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# Two-scale damage modeling of brittle composites

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**Abstract**

This paper is aimed at developing a non-local theory for obtaining a numerical approximation to a boundary-value problem describing damage phenomena in a ceramic composite material. The mathematical homogenization method based on double-scale asymptotic expansion is generalized to account for damage effects in heterogeneous media. A closed-form expression relating local fields to the overall strain and damage is derived. Non-local damage theory is developed by introducing the concept of non-local phase fields (stress, strain, free energy density, damage release rate, etc.). Numerical results of our model were found to be in good agreement with experimental data from 4-point bend tests conducted on composite beams made of Blackglas$^{TM}$/Nextel 5-harness satin weave. © 2001 Published by Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

Damage in composite materials occurs through different mechanisms that are complex and usually involve interaction between microconstituents. During the past two decades, a number of models have been developed to simulate damage and failure processes in ceramic composites, among which the damage-mechanics approach is particularly attractive in the sense that it provides a viable framework for the description of distributed damage including material stiffness degradation, initiation, growth and coalescence of microcracks and voids. Various damage models for brittle composites can be classified into micromechanical (or mesomechanical) and macromechanical approaches. In the macromechanical damage approach, a composite material is idealized (or homogenized) as an anisotropic homogeneous medium and damage is introduced via internal variable whose tensorial nature depends on assumptions about crack orientation [9,15,16,19,21,18]. The micromechanical damage approach, on the other hand, treats each microphase as a statistically homogeneous medium. Local damage variables are defined to represent the state of damage in each phase and phase effective material properties are defined thereafter. The overall response is subsequently obtained by homogenization [1,17,22].

From the mathematical formulation stand point, both approaches can be viewed as a two-step procedure. The main difference between the two approaches is in the chronological order in which the homogenization and evolution of damage are carried out. In the macromechanical approach, homogenization is performed first followed by application of damage mechanics principles to homogenized anisotropic medium, while in the micromechanical approach, damage mechanics is applied to each phase followed by homogenization.

The primary objective of the present manuscript is to simultaneously carry out the two steps (homogenization and evolution of damage) by extending the framework of the classical mathematical homogenization theory [2–4] to account for damage effects. This is accomplished by introducing a double scale asymptotic expansion of damage parameter (or damage tensor in general). This leads to the derivation of the closed form expression relating local fields to overall strains and damage. The second salient feature of our approach is in developing a non-local theory by introducing the concept of non-local phase fields (stress, strain, free energy density, damage release rate, etc.). Non-local phase fields are defined as weighted averages over each phase in the characteristic volume in a manner analogous to that

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currently practiced in concrete [6,7] with the only exception being that the weight functions are taken to be $C^0$ continuous over a single phase and zero elsewhere. On the global (macro) level we limit the finite element size to ensure a valid use of the mathematical homogenization theory and to limit localization. We consider a 4-point bend test conducted on the ceramic composite beam made of Blackglas™/Nextel 5-harness satin weave and compare our numerical simulations to experiments [8].

## 2. Mathematical homogenization for damaged composites

In this section we extend the classical mathematical homogenization theory [2] for statistically homogeneous composite media to account for damage effects. The strain-based continuum damage theory is adopted for constructing constitutive relations at the level of microconstituents. Closed form expressions of local strain and stress fields in a multi-phase composite medium are derived. Attention is restricted to small deformations.

The microstructure of a composite material is assumed to be locally periodic (Y-periodic) with a period defined by the representative volume element (RVE), denoted by $\Theta$. Let $\mathbf{x}$ be a macroscopic coordinate vector in macro domain $\Omega$ and $\mathbf{y}\equiv\mathbf{x}/\varsigma$ be a microscopic position vector in $\Theta$. Here, $\varsigma$ denotes a very small positive number compared with the dimension of $\Omega$, and $\mathbf{y}\equiv\mathbf{x}/\varsigma$ is regarded as a stretched coordinate vector in the microscopic domain. When a solid is subjected to some load and boundary conditions, the resulting deformation, stresses, and internal variables may vary from point to point within the RVE due to the high level of heterogeneity. We assume that all quantities have two explicit dependencies: one on the macroscopic level $\mathbf{x}$, and the other one on the level of microconstituents $\mathbf{y}\equiv\mathbf{x}/\varsigma$. For any Y-periodic response function $f$, we have $f(\mathbf{x},\mathbf{y})=f(\mathbf{x},\mathbf{y}+\mathbf{ky})$ in which vector $\mathbf{y}$ is the basic period of the microstructure and $\mathbf{k}$ is a 3 by 3 diagonal matrix with integer components. Adopting the classical nomenclature, any Y-periodic function $f$ can be represented as $f^{\varsigma}(\mathbf{x})\equiv f(\mathbf{x},\mathbf{y}(\mathbf{x}))$ with superscript $\varsigma$ denoting a Y-periodic function $f$. The indirect macroscopic spatial derivatives of $f^{\varsigma}$ can be calculated by the chain rule as with the comma followed by a subscript variable $x_i$ denoting a partial derivative with respect to the subscript variable (i.e. $f_{,x_i}\equiv\partial f/\partial x_i$). Summation convention for repeated subscripts is employed, except for subscripts $x$ and $y$.

