrt{t}$ for some constant c as well. If, as in Example 1.2, $A(t)$ is a nonhomogeneous Poisson process with deterministic rate function $\lambda(t)$, such that $\phi(t) \equiv \int_0^t \lambda(s)\,ds \to \infty$ as $t \to \infty$, then (4.2) holds with $X = N(0,1)$, $\psi(t) = \sqrt{t}$, $\phi_1(t) = \phi(t)$ and $\phi_2(t) = t$ for all t. Note that $\psi(t) = t^p$ satisfies (4.1) for all $p < 1$, but not for $p \geq 1$.

Remark 4.2. Given that $\psi(t) = t^p$ for $0 < p < 1$, so that (4.1) holds, if $\phi_2(t) = t^q$, then (4.5) always holds, but

$$\frac{\phi_2(n+1) - \phi_2(n)}{\psi(\phi_2(n))} = \frac{(n+1)^q - n^q}{n^{pq}} = \frac{O(n^{q-1})}{n^{pq}} , \tag{4.11}$$

so that (4.4) holds if and only if $p > (q-1)/q$. Hence, (4.4) *always holds for* $q \leq 1$, *but (4.4) does not hold for* $q = 2$ *and* $p = 1/2$.

Remark 4.3. If $\psi(t)/t \to 0$ as $t \to \infty$, then the limits in (4.2) and (4.3) imply

$$\frac{\phi_2(A(t))}{\phi_1(t)} \Rightarrow 1 \ as \ t \to \infty$$

and

$$\frac{\phi_1(\hat{A}_n)}{\phi_2(n)} \Rightarrow -1 \text{ as } n \to \infty ,$$

respectively. These can be regarded as associated weak laws of large numbers (WLLNs).

Example 4.1. To see that, we can have $X(t_n) \Rightarrow X$ as $n \to \infty$ in Theorem 4.1(b) for a sequence $\{t_n\}$ without having $X(t) \Rightarrow X$ in Theorem 4.1(c), let $\phi_1(t) = t$, $\phi_2(t) = t^2$ and $\psi(t) = 1$ for $t \geq 0$. Then (4.1) and (4.5) hold, but (4.4) does not. Let $P(X = 0) = 1$ and let $\hat{A}_n = n^2$, so that

$$\frac{\phi_1(\hat{A}_n) - \phi_2(n)}{\psi(\phi_2(n))} = 0 = -X \text{ w.p.1 for } all\ n .$$

However, $A(t) = \sqrt{\lfloor t \rfloor}$ and $\phi_2(A(t)) = \lfloor t \rfloor$, so that

$$\frac{\phi_2(A(t)) - \phi_1(t)}{\psi(\phi_1(t))} = \lfloor t \rfloor - t ,$$

from which we see that

$$\underline{\lim}_{t \to \infty} \frac{\phi_2(A(t)) - \phi_1(t)}{\psi(\phi_1(t))} = -1 < 0 = \overline{\lim}_{t \to \infty} \frac{\phi_2(A(t)) - \phi_1(t)}{\psi(\phi_1(t))} . \ \blacksquare$$

Paralleling the Corollary to Theorem 2.1, we next establish an equivalence between CLTs without spatial transformations when the growth rate is λt^p.

Theorem 4.2. *Suppose that* $0 < \lambda < \infty$, $0 < p < \infty$ *and* $0 < q < 1$. *Then*

$$X(t) \equiv \frac{A(t) - \lambda t^p}{t^{pq}} \Rightarrow X \quad as \quad t \to \infty , \tag{4.12}$$

if and only if

$$Y(n) \equiv \frac{\hat{A}_n - (n/\lambda)^{1/p}}{n^{1/p - (1-q)}} \Rightarrow \frac{-X}{p\lambda^{q+1/p}} \quad as \quad n \to \infty . \tag{4.13}$$

Proof. The beginning of the proof is just like the proof of Theorem 4.1. If (4.12) holds, then

$$F_t(x-) \equiv P(X(t) < x) \to P(X \le x) \equiv F(x)$$

for all continuity points *x* of *F*. Note that, by Lemma 2.1,

$$\begin{aligned} F_t(x-) &= P(A(t) < n(t)) = P(\hat{A}_{n(t)} > t) \\ &= P\left[\frac{\hat{A}_{n(t)} - (n(t)/\lambda)^{1/p}}{n(t)^{1/p-(1-q)}} > \frac{t - (n(t)/\lambda)^{1/p}}{n(t)^{1/p-(1-q)}}\right] \end{aligned}$$

for

$$n(t) = \lceil \lambda t^p + t^{pq} x \rceil . \tag{4.14}$$

From (4.14), note that it is possible to choose an increasing sequence $\{t_n\}$ such that $n(t_n) = n$ for all n sufficiently large. Hence, as $n \to \infty$,

$$P\left[\frac{\hat{A}_n - (n/\lambda)^{1/p}}{n^{1/p-(1-q)}} > \frac{t_n - (n(t_n)/\lambda)^{1/p}}{n(t_n)^{1/p-1-q}} \right] \to F(x) ,$$

but

$$x_n \equiv \frac{t_n - (n(t_n)/\lambda)^{1/p}}{n(t_n)^{1/p-(1-q)}} = \frac{t_n - (t_n^p + t_n^{pq} x \lambda^{-1})^{1/p}}{(\lambda t_n^p + t_n^{pq} x)^{1/p-(1-p)}} . \tag{4.15}$$

