Multiscale Damage Modeling for Composite Materials: Theory and Computational Framework

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Abstract

A nonlocal multiscale continuum damage model is developed for brittle composite materials. A triple-scale asymptotic analysis is generalized to account for the damage phenomena occurring at micro-, meso- and macro- scales. A closed form expressions relating microscopic, mesoscopic and overall strains and damage is derived. The damage evolution is stated on the smallest scale of interest and nonlocal weighted phase average fields over micro- and meso-phases are introduced to alleviate the spurious mesh dependence. Numerical simulation conducted on a composite beam made of Blackglas/Nextel 2D weave is compared with the test data.

1.0 Introduction

Damage phenomena in composite materials are very complex due to significant heterogeneities and interactions between microconstituents. Typically, damage can be either discrete or continuous and described on at least three different scales: discrete for atomistic voids and lattice defects; and continuous for micromechanical and macromechanical scales, which describe either distributed microvoids and microcracks or discrete cracks whose size is comparable to the structural component. Here, attention is restricted to continuum scales only. On the micromechanical scale, the Representative Volume Element (RVE) is introduced to model the initiation and growth of microscopic damage and their effects on the material behavior. The RVE is defined to be small enough to distinguish microscopic heterogeneities, but sufficiently large to represent the overall behavior of the heterogeneous medium. Most research in this area is focused on the two-scale micro-macro problems with homogeneous microconstituents. For certain composite materials systems, such as woven composites [43], the two-scale model might be insufficient due to strong heterogeneities in one of the microphases. The question arises as to how to account for damage effects in these heterogeneous phases. Most commonly, macroscopic-like point of view [12], [30], [32], [34], [39] is adopted by idealizing the heterogeneous phases as anisotropic homogeneous media. Anisotropic continuum damage theory is then employed to model damage evolution in each phase. As an alternative, which is explored here, is to define smaller scale RVE(s) for the heterogeneous phases and then to carry out *multiple* scale damage analysis with various RVEs at different length scales.

The objective of this paper is to extend the two-scale (macro-micro) nonlocal damage theory developed in [22] to three scales in attempt to account for evolution of damage in heterogeneous microphases. Throughout the manuscript we term the larger scale RVE(s) as *mesoscopic* while RVE(s) comprising heterogeneous meso-phases as *microscopic*. In Section 2, the three-scale (macro-, meso- and micro-) damage theory within the framework of the mathematical homogenization theory is developed. The triple-scale asymptotic expansions of damage and displacements lead to closed form expressions relating local (microscopic and mesoscopic) fields to overall (macroscopic) strains and damage. In Section 3, the nonlocal phase fields for multi-phase composites are defined as weighted averages over each phase in the mesoscopic and microscopic characteristic volumes with piecewise constant weight functions. A more general case of weight functions is discussed in [22]. In Section 4, a simplified variant of the nonlocal damage model for the two-phase composite materials is developed. The computational framework including stress update procedure and consistent tangent stiffness are presented in Section 5. In Section 6, we first study the axial loading capacity of the Blackglas/Nextel 2-D woven composite [11][43]. Numerical results obtained by the present three-scale formulation are compared with those obtained by the two-scale model [22]. We then consider a 4-point bending test conducted on the composite beam made of Blackglas/Nextel 2-D woven composite and compare the simulation results with the experiments data provided by [11]. Discussion and future research directions conclude the manuscript.