The constitutive equation on the microscale is derived from continuum damage theory based on the thermodynamics of irreversible processes and internal state variable theory. We define a scalar damage parameter $\omega^{\varsigma}$ as a function of microscopic and macroscopic position vectors, i.e. $\omega^{\varsigma}=\omega(\mathbf{x},\mathbf{y})$ The anisotropy of damage is introduced through damage shape functions to be formulated later.

Based on the strain-based continuum damage theory, the free energy density has the form of

$$\Psi\left(\omega^{\varsigma},\varepsilon_{ij}^{\varsigma}\right)=(1-\omega^{\varsigma})\Psi_e\left(\varepsilon_{ij}^{\varsigma}\right) \tag{1}$$

where $\omega^{\varsigma}\in[0,1)$ is the damage parameter. For small deformations, elastic free energy density is given as $\Psi_e\left(\varepsilon_{ij}^{\varsigma}\right)=0.5\ L_{ijkl}\varepsilon_{ij}^{\varsigma}\varepsilon_{kl}^{\varsigma}$. The constitutive equation, thermodynamic force (also known as a damage energy release rate) and dissipative inequality follow from (1)

$$\sigma_{ij}^{\varsigma}=\frac{\partial\Psi\left(\omega^{\varsigma},\varepsilon_{ij}^{\varsigma}\right)}{\partial\varepsilon_{ij}^{\varsigma}},\ Y=-\frac{\partial\Psi\left(\omega^{\varsigma},\varepsilon_{ij}^{\varsigma}\right)}{\partial\omega^{\varsigma}},\ Y\dot{\omega}^{\varsigma}\geqslant 0. \tag{2}$$

With this brief glimpse into the constitutive theory, we proceed to outline the strong form of the governing differential equations on the fine scale—the scale of microconstituents. We assume that microconstituents possess homogeneous properties and satisfy equilibrium, constitutive, kinematics and compatibility equations. The corresponding boundary value problem is governed by the following set of equations:

$$\sigma_{ij,j}+b_i=0,\ \ \sigma_{ij}^{\varsigma}=(1-\omega^{\varsigma})L_{ijkl}\varepsilon_{kl}^{\varsigma},\ \ \varepsilon_{ij}^{\varsigma}=u_{(i,j)}^{\varsigma}\ \text{ on }\ \Omega \tag{3}$$

$$u_l^{\varsigma}=\bar{u}_i\quad\text{on}\quad\Gamma_u,\ \ \sigma_{ij}^{\varsigma}n_j=\dot{t}_i\quad\text{on}\quad\Gamma_t. \tag{4}$$

where $\omega^{\varsigma}$ is a scalar damage parameter; $\sigma_{ij}^{\varsigma}$ and $\varepsilon_{ij}^{\varsigma}$ are components of stress and strain tensors; $L_{ijkl}$ represents components of elastic stiffness; $b_i$ is a body force assumed to be independent of $\mathbf{y}$; $u_l^{\varsigma}$ denotes the components of the displacement vector; the subscript pairs with parentheses denote the symmetric gradients; $\Omega$ denotes the macroscopic domain of interest with boundary $\Gamma$; $\Gamma_u$ and $\Gamma_t$ are boundary portions where displacements $\bar{u}_i$ and tractions $\dot{t}_i$ are prescribed, respectively; $n_i$ denotes the normal vector on $\Gamma$.

The mathematical homogenization method based on the double-scale asymptotic expansion is employed to account for microstructural effects on the macroscopic response without explicitly representing the details of the microstructure in the global analysis. As a starting point, we approximate the displacement field, $u_l^{\varsigma}(\mathbf{x})=u_i(\mathbf{x},\mathbf{y})$, and the damage parameter, $\omega^{\varsigma}(\mathbf{x})=\omega(\mathbf{x},\mathbf{y})$, in terms of double-scale asymptotic expansions on $\Omega$ x $\Theta$:

$$\begin{aligned} u_i(\mathbf{x},\mathbf{y}) &\approx u_i^0(\mathbf{x},\mathbf{y})+\varsigma u_i^1(\mathbf{x},\mathbf{y})+\ldots, \\ \omega(\mathbf{x},\mathbf{y}) &\approx \omega^0(\mathbf{x},\mathbf{y})+\varsigma\omega^1(\mathbf{x},\mathbf{y})+\ldots \end{aligned} \tag{5}$$