By doing a Taylor series expansion, we see that the numerator of (4.15) has the asymptotic form

$$t - (t^p + t^{pq} x \lambda^{-1})^{1/p} = t(1 - (1 + t^{-p(1-q)} x \lambda^{-1})^{1/p}$$

$$= \frac{-x t^{1-p(1-q)}}{\lambda p} + o(t^{1-p(1-q)}) \quad as \quad t \to \infty ,$$

so that

$$x_n \to \frac{-x}{p\ \lambda^{(1/p)-q}} \quad as \quad n \to \infty .$$

Hence, $-p\ \lambda^{1/p+q}\ Y(n) \Rightarrow X$ as $n \to \infty$, which is equivalent to the desired conclusion (4.13). As in the proof of Theorem 4.1, the argument the other way is similar, yielding $X(t_n) \Rightarrow X$ for the sequence

$$t_n = (n/\lambda)^{1/p} + n^{(1/p)-(1-q)} x . \tag{4.16}$$

Next choose $t_n \le t \le t_{n+1}$. As in the proof of Theorem 4.1, since $A(t_n) \le A(t) \le A(t_{n+1})$, it suffices to have

$$\frac{\lambda\ t_{n+1}^p - \lambda\ t_n^p}{t_n^{pq}} \to 0 \quad as \quad n \to \infty \tag{4.17}$$

and

$$\frac{t_{n+1}^{pq}}{t_n^{pq}} \to 1 \quad as \quad n \to \infty \ . \tag{4.18}$$

From (4.16), it follows that $t_n / n^{1/p} \to \lambda^{1/p}$ as $n \to \infty$, which implies (4.18). We now establish (4.17). Note that

$$\begin{aligned}\lambda\, t_{n+1}^p &= ((n+1)^{1/p} + (n+1)^{1/p-(1-q)} x\, \lambda^{1/p})^p \\ &= (n+1)(1 + (n+1)^{-(1-q)} x\, \lambda^{1/p})^p \\ &= (n+1)(1 + p\, x\, \lambda^{1/p} (n+1)^{-(1-q)} + o(n+1)^{-(1-q)}) \ .\end{aligned}$$

Since $1-q > 0$, $\lambda\, t_{n+1}^p - \lambda\, t_n^p < 2$ for all n sufficiently large, which implies (4.17), since $t_n^{pq} \to \infty$ as $n \to \infty$. ■

5. FCLTs for Nonstationary Counting Processes and Their Inverses

In §4 we established a CLT version of the SLLN-equivalence in Theorem 2.2. Now we establish a FCLT version. The FCLT version is of interest, in part, because we obtain a stronger duality than in Theorem 4.1; i.e., we do not need the extra conditions in Theorem 4.1(c). In fact, the result here is an easy application of Section 7 of Whitt (1980).

We thus begin by reviewing the more general framework in Section 7 of Whitt (1980); it is also used in Glynn and Whitt (1989). We consider nondecreasing right-continuous nonnegative real-valued functions x on $\mathbb{R}_+$ with $x(t) \to \infty$ as $t \to \infty$. For such functions, we define an *inverse function* x^{-1} by

$$x^{-1}(t) = \inf\{s \ge 0 : x(s) > t\} \ , \ t \ge 0 \ . \tag{5.1}$$

Our point process setting is covered as the special case in which $x(t) = \hat{A}_{\lfloor t \rfloor}$ with $\hat{A}_0 = 0$ and $x^{-1}(t) = A(t) + 1$ for $t \ge 0$.

This framework is easily understood by considering the *completed graph* Γ_x in $\mathbb{R}_+^2$ associated with x, i.e.,

$$\Gamma_x = \{(0,x) : x \le x(0)\} \cup \{(t,x) : x(t-) \le x \le x(t)\} . \tag{5.2}$$

Then

$$\Gamma_{x^{-1}} = \{(x,t) : (t,x) \in \Gamma_B\} . \tag{5.3}$$

Given Γ_x, we construct x as the unique right-continuous function with graph Γ_x. Since $\Gamma_{(x^{-1})^{-1}} = \Gamma_x$, we see that $(x^{-1})^{-1} = x$.

However, we do not necessarily have $x^{-1}(x(t)) = t$. We do have $x^{-1}(x(t)) \ge t$. Moreover, if $x(s) \le t$, then $x^{-1}(t) \ge s$, and if $x(s) > t$, then $x^{-1}(t) \le s$. These relations serve as the basis for extensions of Theorem 2.1 and 2.2.