2.0 Mathematical Homogenization for Damaged Composites

As shown in Figure 1 , the composite material is represented by two locally periodic RVEs on the meso-scale (Y-periodic) and the micro-scale (Z-periodic), denoted by Θ_y and Θ_z, respectively. Let $\boldsymbol{x}$ be the macroscopic coordinate vector in the macro domain Ω; $\boldsymbol{y} \equiv \boldsymbol{x}/\varsigma$ be the mesoscopic position vector in Θ_y and $\boldsymbol{z} \equiv \boldsymbol{y}/\varsigma$ be the microscopic position vector in Θ_z. Here, ς denotes a very small positive number; $\boldsymbol{y} \equiv \boldsymbol{x}/\varsigma$ and $\boldsymbol{z} \equiv \boldsymbol{y}/\varsigma$ are regarded as the stretched local coordinate vectors. When a solid is subjected to some load and boundary conditions, the resulting deformation, stresses, and internal variables may vary from point to point within the RVE(s) due to a high level of heterogeneity. We assume that all quantities on the meso-scale have two explicit dependences: one on the macro-scale $\boldsymbol{x}$ and the other on the meso-scale $\boldsymbol{y}$. For the quantities on the micro-scale, additional dependence on the micro-scale $\boldsymbol{z}$ is introduced. For any microscopically periodic response function f, we have $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{k}\hat{\boldsymbol{y}}, \boldsymbol{z} + \boldsymbol{k}\hat{\boldsymbol{z}})$ in which vectors $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are the basic periods in the meso- and micro- structures and $\boldsymbol{k}$ is a 3 by 3 diagonal matrix with integer components. Adopting the classical nomenclature, any locally periodic function f can be represented as

$$f^{\varsigma}(\boldsymbol{x}) \equiv f(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y})) \tag{1}$$

where superscript ς indicates that the corresponding function f is locally periodic and is a function of macroscopic spatial variables. The indirect macroscopic spatial derivative of f^{ς} is calculated by the chain rule as

$$f^{\varsigma}_{,x_i}(\boldsymbol{x}) = f_{,x_i}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \frac{1}{\varsigma} f_{,y_i}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \frac{1}{\varsigma^2} f_{,z_i}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \tag{2}$$

where the comma followed by a subscript variable denotes the partial derivative (i.e. $f_{,x_i} \equiv \partial f / \partial x_i$). Summation convention for repeated subscripts is employed, except for subscripts x, y and z.

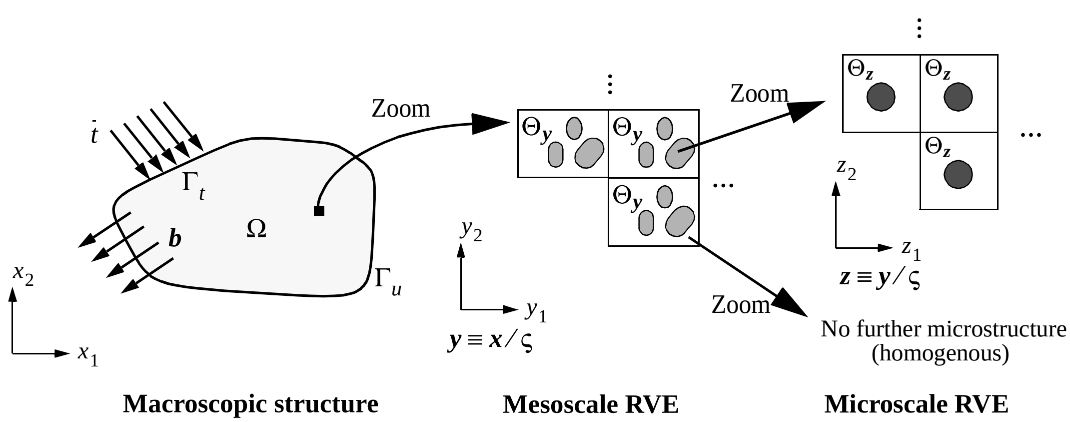

FIGURE 1. Three-Scale Composite Materials

To model the isotropic damage process in meso- and micro- constituents, we define a scalar damage variable $\omega^{\varsigma} = \omega(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. The constitutive equation can be derived from the strain-based continuum damage theory based on the thermodynamics of irreversible processes and internal state variable theory. We assume that micro-constituents possess homogeneous properties and satisfy equilibrium, constitutive, kinematics and compatibility equations as well as jump conditions at the interface. The corresponding boundary value problem on the smallest (micro) scale of interest is described by the following set of equations:

$$\sigma^{\varsigma}_{ij, x_j} + b_i = 0 \qquad \text{in} \quad \Omega \tag{3}$$

$$\sigma^{\varsigma}_{ij} = (1 - \omega^{\varsigma}) L^{\varsigma}_{ijkl} \varepsilon^{\varsigma}_{kl} \qquad \text{in} \quad \Omega \tag{4}$$