Strain expansions on the composite domain $\Omega$ x $\Theta$ can be obtained by substituting (5) into (3) with consideration of the indirect differentiation rule

$$\varepsilon_{ij}(\mathbf{x},\mathbf{y}) \approx \frac{1}{\varsigma}\varepsilon_{ij}^{-1}(\mathbf{x},\mathbf{y}) + \varepsilon_{ij}^{0}(\mathbf{x},\mathbf{y}) + \varsigma\varepsilon_{ij}^{1}(\mathbf{x},\mathbf{y}) + \ldots \tag{6}$$

$$\begin{aligned} &\varepsilon_{ij}^{-1} = \varepsilon_{yij}(\mathbf{u}^0), \quad \varepsilon_{ij}^{s} = \varepsilon_{xij}(\mathbf{u}^s) + \varepsilon_{yij}(\mathbf{u}^{s+1}), \\ &\quad \varepsilon_{xij}(\mathbf{u}^s) = u^s_{(i,x_j)}, \quad \varepsilon_{yij}(\mathbf{u}^s) = u^s_{(i,y_j)} \end{aligned} \tag{7}$$

Stresses and strains for different orders of $\varsigma$ are related by the constitutive Eq. (3)

$$\begin{aligned} &\sigma_{ij}^{-1} = (1-\omega^0)L_{ijkl}\varepsilon_{kl}^{-1}, \\ &\quad \sigma_{ij}^{s} = (1-\omega^0)L_{ijkl}\varepsilon_{kl}^{s} + \sum_{r=0}^{s}\omega^{s-r+1}L_{ijkl}\varepsilon_{kl}^{r-1}. \end{aligned} \tag{8}$$

The resulting asymptotic expansion of stress is given as

$$\sigma_{ij}(\mathbf{x},\mathbf{y}) \approx \frac{1}{\varsigma}\sigma_{ij}^{-1}(\mathbf{x},\mathbf{y}) + \sigma_{ij}^{0}(\mathbf{x},\mathbf{y}) + \varsigma\sigma_{ij}^{1}(\mathbf{x},\mathbf{y}) + \ldots \tag{9}$$

Inserting the stress expansion (9) into equilibrium Eq. (3) and making use of the equation yield the following equilibrium equations for various orders:

$$\begin{aligned} &O(\varsigma^{-2}) : \sigma^{-1}_{ij,y_j} = 0, \quad O(\varsigma^{-1}) : \sigma^{-1}_{ij,x_j} + \sigma^{0}_{ij,y_j} = 0, \\ &\quad O(\varsigma^{0}) : \sigma^{0}_{ij,x_j} + \sigma^{1}_{ij,y_j} + b_i = 0. \end{aligned} \tag{10}$$

**Remark.** For modeling damage in woven or textile composites it is often necessary to introduce additional scale to model microcracking of matrix in the bundle. In this case we can approximate the microscopic displacement field, $\mathbf{u}^{\varsigma}(\mathbf{x}) \equiv \mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{z})$, and the damage variable, $\omega^{\varsigma}(\mathbf{x}) \equiv \omega(\mathbf{x},\mathbf{y},\mathbf{z})$, in terms of the triple-scale asymptotic expansions on $\Omega$x$\Theta_y$x$\Theta_z$:

$$\mathbf{u}_{\text{mic}} \approx \mathbf{u}^0(\mathbf{x}) + \varsigma\mathbf{u}^1(\mathbf{x},\mathbf{y}) + \varsigma^2\mathbf{u}^2(\mathbf{x},\mathbf{y},\mathbf{z}) \tag{11}$$

$$\omega_{\text{mic}} \approx \omega^0(\mathbf{x},\mathbf{y},\mathbf{z}) + \varsigma\omega^1(\mathbf{x},\mathbf{y},\mathbf{z}) + \varsigma^2\omega^2(\mathbf{x},\mathbf{y},\mathbf{z}) \tag{12}$$

where $\mathbf{z}$ represents the microscale. Asymptotic expansion on $\Omega$x$\Theta_y$x$\Theta_z$ can be then obtained in a similar manner as for two scales. For more details we refer to [23].

From the $O(\varsigma^{-2})$ equilibrium Eq. (10) we arrive at the classical result $u_i^0 = u_i^0(\mathbf{x})$.

We proceed to the $O(\varsigma^{-1})$ equilibrium Eq. (10). From (7) and (8) follow

$$\left\{(1-\omega^0)L_{ijkl}\left(\varepsilon_{xkl}(\mathbf{u}^0) + \varepsilon_{ykl}(\mathbf{u}^1)\right)\right\}_{,y_j} = 0 \quad \text{on} \quad \Theta \tag{13}$$

To solve for (13) up to a constant we introduce the following separation of variables

$$u_i^1(\mathbf{x},\mathbf{y}) = H_{ikl}(\mathbf{y})\left\{\varepsilon_{xkl}(\mathbf{u}^0) + d_{kl}^{\omega}(\mathbf{x})\right\} \tag{14}$$

where $H_{ikl}$ is a Y-periodic function. We assume that $d_{kl}^{\omega}(\mathbf{x})$ is macroscopic damage-induced strain driven by the macroscopic strain $\bar{\varepsilon}_{kl} \equiv \varepsilon_{xkl}(\mathbf{u}^0)$. More specifically we can state that if $\bar{\varepsilon}_{kl} = 0$, then $d_{kl}^{\omega}(\mathbf{x})$ and $\omega^0(\mathbf{x},\mathbf{y}) = 0$.