For the functional limit theorems of interest, we consider the function space $D[0,\infty)$ endowed with the Skorohod (1956) M_1 topology; see Pomarede (1976) and Section 7 of Whitt (1980). As with the usual Skorohod (1956) J_1 topology in Billingsley (1968), at continuous limit functions convergence in the M_1 topology is equivalent to uniform convergence over bounded intervals. Convergence in distribution of random elements in $D[0,\infty)$ is understood to be weak convergence of the probability measures on this function space, as in Billingsley (1968). As in Whitt (1980), the following result covers all forms of functional limit theorems (e.g., FWLLNs and FSLLNs) as well as FCLTs. So before, let e be the *identity map* in D, i.e., $e(t) = t$ for $t \ge 0$, and let Let +denote composition, i.e., $(x+y)(t) = x(y(t))$.

Theorem 5.1. *Let* $\{x_n : n \ge 1\}$ *be a sequence of nondecreasing right-continuous nonnegative real-valued functions that are unbounded above; let* $\{\phi_{1n} : n \ge 1\}$ *and* $\{\phi_{2n} : n \ge 1\}$ *be two sequences of homeomorphisms of* $\mathbb{R}_+$*; and let* $\{c_n\}$ *be a sequence of scalars with* $c_n \to \infty$ *as* $n \to \infty$*. Then*

$$c_n(\phi_{2n} \circ x_n \circ \phi_{1n}^{-1} - e) \to z \quad \textit{as } n \to \infty \quad \textit{in} \quad (D[0,\infty), M_1)$$

if and only if

$$c_n(\phi_{1n} \circ x_n^{-1} \circ \phi_{2n}^{-1} - e) \to -z \quad \textit{as } n \to \infty \ \textit{in} \quad (D[0,\infty), M_1) \ .$$

Proof. Apply Theorem 7.5 of Whitt (1980) after noting (e.g., Lemma 7.6 in Whitt (1980)) that $(\phi_{2n} \circ x_n \circ \phi_{1n}^{-1})^{-1} = \phi_{1n} \circ x_n^{-1} \circ \phi_{2n}^{-1}$. ∎

We now apply Theorem 5.1 to obtain a specific relation among FCLTs for stochastic processes with the general nondecreasing sample paths specified in the beginning of this section.

Theorem 5.2. *Let* $\{B(t) : t \geq 0\}$ *be a stochastic process with sample paths that are nondecreasing, nonnegative, unbounded above and in* $D[0,\infty)$; *let* ϕ_1 *and* ϕ_2 *be homeomorphisms of* $\mathbb{R}_+$; *and let* ψ *be a function on* $\mathbb{R}_+$ *such that* $t/\psi(t) \to \infty$ *as* $t \to \infty$. *Then*

$$\frac{\phi_2(B(\phi_1^{-1}(st))) - st}{\psi(s)} \Rightarrow X(t) \ \text{in} \ (D[0,\infty), M_1) \ \text{as} \ s \to \infty \tag{5.7}$$

if and only if

$$\frac{\phi_1(B^{-1}(\phi_2^{-1}(st))) - st}{\psi(s)} \Rightarrow -X(t) \ \text{in} \ (D[0,\infty), M_1) \ \text{as} \ s \to \infty \ , \tag{5.8}$$

in which case they hold jointly.

Proof. Apply Theorem 5.1, setting $x_n(t) = B(s_n t)/s_n$, $\phi_{in}(t) = \phi_i(s_n t)/s_n$ and $c_n = s_n/\psi(s_n)$ for any sequence $\{s_n : n \geq 1\}$ with $s_n \to \infty$ as $n \to \infty$. Note that $x_n^{-1}(t) = B^{-1}(s_n t)/s_n$ and $\phi_{in}^{-1}(t) = \phi_i^{-1}(s_n t)/s_n$. By above,

$$\frac{\phi_2(B(\phi_1^{-1}(s_n t))) - s_n t}{\psi(s_n)} = c_n((\phi_{2n} \circ x_n \circ \phi_{1n}^{-1})(t) - t) \ , \ t \geq 0 \ ,$$

and

$$\frac{\phi_1(B(\phi_2(s_n t))) - s_n t}{\psi(s_n)} = c_n((\phi_{1n} \circ x_n \circ \phi_{2n}^{-1})(t) - t) \ , \ t \geq 0 \ ,$$

where $c_n = s_n/\psi(s_n)$. As discussed in Section 1 of Whitt (1980), use the Skorohod representation theorem to convert the stated convergence in distribution to the w.p.1 convergence needed in Theorem 5.1. ■

We now show that these FCLTs provide the CLTs in §4.

Corollary 1. *Under the conditions of Theorem 5.1, if either of the limits in (5.7) or (5.8) hold and if* $P(X(1-) = X(1)) = 1$, *then*

$$\frac{\phi_2(B(t)) - \phi_1(t)}{\psi(\phi_1(t))} \Rightarrow X(1) \text{ in } \mathbb{R} \text{ as } t => \infty \tag{5.9}$$

and

$$\frac{\phi_1(B^{-1}(t)) - \phi_2(t)}{\psi(\phi_2(t))} \Rightarrow -X(1) \text{ in } \mathbb{R} \text{ as } t \to \infty \ . \tag{5.10}$$

Proof. By the continuous mapping theorem with the projection map, we obtain

$$\frac{\phi_2(B(\phi_1^{-1}(t)) - t}{\psi(t)} \Rightarrow X(1) \text{ in } \mathbb{R} \text{ as } t \to \infty$$

from (5.7). Then, by a time change, we obtain (5.9). A similar argument applies with (5.8) to yield (5.10). ■