$$\varepsilon^{\varsigma}_{ij} = u^{\varsigma}_{(i, x_j)} \qquad \text{in} \quad \Omega \tag{5}$$

$$u^{\varsigma}_i = \bar{u}_i \qquad \text{on} \quad \Gamma_u \tag{6}$$

$$\sigma^{\varsigma}_{ij} n_j = \bar{t}_i \qquad \text{on} \quad \Gamma_t \tag{7}$$

where $\omega^{\varsigma} \in [0, 1)$ is a scalar damage variable governed by a strain history parameter (see Section 4); σ^{ς}_{ij} and $\varepsilon^{\varsigma}_{ij}$ are components of stress and strain tensors; L^{ς}_{ijkl} represents components of elastic stiffness satisfying symmetry

$$L^{\varsigma}_{ijkl} = L^{\varsigma}_{jikl} = L^{\varsigma}_{ijlk} = L^{\varsigma}_{klij} \tag{8}$$

and positivity

$$\exists \ C_0 > 0, \qquad L^{\varsigma}_{ijkl} \xi^{\varsigma}_{ij} \xi^{\varsigma}_{kl} \geq C_0 \xi^{\varsigma}_{ij} \xi^{\varsigma}_{ij} \qquad \forall \xi^{\varsigma}_{ij} = \xi^{\varsigma}_{ji} \tag{9}$$

conditions; b_i is a body force assumed to be independent of local position vectors $\boldsymbol{y}$ and $\boldsymbol{z}$; u^{ς}_i denotes the components of the displacement vector; the subscript pairs with parentheses denote the symmetric gradients defined as

$$u^{\varsigma}_{(i, x_j)} \equiv \frac{1}{2}(u^{\varsigma}_{i, x_j} + u^{\varsigma}_{j, x_i}) \tag{10}$$

Ω denotes the macroscopic domain of interest with boundary Γ; Γ_u and Γ_t are boundary portions where displacements $\bar{u}_i$ and tractions $\bar{t}_i$ are prescribed, respectively, such that $\Gamma_u \cap \Gamma_t = \emptyset$ and $\Gamma = \Gamma_u \cup \Gamma_t$; n_i denotes the normal vector on Γ. We assume that the interface between the phases is perfectly bonded, i.e. $[\sigma^{\varsigma}_{ij} \hat{n}_j] = 0$ and $[u^{\varsigma}_i] = 0$ at the interface, Γ_{int}, where $\hat{n}_i$ is the normal vector to Γ_{int} and $[\bullet]$ is a jump operator.

In the following, the contracted (matrix) notation is adopted. The indices of the contracted notation follow the definition $11 \equiv 1$, $22 \equiv 2$, $33 \equiv 3$, $12 \equiv 4$, $13 \equiv 5$ and $23 \equiv 6$. The scalar quantities and tensor components are denoted by lightface letters. $\boldsymbol{X}^T$ and $\boldsymbol{X}^{-1}$ denote the transpose and inverse of matrix $\boldsymbol{X}$ (or vector), respectively; the subscripts in $\boldsymbol{X}_{mic}$, $\boldsymbol{X}_{meso}$ and $\boldsymbol{X}_{mac}$ represent the scale on which the quantities are measured. i.e. $\boldsymbol{X}_{mic} \equiv \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, $\boldsymbol{X}_{meso} \equiv \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{X}_{mac} \equiv \boldsymbol{X}(\boldsymbol{x})$.