Based on the decomposition given in (14), the $O(\varsigma^{-1})$ equilibrium equation takes the following form:

$$\begin{aligned} &\Big\{(1-\omega^0)L_{ijkl}\Big[(I_{klmn} + G_{klmn})\varepsilon_{xmn}(\mathbf{u}^0) \\ &\quad + G_{klmn}d_{mn}^{\omega}(\mathbf{x})\Big]\Big\}_{,y_j} = 0 \ \text{ in } \ \Theta \end{aligned} \tag{15}$$

where

$$\begin{aligned} &I_{klmn} = \frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{nk}\delta_{ml}), \\ &\quad G_{klmn}(\mathbf{y}) = H_{(k,y_l)mn}(\mathbf{y}) \end{aligned} \tag{16}$$

and $\delta_{mk}$ is the Kronecker delta, while $G_{klmn}$ is known as a polarization function. It can be shown that the integrals of the polarization functions in $\Theta$ vanish due to periodicity conditions. Since Eq. (15) should be valid for arbitrary macroscopic fields, we may first consider the case of $d_{kl}^{\omega}(\mathbf{x}) = 0$ (and $\omega^0 = 0$) but $\bar{\varepsilon}_{kl} \neq 0$, which yields the following equation in $\Theta$:

$$\left\{L_{ijkl}\left(I_{klmn} + H_{(k,y_l)mn}\right)\right\}_{,y_j} = 0 \tag{17}$$

Eq. (17) together with the Y-periodic boundary conditions is a linear boundary value problem in $\Theta$. By exploiting the symmetry with respect to the indexes *(m, n),* the weak form of (17) is solved for 3 right hand side vectors in 2-D and 6 right hand side vectors in 3-D (see for example [11,14]). In the absence of damage, the asymptotic expansion of strain (6) can be expressed in terms of the macroscopic strain $\bar{\varepsilon}_{ij}$ as follows

$$\varepsilon_{ij} = A_{ijkl}\bar{\varepsilon}_{kl} + O(\varsigma), \quad A_{ijkl} = I_{ijkl} + G_{ijkl}. \tag{18}$$

The elastic homogenized stiffness $\bar{L}_{ijkl}$ follows from the $O(\varsigma^0)$ equilibrium equation [17]:

$$\begin{aligned} \bar{L}_{ijkl} &\equiv \frac{1}{|\Theta|}\int_{\Theta} L_{ijmn}A_{mnkl}d\Theta \\ &= \frac{1}{|\Theta|}\int_{\Theta} A_{mnij}L_{mnst}A_{stkl}d\Theta \end{aligned} \tag{19}$$

where $|\Theta|$ is the volume of a RVE.

After solving (17) for $H_{imn}$, we proceed to find $d_{mn}^{\omega}$ from (15). Premultiplying it by $H_{ist}$ and integrating it by parts with consideration of Y-periodic boundary conditions yields

$$\int_{\Theta}\left(1-\omega^{0}\right)G_{ijst}L_{ijkl}\left(A_{klmn}\varepsilon_{xmn}\left(\mathbf{u}^{0}\right)+G_{klmn}d^{\omega}_{mn}(\mathbf{x})\right)d\Theta=0 \tag{20}$$

from where the expression of the macroscopic damaged induced strain can be shown to be

$$\begin{aligned}d^{\omega}_{mn}(\mathbf{x})=-&\left\{\int_{\Theta}\left(1-\omega^{0}\right)G_{ijst}L_{ijkl}G_{klmn}\;\; d\Theta\right\}^{-1}\\&\left\{\int_{\Theta}\left(1-\omega^{0}\right)G_{ijst}L_{ijkl}A_{klpb}\;\; d\Theta\right\}\bar{\varepsilon}_{pb}\end{aligned} \tag{21}$$

Let $\vec{\psi}\equiv\left\{\psi^{(\eta)}(\mathbf{y})\right\}_{1}^{n}$ be a set of $C^{-1}$ continuous functions, then the damage parameter $\omega^{0}(\mathbf{x},\mathbf{y})$ is assumed to have the following decomposition

$$\omega^{0}(\mathbf{x},\mathbf{y})=\sum_{\eta=1}^{n}\psi^{(\eta)}(\mathbf{y})\omega^{(\eta)}(\mathbf{x}) \tag{22}$$

where $\psi^{(\eta)}(\mathbf{y})$ is a damage shape function on the microscale. If experimental data is available damage directionality can be introduced through these shape functions. Rewriting (21) in terms of strain concentration function $A_{ijkl}$ and manipulating it with (19) and (22) yields

$$d^{\omega}_{mn}(\mathbf{x})=D_{klmn}(\mathbf{x})\bar{\varepsilon}_{mn} \tag{23}$$