Remark 5.1. Recall that (5.1) makes $A(t) + 1$ the inverse of $\hat{A}_{\lfloor t \rfloor}$. Hence, Corollary 1 to Theorem 5.2 does not apply directly to the counting process $A(t)$. In addition, we need

$$\frac{\phi_2(A(t) + 1)) - \phi_2(A(t))}{\psi(\phi_1(t))} \Rightarrow 0 \text{ as } t \to \infty \tag{5.11}$$

Of course, the limit (5.11) always holds when $\phi_2(t) = t$ and $\psi(t) \to \infty$ as $t \to \infty$. ■

We now note that we can deduce something about the limit process from the convergence in Theorem 5.2. Recall that a positive monotone function ψ is *regularly varying* at infinity if $\psi(ct)/\psi(t) \to \xi(c)$ as $t \to \infty$, in which case $\xi(c) = c^p$ for some $p > 0$; see p. 275 of Feller

(1971).

Lemma 5.1. *If* $Y(st)/\psi(s) \Rightarrow X(t)$ *in* $\mathbb{R}$ *as* $s \to \infty$ *for each* t *and* $\psi(sc)/\psi(s) \to c^p$ *as* $s \to \infty$ *for* $p > 0$, *then*

$$X(ct) \stackrel{d}{=} c^p X(t) \ . \tag{5.12}$$

Proof. Replace t by ct and s by cs to obtain

$$\frac{Y(sct)}{\psi(s)} \Rightarrow X(ct) \text{ and } \frac{\psi(cs)\,Y(sct)}{\psi(s)\,\psi(cs)} \Rightarrow c^p X(t) \ . \ \blacksquare$$

Remark 5.2 Of course, usually $\psi(t) = \sqrt{t}$ so that $p = 1/2$. Moreover, often X is Brownian motion, which is well known to satisfy (5.12) with $p = 1/2$. $\blacksquare$

The following is an elementary consequence of Theorem 5.2, which we will apply in Section 6.

Corollary 2. *If* $P(X(t-) = X(t)) = 1$ *for all* t *and*

$$\frac{\hat{A}_{\lfloor nt \rfloor} - \lambda^{-1} nt}{\sqrt{n}} \Rightarrow X(t) \ \text{ in } (D[0,\infty), M_1) \text{ as } n \to \infty , \tag{5.13}$$

Then

$$\left[\frac{A(nt) - nt}{\sqrt{n}} , \frac{\hat{A}_{\lfloor \lambda nt \rfloor} - nt}{\sqrt{n}} , \frac{\hat{A}_{\lfloor \mu nt \rfloor} - \mu nt/\lambda}{\sqrt{n}} \right]$$

$$\Rightarrow (-\lambda X(\lambda t) , X(\lambda t), X(\mu t)) \text{ in } (D[0,\infty), M_1)^3 \text{ as } n \to \infty$$

Proof. Let $B(t) = A_{\lfloor t \rfloor}$, $\phi_1(t) = \lambda t$, $\phi_2(t) = t$ and $\psi(t) = \sqrt{t}$ in Theorem 5.2. Note that $B^{-1}(t) = A(t) + 1$, but the difference is asymptotically negligible; apply Theorem 4.1 of Billingsley (1968). $\blacksquare$

6. CLTs for Unstable Nonstationary Queues

The CLTs and FCLTs for nonstationary counting processes and their inverses have easy applications to unstable nonstationary queues. We illustrate by stating one CLT; the corresponding FCLT is similar. Unfortunately, however, the nonlinear functions ϕ_2 and ϕ_1 in (4.2) and (4.3) do not combine well with the linear relations (1.5) and (1.6). We obtain useful results for $Q(t)$ and W_n by assuming at least one of ϕ_i is the identity.

Proposition 6.1. *Consider a queue with the FIFO discipline as in §3. Let* ϕ_a, ψ_a, ϕ_d *and* ψ_d *be homeomorphisms of* $\mathbb{R}_+$. *If* $\psi_d(t)/\psi_a(t) \to \gamma < \infty$ *and*

$$\left[\frac{A(t) - \phi_a(t)}{\psi_a(t)} , \frac{D(t) - \phi_d(t)}{\psi_d(t)} \right] \Rightarrow (X,Y) \text{ in } \mathbb{R}^2 ,$$

then

$$\frac{Q(t) - [\phi_a(t) - \phi_d(t)]}{\psi_a(t)} \Rightarrow (X - \gamma Y) \text{ in } \mathbb{R} .$$

Proof. Apply the continuous mapping theorem, writing

$$\frac{Q(t) - [\phi_a(t) - \phi_d(t)]}{\psi_a(t)} = \frac{A(t) - \phi_a(t)}{\psi_a(t)} - \left[\frac{\psi_d(t)}{\psi_a(t)} \right] \frac{(D(t) - \phi_d(t))}{\psi_d(t)} . \quad \blacksquare$$

Remark 6.1. As before, the joint convergence condition in Proposition 6.1 are often easy to establish because the arrival process is often independent of the service process, and the departure process is asymptotically equivalent to the service process when the server is almost always busy. ■

The following is based on Theorem 4.2.