Clearly, the straightforward approach attempting at discretization of the entire macro domain with a grid spacing comparable to that of the microconstituents or even meso-scale features is not computationally feasible. Instead, a mathematical homogenization method based on the triple-scale asymptotic expansion is employed to account for meso- and micro- mechanical effects without explicitly representing the details of local structures in the global analysis. As a starting point, we approximate the microscopic displacement field, $\boldsymbol{u}^{\varsigma}(\boldsymbol{x}) \equiv \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, and the damage variable, $\omega^{\varsigma}(\boldsymbol{x}) \equiv \omega(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, in terms of the triple-scale asymptotic expansions on $\Omega \times \Theta_y \times \Theta_z$:

$$\boldsymbol{u}_{mic} \approx \boldsymbol{u}^0(\boldsymbol{x}) + \varsigma \boldsymbol{u}^1(\boldsymbol{x}, \boldsymbol{y}) + \varsigma^2 \boldsymbol{u}^2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \tag{11}$$

$$\omega_{mic} \approx \omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma \omega^1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma^2 \omega^2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \tag{12}$$

where the superscripts on $\boldsymbol{u}$ and ω denote the length scale of each term in the asymptotic expansions. The strain expansion on $\Omega \times \Theta_y \times \Theta_z$ can be obtained by substituting (11) into (5) with consideration of the indirect differentiation rule (2)

$$\boldsymbol{\varepsilon}_{mic} \equiv \varepsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{\varepsilon}^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma \boldsymbol{\varepsilon}^1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma^2 \boldsymbol{\varepsilon}^2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \tag{13}$$

where the strain components for various orders of ς are given as

$$\boldsymbol{\varepsilon}^0 = \boldsymbol{u}^0_{,\boldsymbol{x}} + \boldsymbol{u}^1_{,\boldsymbol{y}} + \boldsymbol{u}^2_{,\boldsymbol{z}} \tag{14}$$

$$\boldsymbol{\varepsilon}^1 = \boldsymbol{u}^1_{,\boldsymbol{x}} + \boldsymbol{u}^2_{,\boldsymbol{y}} \tag{15}$$

$$\boldsymbol{\varepsilon}^2 = \boldsymbol{u}^2_{,\boldsymbol{x}} \tag{16}$$

We further define scale-average strains by integrating (13) in Θ_z and $\Theta_y \times \Theta_z$ with consideration of Y- and Z- periodic conditions, respectively:

$$\boldsymbol{\varepsilon}_{meso}(\boldsymbol{x}, \boldsymbol{y}) \equiv \frac{1}{|\Theta_z|} \int_{\Theta_z} \boldsymbol{\varepsilon}^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \, d\Theta = \boldsymbol{u}^0_{,\boldsymbol{x}} + \boldsymbol{u}^1_{,\boldsymbol{y}} \tag{17}$$

$$\boldsymbol{\varepsilon}_{mac}(\boldsymbol{x}) \equiv \frac{1}{|\Theta_y|} \frac{1}{|\Theta_z|} \int_{\Theta_y} \int_{\Theta_z} \boldsymbol{\varepsilon}^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \, d\Theta \, d\Theta = \frac{1}{|\Theta_y|} \int_{\Theta_y} \boldsymbol{\varepsilon}_{meso} \, d\Theta = \boldsymbol{u}^0_{,\boldsymbol{x}} \tag{18}$$

Following (17) and (18), the mesoscopic strain obeys the decomposition

$$\boldsymbol{\varepsilon}_{meso} = \boldsymbol{\varepsilon}_{mac} + \boldsymbol{u}^1_{,\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) \tag{19}$$

where the macroscopic strain $\boldsymbol{\varepsilon}_{mac}(\boldsymbol{x})$ in Ω, represents the average strain on meso-scale and $u^1_{,\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})$ is the local oscillatory part of mesoscopic strain in Θ_y. The asymptotic expansion of the microscopic strain (13) can be expressed in a similar fashion as

$$\boldsymbol{\varepsilon}_{mic} = \boldsymbol{\varepsilon}_{meso} + \boldsymbol{u}^2_{,\boldsymbol{z}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + O(\varsigma) \tag{20}$$

where the mesoscopic strain in Θ_y, $\boldsymbol{\varepsilon}_{meso}(\boldsymbol{x}, \boldsymbol{y})$, represents the average strain on the micro-scale, whereas $u^2_{,\boldsymbol{z}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is the local oscillatory part of the microscopic strain in Θ_z.