where

$$D_{klmn}(\mathbf{x})=\left(I_{klst}-\sum_{\eta=1}^{n}B^{(\eta)}_{klst}\omega^{(\eta)}(\mathbf{x})\right)^{-1}\left(\sum_{\eta=1}^{n}B^{(\eta)}_{stmn}\omega^{(\eta)}(\mathbf{x})\right) \tag{24}$$

$$\begin{aligned}B^{(\eta)}_{ijkl}=&\frac{1}{|\Theta|}\\&\times\left(\tilde{L}_{ijmn}-\bar{L}_{ijmn}\right)^{-1}\int_{\Theta}\psi^{(\eta)}G_{stmn}L_{stpq}G_{pqkl}\;\; d\Theta\end{aligned} \tag{25}$$

$$\begin{aligned}C^{(\eta)}_{ijkl}=&\frac{1}{|\Theta|}\\&\times\left(\tilde{L}_{ijmn}-\bar{L}_{ijmn}\right)^{-1}\int_{\Theta}\psi^{(\eta)}G_{stmn}L_{stpq}A_{pqkl}\;\; d\Theta\end{aligned} \tag{26}$$

$$\tilde{L}_{ijmn}=\frac{1}{|\Theta|}\int_{\Theta}L_{ijmn}\;\; d\Theta \tag{27}$$

In conjunction with (14) and (23), the asymptotic expansion of strain field (6) can be finally cast as

$$\varepsilon_{ij}(\mathbf{x},\mathbf{y})=A_{ijmn}(\mathbf{y})\bar{\varepsilon}_{mn}(\mathbf{x})+G_{ijkl}(\mathbf{y})D_{klmn}(\mathbf{x})\bar{\varepsilon}_{mn}(\mathbf{x})+O(\varsigma) \tag{28}$$

where $G_{ijkl}(\mathbf{y})$ can be interpreted as a damage strain influence function. Note that the asymptotic expansion of the strain field is given as a sum of mechanical fields induced by the macroscopic strain via elastic strain concentration function and thermodynamical fields governed by damage-induced strain, $d^{\omega}_{kl}(\mathbf{x})=D_{klmn}(\mathbf{x})\bar{\varepsilon}_{mn}(\mathbf{x})$, through the damage strain influence function.

Finally, we integrate the $O(\varsigma^{0})$ equilibrium Eq. (10) over $\Theta$. The $\int_{\Theta}\sigma^{1}_{ij,y_j}d\Theta$ term vanishes due to periodicity which yields the macroscopic equilibrium equation

$$\bar{\sigma}_{ij,x_j}+b_i=0 \quad \text{and} \quad \left(L_{ijmn}\bar{\varepsilon}_{mn}\right)_{,x_j}+b_i=0 \tag{29}$$

where $L_{ijmn}$ is an instantaneous secant stiffness given as

$$\begin{aligned}L_{ijmn}=&\left(\bar{L}_{ijkl}+\sum_{\eta=1}^{n}\frac{\omega^{(\eta)}}{|\Theta|}\int_{\Theta}\psi^{(\eta)}L_{ijst}A_{stkl}d\Theta\right)\\&\cdot\left(I_{klmn}+D_{klmn}\right)-\left(\tilde{L}_{ijkl}+\sum_{\eta=1}^{n}\frac{\omega^{(\eta)}}{|\Theta|}\int_{\Theta}\psi^{(\eta)}L_{ijkl}d\Theta\right)\cdot D_{klmn}\end{aligned} \tag{30}$$

and the macroscopic stress $\bar{\sigma}_{ij}$ is defined as $\bar{\sigma}_{ij}\equiv\frac{1}{|\Theta|}\int_{\Theta}\sigma^{0}_{ij}\;\, d\Theta$.

Accumulation of damage leads to strain softening and loss of ellipticity. The local approach, stating that in the absence of thermal effects, stresses in a material at a point are completely determined by the deformation and the deformation history at that point, may result in a physically unacceptable localization of the deformation [5]. The principal fault of the local approach, as indicated in [4,5,7,20], is that the energy dissipation at failure is incorrectly predicted to be zero and the corresponding finite element solution converges to this spurious solution as the mesh is refined. To remedy the situation, a number of approaches have been devised to limit strain localization and to circumvent mesh sensitivity associated with strain softening [10]. One of these approaches is based on the non-local damage theory [4,7], the essence of which is to smear solution variables causing strain softening over the characteristic volume of the material.

Following [5] and [7], the non-local damage parameter $\bar{\omega}(\mathbf{x})$ is defined as:

$$\bar{\omega}(\mathbf{x})=\frac{1}{|\Theta_C|}\int_{\Theta_C}\varphi(\mathbf{y})\omega^{0}(\mathbf{x},\mathbf{y})\;\; d\Theta \tag{31}$$

where $\varphi(\mathbf{y})$ is a weight function; $\Theta_C$ is the characteristic volume [6,13]. In the present manuscript, we redefine the representative volume element (RVE) as the maximum required from the statistically homogeneity point of view for which the local periodicity assumption is valid and the characteristic volume.