Proposition 6.2. *Suppose that* $0 < \lambda < \infty$, $0 < \mu < \infty$, $0 < r < p_1 < \infty$, $0 < p_2 \le p_1 < \infty$ *and*

$$\left[\frac{A(t) - \lambda t^{p_1}}{t^r} \; , \; \frac{D(t) - \mu t^{p_2}}{t^r} \right] \Rightarrow (X,Y) \; in \; \mathbb{R}^2 \; .$$

(a) If $p_1 = p_2 = p$, then

$$\left[\frac{\hat{A}_n - (n/\lambda)^{1/p}}{n^{(1+r)/p} - 1} \; , \; \frac{\hat{D}_n - (n/\mu)^{1/p}}{n^{(1+r)/p} - 1} \; , \; \frac{W_n - (n/\mu)^{1/p} + (n/\lambda)^{1/p}}{n^{(1+r)/p} - 1} \right]$$

$$\Rightarrow \left[\frac{-X}{p\lambda^{(r+1)/p}} \; , \; \frac{-Y}{p\mu^{(r+1)/p}} \; , \; \frac{X}{p\lambda^{(r+1)/p}} - \frac{Y}{p\mu^{(r+1)/p}} \right] \; in \; \mathbb{R}^3 \; .$$

(b) If $p_1 > p_2$, then

$$\left[\frac{\hat{A}_n - (n/\lambda)^{1/p_1}}{n^{(1+r)/p_1} - 1} \; , \; \frac{\hat{D}_n - (n/\mu)^{1/p_2}}{n^{(1+r)/p_2} - 1} \; , \; \frac{W_n - (n/\mu)^{1/p_2} + (n/\lambda)^{1/p_1}}{n^{(1+r)/p_2} - 1} \right]$$

$$\Rightarrow \left[\frac{-X}{p_1\lambda^{(r+1)/p_1}} \; , \; \frac{-Y}{p_2\mu^{(r+1)/p_2}} \; , \; \frac{Y}{p_2\mu^{(r+1)/p_2}} \right] \; in \; \mathbb{R}^3 \; .$$

Proof. Apply Theorem 4.2 with $q_i = r/p_i$ and then apply the continuous mapping theorem. ■

The following is an unstable stationary CLT version of $L = \lambda W$; see Whitt (1971), Szczotka (1986, 1992) and Serfozo, Szczotka and Topolski (1992) for related results.

Proposition 6.3. *If the discipline is FIFO, $\lambda \geq \mu$ and*

$$\left[\frac{\hat{A}_{\lfloor nt \rfloor} - \lambda^{-1} nt}{\sqrt{n}} \; , \; \frac{\hat{D}_{\lfloor nt \rfloor} - \mu^{-1} nt}{\sqrt{n}} \right] \Rightarrow (X(t), \; Y(t)) \; in \; (D[0,\infty), \; M_1)^2 \; as \; n \to \infty \; ,$$

where at least one of $X(t)$ or $Y(t)$ has continuous sample paths w.p.1, then

$$\left[\frac{Q(t) - (\lambda - \mu)t}{\sqrt{t}} \; , \; \frac{\lambda W_{\lfloor \mu t \rfloor} - (\lambda - \mu)t}{\sqrt{t}} \right]$$

$$\Rightarrow (-\lambda X(\lambda) + \mu Y(\mu) \; , \; \lambda Y(\mu) + \lambda X(\mu)) \; in \; \mathbb{R}^2 \; as \; t \to \infty$$

and

$$\frac{(Q(t) - \lambda W_{\lfloor \mu t \rfloor})}{\sqrt{t}} \Rightarrow (\mu - \lambda) Y(\mu) - \lambda (X(\lambda) - X(\mu)) \ \ as \ t \to \infty \ . \tag{6.1}$$

Proof. Apply Corollary 2 to Theorem 5.2 and the continuous mapping theorem. ■

Note that the right side of (6.1) is 0 when $\lambda = \mu$. Generalizations of Proposition 6.3 with other normalizations also follow from Theorem 5.2.

7. Constructing Nonstationary Point Processes

In this section we consider how to construct nonstationary point processes for which SLLNs and CLTs can be shown to hold. We propose using a random or deterministic time transformation of a familiar counting process $\{S(t) : t \geq 0\}$ for which the SLLN, CLT and FCLT are known. For example, $\{S(t) : t \geq 0\}$ might be a renewal counting process or a counting process with stationary increments. Let $\{C(t) : t \geq 0\}$ be a stochastic process with nondecreasing nonnegative sample paths that is independent of the given counting process $\{S(t) : t \geq 0\}$. We then define our general nonnegative counting process A by setting

$$A(t) = S(C(t)) \ , \ t \geq 0 \ . \tag{7.1}$$

A familiar special case occurs when S is a homogeneous Poisson process with unit intensity and $C(t) = \int_0^t \lambda(s)\, ds$ for a deterministic positive function λ; then A is a nonhomogeneous Poisson process with rate function λ. If S is Poisson and C is random, then A is a *doubly-stochastic Poisson process, Cox process* or *Poisson process in a random environment;* e.g., see Serfozo (1990). If, in addition, the process C is Markov, then A is a *Markov modulated Poisson process* (MMPP). However, (7.1) can be used without S being a Poisson process. We are primarily interested in the case in which C is nonstationary.

From (7.1) we can easily obtain a SLLN for A, which is of the form of §2.

Proposition 7.1. *If* $S(t)/t \to 1$ *and* $C(t)/\phi(t) \to \lambda$ *w.p.1 as* $t \to \infty$, *then* $A(t)/\phi(t) \to \lambda$ *w.p.1 as* $t \to \infty$.

Proof. Note that

$$\frac{A(t)}{\phi(t)} = \frac{S(C(t))}{C(t)} \cdot \frac{C(t)}{\phi(t)} \to 1 \cdot \lambda \quad as \quad t \to \infty . \ \blacksquare$$

We now turn to FCLTs. We continue to use the M_1 topology because of its role in Theorem 5.1. Note that the conclusions in Theorem 7.2 are in the form of one of the limits in Theorem 5.2. Thus, Theorems 5.2 and 7.2 and Proposition 6.1 combine to yield CLTs for queueing processes.

Theorem 7.2. *Let* ϕ *and* ψ *be homeomorphisms of* $\mathbb{R}_+$ *such that* $\psi(s)/s \to 0$ *as* $s \to \infty$. *Let* A *be defined by (7.1). Suppose that*

$$\frac{S(st) - st}{s^p} \Rightarrow X(t) \ \ in \ \ (D[0,\infty), M_1) \ \ as \ \ s \to \infty \tag{7.2}$$

for $p > 0$ *and*

$$\frac{C(\phi^{-1}(st)) - st}{\psi(s)} \Rightarrow Y(t) \ \ in \ \ (D[0,\infty), M_1) \ \ as \ \ s \to \infty \ . \tag{7.3}$$

(a) If $\psi(s)/s^p \to 0$ *as* $s \to \infty$, *then*

$$\frac{A(\phi^{-1}(st)) - st}{s^p} \Rightarrow X(t) \ \ in \ \ (D[0,\infty), M_1) \ \ as \ \ s \to \infty \ .$$

(b) If $\psi(s)/s^p \to \gamma$ *as* $s \to \infty$, $0 < \gamma < \infty$ *and either* X *or* Y *as continuous sample paths, then*

$$\frac{A(\phi^{-1}(st)) - st}{s^p} \Rightarrow X(t) + \gamma Y(t) \ \ in \ \ (D[0, \infty), M_1) \ \ as \ \ s \to \infty \ .$$

(c) If $\psi(s)/s^p \to \infty$ *as* $s \to \infty$, *then*

$$\frac{A(\phi^{-1}(st)) - st}{\psi(s)} \Rightarrow Y(t) \ \ in \ \ (D[0,\infty), M_1) \ \ as \ \ s \to \infty \ .$$

Proof. By Theorems 3.2 and 4.4 of Billingsley (1968),

$$\left[\frac{S(st) - st}{s^p} , \frac{C(\phi^{-1}(st)) - st}{\psi(s)} , \frac{C(\phi^{-1}(st))}{s} , \frac{\psi(s)}{s^p} \right]$$
$$\Rightarrow (X(t), Y(t), Z(t), \gamma) \ \ in \ \ (D[0,\infty), M_1)^3 \times \mathbb{R} \ \ as \ \ s \to \infty \ .$$

For (a) and (b), apply the continuous mapping theorem with the function $h: D^3 \times \mathbb{R} \to D$ defined by $h(x,y,z,w)(t) = x(z(t)) + wy(t)$ using the path-continuity condition in (b) to assure that addition is continuous almost surely; see §4 of Whitt (1980). For (c) use the continuous mapping theorem with $h(x,y,z,w)(t) = x(z(t))/w + y(t)$. ■

Remark 7.1. The crucial condition in Theorem 7.2 is (7.3). Note that (7.3) will hold if we *choose* C to be of the form $C(t) = Z(\phi(t))$, where Z is a stochastic process known to satisfy a FCLT. Typically we will have $\frac{Z(st) - st}{\sqrt{s}} \Rightarrow Y(t)$ as $s \to \infty$, where Y is Brownian motion, so that $\psi(s) = s^{1/2}$ in (7.3). Also, typically $p = 1/2$ in (7.2), so that case (b) results. In the deterministic case $C = \phi$ and case (a) results.

Remark 7.2. We can also apply a corresponding procedure with the arrival epochs, writing

$$\hat{A}_n = \hat{S}_{\hat{C}_n} \ , \ n \geq 1 \ , \tag{7.4}$$

where $\{\hat{S}_n\}$ is a familiar stochastic point sequence and $\{\hat{C}_n\}$ is a nondecreasing nonnegative sequence. Analogs of Proposition 7.1 and Theorem 7.2 are easy. They yield conditions for the convergence of $\hat{A}_n/\phi^{-1}(n)$ and $[\hat{A}_n - \phi^{-1}(n)]/\psi(n)$ as $n \to \infty$.