We further assume that the microscopic damage distribution function $\psi^{(\eta)}(\mathbf{y})$ introduced in (22) is a piecewise function, i.e. it is continuous within the domain of microphase, $\Theta^{(\eta)} \subset \Theta_C \subset \Theta$, but vanishes elsewhere, i.e.

$$\psi^{(\eta)}(\mathbf{y}) = \begin{cases} g^{(\eta)}(\mathbf{y}) & \text{if } \mathbf{y} \in \Theta^{(\eta)} \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

where $\bigcup\limits^{n}\Theta^{(\eta)} = \Theta$ and $\Theta^{(\lambda)} \bigcap \Theta^{(\eta)} = \varnothing$ for $\lambda \neq \eta$, $\eta = 1, 2, \ldots, \bar{n};^1 n$ is the product of the number of different microphases and the number of characteristic volumes in RVE; $\psi^{(\eta)}(\mathbf{y})$ is a distribution function; $g^{(\eta)}(\mathbf{y})$ is a $C^0$ continuous function in $\Theta^{(\eta)}$; and $\omega^{(\eta)}(\mathbf{x})$ is a macroscopically variable amplitude. We further define the weight function in (31) as

$$\varphi(\mathbf{y}) = \mu^{(\eta)}\psi^{(\eta)}(\mathbf{y}) \tag{33}$$

where the constant $\mu^{(\eta)}$ is determined by the orthogonality condition

$$\frac{\mu^{(\eta)}}{|\Theta_C|}\int_{\Theta_C} g^{(\lambda)}(\mathbf{y})g^{(\eta)}(\mathbf{y})\ d\Theta = \delta_{\lambda\eta}, \quad \lambda, \eta = 1, 2, \ldots, n \tag{34}$$

and $\delta_{\lambda\eta}$ is Kronecker delta. Substituting (22) and (32)–(34) into (31) yields

$$\bar{\omega}(\mathbf{x}) = \frac{\mu^{(\eta)}}{\Theta_C}\int_{\Theta_C}\left(g^{(\eta)}(\mathbf{y})\right)^2\omega^{(\eta)}(\mathbf{x})\ d\Theta = \omega^{(\eta)}(\mathbf{x}) \tag{35}$$

which provides the motivation for the specific choice of the weight function. It can be seen that $\omega^{(\eta)}$ has a meaning of the non-local phase damage parameter.

The average strains in each subdomain in RVE are obtained by integrating (28) over $\Theta^{(\eta)}$:

$$\begin{aligned} \varepsilon_{ij}^{(\eta)} &= \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}}\varepsilon_{ij}\ d\Theta \\ &= A_{ijkl}^{(\eta)}\bar{\varepsilon}_{kl} + G_{ijkl}^{(\eta)}D_{klmn}\bar{\varepsilon}_{mn} + O(\varsigma) \end{aligned} \tag{36}$$

where

$$A_{ijkl}^{(\eta)} = \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}} A_{ijkl}\ d\Theta, \quad G_{ijkl}^{(\eta)} = \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}} G_{ijkl}\ d\Theta. \tag{37}$$

To construct the non-local constitutive relation between the phase averages we define the local average stress in $\Theta^{(\eta)}$ as:

$$\begin{aligned} \sigma_{ij}^{(\eta)} &\equiv \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}}\sigma_{ij}^{0}\ d\Theta \\ &= \left(I_{klmn} - \omega^{(\eta)}N_{klmn}^{(\eta)}\right)L_{ijkl}^{(\eta)}\varepsilon_{mn}^{(\eta)} \end{aligned} \tag{38}$$

where

$$N_{klmn}^{(\eta)} = \left(\bar{A}_{klst}^{(\eta)} + \bar{G}_{klpq}^{(\eta)}D_{pqst}\right)\cdot\left(A_{mnst}^{(\eta)} + G_{mnij}^{(\eta)}D_{ijst}\right)^{-1} \tag{39}$$

$$\bar{A}_{ijkl}^{(\eta)} = \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}} g^{(\eta)}A_{ijkl}\ d\Theta, \quad \bar{G}_{ijkl}^{(\eta)} = \frac{1}{\left|\Theta^{(\eta)}\right|}\int_{\Theta^{(\eta)}} g^{(\eta)}G_{ijkl}\ d\Theta. \tag{40}$$

The constitutive Eq. (38) has a non-local character in the sense that it represents the relation between phase averages. The response characteristics between the phases are not smeared as the damage evolution law and thermomechanical properties of phases might be considerably different, in particular when damage occurs in a single phase.