Remark 7.3. We could also consider additive modifications, e.g.,

$$A(t) = S(t) + C(t) \ , \ t \geq 0 \ , \tag{7.5}$$

or

$$\hat{A}_n = \hat{S}_n + \hat{C}_n \,, \; n \geq 1 \tag{7.6}$$

Again, analogs of Proposition 7.1 and Theorem 7.1 are easy. ■

8. Exploiting the Compensator

Given a nonstationary counting process $\{A(t) : t \geq 0\}$, we can obtain LLNs and CLTs by exploiting its compensator; see Brémaud (1981) and Ethier and Kurtz (1986). In particular, now assume that $\{A(t) : t \geq 0\}$ is a stochastic counting process on $\mathbb{R}_+$ that is adapted to a history $\{\wedge_t : t \geq 0\}$. Let $C \equiv \{C(t) : t \geq 0\}$ be the compensator of A. We are thinking of A growing nonlinear manner. Hence, we assume that

$$C(t)/\phi(t) \rightarrow 1 \quad \text{w.p.1 } as \; t \rightarrow \infty \,. \tag{8.1}$$

where ϕ is a deterministic homeomorphism of $\mathbb{R}_+$. Moreover, we assume that $\{C(t) : t \geq 0\}$ is a random homeomorphism of $\mathbb{R}^+$. For example, this occurs in the familiar case in which $A(t)$ has a stochastic intensity $\{\Lambda(t) : t \geq 0\}$ with

$$\Lambda(t) > 0 \text{ for all } t \text{ and } \int_0^\infty \Lambda(t)\, dt = \infty \text{ w.p.1.} \tag{8.2}$$

Then

$$C(t) = \int_0^t \Lambda(s)\, ds \,, \; t \geq 0 \,, \tag{8.3}$$

and $C^{-1}(t)$ is defined by

$$\int_0^{C^{-1}(t)} \Lambda(s)\, ds = t \,, \; t \geq 0 \,. \tag{8.4}$$

An important property for our purposes is the random-time-change theorem that allows us to transform A by C^{-1} in order to obtain a stationary Poisson process; see p. 40 of Brémaud (1981).

Lemma 8.1. *The process* $A \circ C^{-1} \equiv \{A(C^{-1}(t)) : t \geq 0\}$ *is a homogeneous Poisson process with unit rate.*

We now apply Lemma 8.1 to obtain sufficient conditions for A to satisfy a FCLT.

Theorem 8.1. *Suppose that C is a random homeomorphism of $\mathbb{R}_+$ and the compensator of A. Suppose that*

$$\frac{C(\phi^{-1}(st)) - st}{\psi(s)} \Rightarrow X(t) \text{ in } (D[0,\infty), M_1) \text{ as } s \to \infty \tag{8.5}$$

where $\psi(s)/s \to 0$ *as* $s \to \infty$.

(a) If $\psi(s)/\sqrt{s} \to 0$ *as* $s \to \infty$, *then*

$$\frac{A(\phi^{-1}(st)) - st}{\sqrt{s}} \Rightarrow B(t) \text{ in } (D[0,\infty), M_1) \text{ as } s \to \infty\ ,$$

where B is standard Brownian motion.

(b) If $\sqrt{s}/\psi(s) \to 0$ *as* $s \to \infty$, *then*

$$\frac{A(\phi^{-1}(st)) - st}{\psi(s)} \Rightarrow X(t) \text{ in } (D[0,\infty), M_1) \text{ as } s \to \infty\ .$$

Proof. By Lemma 8.1 and the FCLT for a Poisson process,

$$\frac{A(C^{-1}(st)) - st}{\sqrt{s}} \Rightarrow B(t) \text{ in } (D[0,\infty), M_1) \text{ as } s \to \infty \tag{8.6}$$

In (a), the conditions imply that

$$\left[\frac{C(\phi^{-1}(st)) - st}{\sqrt{s}}\ ,\ \frac{C(\phi^{-1}(st))}{s} \right] \Rightarrow (0,t) \text{ in } (D[0,\infty),\ M)^2 \quad \text{as } s \to \infty\ . \tag{8.7}$$

Since the limit in (8.7) is deterministic, it holds jointly with (8.6) by Theorem 4.4 of Billingsley (1968). Hence, we can apply the continuous mapping theorem as in the proof of Theorem 7.2 to obtain the conclusion. The argument is essentially the same for (b), dividing by $\psi(s)$ instead of $\sqrt{s}$. ■

Remark 8.1. Theorem 8.1 says nothing about the difficult case in which $\psi(s)/\sqrt{s} \to \gamma$ as

$s \to \infty$ where $0 < \gamma < \infty$. Then we need the joint convergence of (8.5) and (8.6), which seems difficult to verify; see Glynn, Melamed and Whitt (1993) for related problems in a stationary setting. However, in our nonstationary setting, the cases in Theorem 8.1 are natural. ■

References

[1] Billingsley, P. (1968) *Convergence of Probability Measures*, Wiley, New York.

[2] Brémaud, P. (1981) *Point Processes and Queues*, Springer-Verlag, New York.

[3] Brémaud, P. (1992) A Swiss army formula of Palm calculus, *J. Appl. Prob.*, to appear.

[4] El-Taha, M. and Stidham, Jr., S. (1991) Deterministic analysis of queueing systems with heterogeneous servers. *Theoret. Comput. Sci.* 109, to appear.

[5] Ethier, S. N. and Kurtz, T. G. (1986) *Markov Processes, Characterization and Convergence*, Wiley, New York.

[6] Feller, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. II, second ed., Wiley, New York.

[7] Gelenbe, E. (1983) Stationary deterministic flows in discrete systems, I, *Theoret. Comput. Sci.* 23, 107-128.

[8] Gelenbe, E. and Finkel, D. (1987) Stationary deterministic flows II: the single server queue, *Theoret. Comput. Sci.* 52, 269-280.

[9] Glynn, P. W., Melamed, B. and Whitt, W. (1993) Estimating customer and time averages. *Oper. Res.*, to appear.

[10] Glynn, P. W. and Whitt, W. (1986) A central-limit-theorem version of $L = \lambda W$, *Queueing Systems* 1, 191-215.

[11] Glynn, P. W. and Whitt, W. (1988) Ordinary CLT and WLLN versions of $L = \lambda W$, *Math. Oper. Res.* 13, 674-692.

[12] Glynn, P. W. and Whitt, W. (1989) Extensions of the queueing relations $L = \lambda W$ and $H = \lambda G$, *Oper. Res.* 37, 634-644.

[13] Glynn, P. W. and Whitt, W. (1992) The asymptotic efficiency of simulation estimators. *Oper. Res.*, 40, 505-520.

[14] Grier, N., Massey, W. A. and Whitt, W. (1992) Uniform acceleration approximations for the time-dependent Erlang loss model, in preparation.

[15] Heyman, D. P. and Whitt, W. (1984) The asymptotic behavior of queues with time-varying arrival rates, *J. Appl. Prob.* 21, 143-156.

[16] Iglehart, D. L. and Whitt, W. (1970a) Multichannel queues in heavy traffic, *I. Adv. Appl. Prob.* 2, 150-177.

[17] Iglehart, D. L. and Whitt, W. (1970b) Multiple channel queues in heavy traffic, II: sequences, networks and batches. *Adv. Appl. Prob.* 2, 355-369.

[18] Lemoine, A. J. (1989) Waiting time and workload in queues with periodic Poisson input, *J. Appl. Prob.* 26, 390-397.

[19] Little, J. D. C. (1961) A proof of the queueing formula: $L = \lambda W$, *Oper. Res.* 9, 383-387.

[20] Mandelbaum, A. and Massey, W. A. (1992) Strong Approximations for time dependent queues, AT&T Bell Laboratories, Murray Hill, NJ.

[21] Massey, W. A. (1985) Asymptotic analysis of the time-dependent M/M/1 queue, *Math. Oper. Res.* 10, 305-327.

[22] Massey, W. A. and Whitt, W. (1993) Networks of infinite-server queues with nonstationary Poisson input, *Queueing Systems,* to appear.

[23] Pomarede, J. L. (1976) *A unified Approach via Graphs to Skorohod's Topologies on the Function Space* D, Ph.D. dissertation, Department of Statistics, Yale University.

[24] Rolski, T. (1989) Queues with nonstationary inputs, *Queueing Systems*, 5, 113-130.

[25] Serfozo, R. F. (1990) Point processes. In *Stochastic Models*, D. P. Heyman and M. J. Sobel (eds.), North-Holland, Amsterdam, 1-93.

[26] Serfozo, R. F., Szczotka, W. and Topolski, K. (1992) Relating the waiting time in a heavy-traffic queueing system to the queue length, Georgia Institute of Technology.

[27] Skorohod, A. V. (1956) Limit theorems for stochastic processes, *Theor. Probab. Appl.* 1, 261-290.

[28] Stidham, Jr., S. (1974) A last word on $L = \lambda W$, *Oper. Res.* 22, 417-421.

[29] Stidham, Jr., S. and El-Taha, M. (1989) Sample-path analysis of processes with imbedded point processes. *Queueing Systems* 5, 131-166.

[30] Szczotka, W. (1986) Joint distribution of waiting time and queue size for single server queues. *Dissertationes Mathematicae,* 248.

[31] Szczotka, W. (1992) A distributional form of Little's law in heavy traffic. *Ann. Probab.* 20, 790-800.

[32] Szczotka, W. and Topolski, K. (1991) Conditioned limit theorems for the difference of the waiting time and queue line processes of the GI/G/1 queue. Institute of Mathematics, Wroclaw University.

[33] Whitt, W. (1971) Weak convergence theorems for priority queues: preemptive-resume discipline. *J. Appl. Prob.* 8, 74-94.

[34] Whitt, W. (1980) Some useful functions for functional limit theorems. *Math. Oper. Res.* 5, 67-85.

[35] Whitt, W. (1991a) The pointwise stationary approximation for $M_t/M_t/s$ queues is asymptotically correct as the rates increase, *Management Sci.* 37, 307-314.

[36] Whitt, W. (1991b) A review of $L = \lambda W$ and extensions. *Queueing Systems* 9, 235-268.

[37] Whitt, W. (1992) Correction note on $L = \lambda W$. *Queueing Systems* 10.