For the isotropic strain-based damage model adopted in this paper, the phase free energy density corresponding to the non-local constitutive Eq. (38) is given as

$$\Psi^{(\eta)}\left(\omega^{(\eta)}, \varepsilon_{ij}^{(\eta)}\right) = \frac{1}{2}\left(I_{klmn} - \omega^{(\eta)}N_{klmn}^{(\eta)}\right)L_{ijkl}^{(\eta)}\varepsilon_{mn}^{(\eta)}\varepsilon_{ij}^{(\eta)} \tag{41}$$

and the corresponding non-local phase damage energy release rate can be expressed as

$$Y^{(\eta)} = -\frac{\partial\Psi^{(\eta)}}{\partial\omega^{(\eta)}} = \frac{1}{2}N_{klmn}^{(\eta)}L_{ijkl}^{(\eta)}\varepsilon_{mn}^{(\eta)}\varepsilon_{ij}^{(\eta)} \tag{42}$$

As a special case we consider a composite material consisting of two phases, matrix and reinforcement, denoted by $\Theta^{(m)}$ and $\Theta^{(f)}$ such that $\Theta = \Theta^{(m)} \bigcup \Theta^{(f)}$. Superscripts $m$ and $f$ represent matrix and reinforcement phases, respectively. For simplicity, we assume that damage occurs in the matrix phase only, i.e. $\omega^{(f)} \equiv 0$. The volume fractions for matrix and reinforcement are denoted as $\nu^{(m)}$ and $\nu^{(f)}$, respectively, such that $\nu^{(m)} + \nu^{(f)} = 1$. To further simplify the matters, we define the microscopic damage distribution function $\psi^{(\eta)}(\mathbf{y})$ (22) as a piecewise constant function. The corresponding weight function becomes piecewise constant function with $\mu^{(\eta)} = \left|\Theta^{(\eta)}\right|/\Theta_C$.

The non-local isotropic damage state variable $\omega^{(m)}$ is assumed to be a monotonically increasing function of non-local phase deformation history parameter $\kappa^{(m)}$ [9,13,15,16] which characterizes the ultimate deformation experienced throughout the loading history. In general, the evolution of matrix damage at time $t$ can be expressed as

$$\omega^{(m)}(\mathbf{x}, t) = f\big(\kappa^{(m)}(\mathbf{x}, t)\big) \tag{43}$$

The non-local phase deformation history parameter $\kappa^{(m)}$ is determined by the evolution of non-local phase damage equivalent strain, denoted by $\bar{\vartheta}^{(m)}$, as follows

$$\kappa^{(m)}(\mathbf{x}, t) = \max\left\{\bar{\vartheta}^{(m)}(\mathbf{x}, \tau)|(\tau \leqslant t), \kappa_{ini}^{(m)}\right\} \tag{44}$$

where the threshold value for damage initiation in the matrix, $\kappa_{ini}^{(m)}$, represents the extreme value of the equivalent strain prior to the initiation of damage. The non-local phase damage equivalent strain, $\bar{\vartheta}^{(m)}$, is defined as square root of the non-local phase damage energy release rate

$$\bar{\vartheta}^{(m)} \equiv \sqrt{Y^{(m)}} = \sqrt{\frac{1}{2} L_{ijkl}^{(m)} \varepsilon_{ij}^{(m)} \varepsilon_{kl}^{(m)}} \tag{45}$$

We adopt an arctangent form of damage evolution law

$$\Phi^{(m)}\left(\alpha, \beta, \omega^{(m)}, \kappa^{(m)}, \kappa_0^{(m)}\right) = \omega^{(m)} - \frac{a\tan\left(\alpha \dfrac{\kappa^{(m)}}{\kappa_0^{(m)}} - \beta\right) + a\tan(\beta)}{\dfrac{\pi}{2} + a\tan(\beta)} = 0 \tag{46}$$

where $\alpha$, $\beta$ are material parameters; and $\kappa_0^{(m)}$ denotes the threshold of the strain history parameter beyond which the damage will develop very quickly. Computational aspects of the non-local piecewise constant damage model for two-phase materials have described in [12].

## 3. Numerical examples

We consider a 4-point bending problem carried out on a composite beam made of Blackglas$^{\mathrm{TM}}$/ Nextel 5-harness satin weave. The fabric designs used 600 denier bundles of Nextel$^{\mathrm{TM}}$ 312 fibers, spaced at 46 threads per inch, and surrounded by Blackglas$^{\mathrm{TM}}$ matrix material. The bundle is assumed to be linear elastic throughout the analysis. The average transversely isotropic elastic properties of the bundle were computed by the Mori–Tanaka method. We will refer to this material system as AF1O. The phase properties of RVE are: Blackglas$^{\mathrm{TM}}$ Matrix: volume fraction = 0.548; Young's modulus = 9.653 GPa; Poisson's ratio = 0.244; Nextel$^{\mathrm{TM}}$ 312 Fiber: volume fraction = 0.452; Young's modulus = 151.7 GPa; Poisson's ratio = 0.26.

The microstructure of RVE is discretized with 6857 elements totaling 10,608 degrees of freedom as shown in Fig. 1. The configuration of the composite beam is shown in Fig. 1 where the loading direction (normal to the plane of the weave) is aligned along the Y axis. The finite element model of the beam (macrostructure) is composed of 1856 brick elements totaling 7227 degrees of freedom. The parameters for the damage evolution law for Blackglas$^{\mathrm{TM}}$ matrix are taken as $\alpha = 7.1$, $\beta = 10.1$ and $\kappa_0^{(m)} = 0.22$, which were chosen based on calibration to the tensile and shear test data.

Comparison between tensile test data and the numerical simulation for the uniaxial tension is shown in

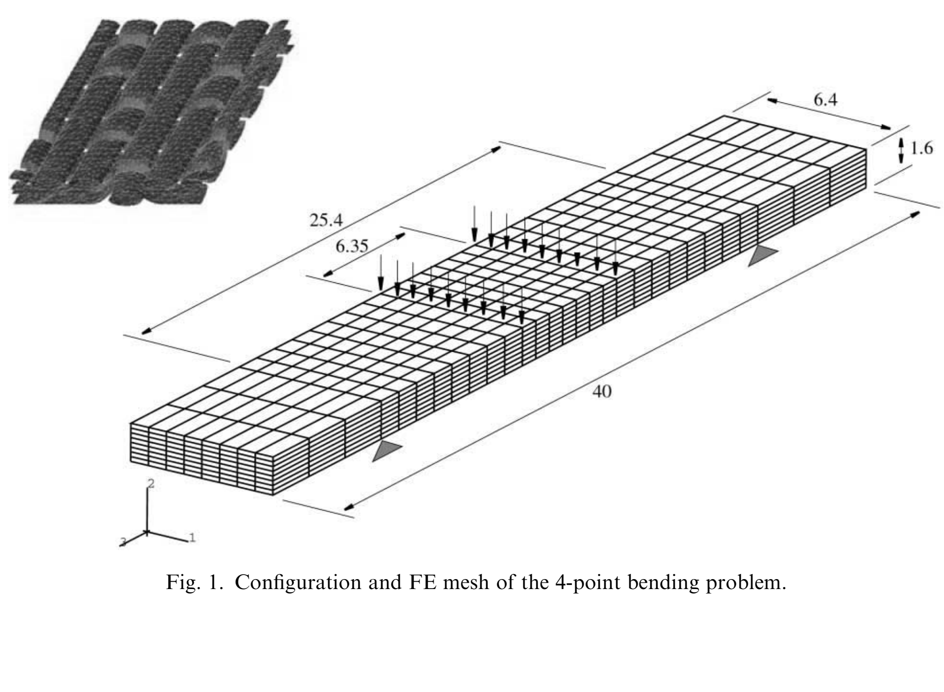


Fig. 1. Configuration and FE mesh of the 4-point bending problem.

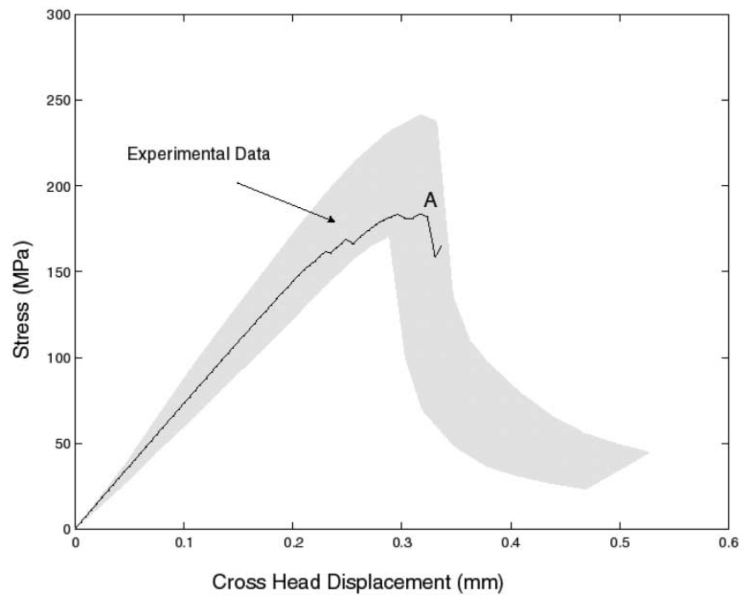


Fig. 2. Force versus displacement for the 4-point bending problem.

Fig. 2. It can be seen that the ultimate experimental stress/strain values in the uniaxial tension test are $\sigma_u = 150 \pm 7$ MPa and $\varepsilon_u = 2.5\times10^{-3} \pm 0.3\times10^{-3}$, while the numerical simulation gives $\sigma_u = 152$ MPa at $\varepsilon_u = 3.2\times10^{-3}$.

Experiments have been conducted on five identical beams and the scattered experimental data of force versus the displacement at the point of load application in the beam are shown by the gray area in Fig. 2. It can be seen that the numerical simulation results are in good agreement with the experimental data in terms of predicting the overall behavior. Both numerical simulation and experimental data predict that the dominant failure mode is tension/compression (so-called bending induced failure)

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