Stresses and strains for different orders of ς are related by the constitutive equation (4) and the expansion of the damage variable (12):

$$\sigma^0 = (1 - \omega^0)\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^0 \tag{21}$$

$$\sigma^1 = (1 - \omega^0)\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^1 - \omega^1\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^0 \tag{22}$$

$$\boldsymbol{\sigma}^2 = (1 - \omega^0)\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^2 - \omega^1\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^1 - \omega^2\boldsymbol{L}_{mic}\boldsymbol{\varepsilon}^0 \tag{23}$$

The resulting asymptotic expansion of stress is given as

$$\sigma_{mic} \equiv \sigma(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sigma^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma^1\sigma^1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \varsigma^2\sigma^2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + O(\varsigma^3) \tag{24}$$

Once again, we define the scale-average stresses by integrating (24) in Θ_z and $\Theta_y \times \Theta_z$:

$$\sigma_{meso}(\boldsymbol{x}, \boldsymbol{y}) \equiv \frac{1}{|\Theta_z|}\int_{\Theta_z} \sigma^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\ d\Theta \tag{25}$$

$$\sigma_{mac}(\boldsymbol{x}) \equiv \frac{1}{|\Theta_y|}\frac{1}{|\Theta_z|}\int_{\Theta_y}\int_{\Theta_z} \sigma^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\ d\Theta\ d\Theta = \frac{1}{|\Theta_y|}\int_{\Theta_y} \sigma_{meso}\ d\Theta \tag{26}$$

Inserting the stress expansion (24) into the equilibrium equation (3) and making use of indirect differentiation rule (2) yields the following equilibrium equations for various orders:

$$O(\varsigma^{-2}): \quad \sigma^0_{,\boldsymbol{z}} = 0 \tag{27}$$

$$O(\varsigma^{-1}): \quad \sigma^0_{,\boldsymbol{y}} + \sigma^1_{,\boldsymbol{z}} = 0 \tag{28}$$

$$O(\varsigma^0): \quad \sigma^0_{,\boldsymbol{x}} + \sigma^1_{,\boldsymbol{y}} + \sigma^2_{,\boldsymbol{z}} + \boldsymbol{b} = 0 \tag{29}$$

Consider the $O(\varsigma^{-2})$ equilibrium equation (27) first. From (13), (14), (17) and (21) follow

$$\{\mathrm{L}_{mic}\varepsilon_{mic}\}_{,\boldsymbol{z}} = \{\mathrm{L}_{mic}(\varepsilon_{meso} + \boldsymbol{u}^2_{,\boldsymbol{z}})\}_{,\boldsymbol{z}} = 0 \qquad \text{in} \quad \Theta_z \tag{30}$$

$$\mathrm{L}_{mic} = \{1 - \omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\}\boldsymbol{L}_{mic} \tag{31}$$

where L_{mic} is a history-dependent microscopic stiffness.

To solve for (30) we introduce the following decomposition:

$$\boldsymbol{u}^2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = H(\boldsymbol{z})\{\varepsilon_{meso}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{d}_{meso}(\boldsymbol{x}, \boldsymbol{y})\} \tag{32}$$

where the third order tensor $H(\boldsymbol{z}) \equiv H_{ikl}(\boldsymbol{z})$ is symmetric with respect to indices k and l [19][22] and Z-periodic in Θ_z. We assume that $\boldsymbol{d}_{meso}(\boldsymbol{x}, \boldsymbol{y})$ is mesoscopic damage-induced strain driven by mesoscopic strain $\varepsilon_{meso}(\boldsymbol{x}, \boldsymbol{y})$. More specifically, we can state that if $\varepsilon_{meso} = 0$, then $\boldsymbol{d}_{meso} = 0$ and $\omega^0 = 0$. However, vice versa is not true, i.e., if $\boldsymbol{d}_{meso} = 0$ or $\omega^0 = 0$, the mesoscopic strain ε_{meso} may not be necessarily zero.

Based on the decomposition given in (32), $O(\varsigma^{-2})$ equilibrium equation takes the following form:

$$\{L_{mic}\{(\boldsymbol{I} + \boldsymbol{G}_z)\varepsilon_{meso} + \boldsymbol{G}_z\boldsymbol{d}_{meso}\}\}_{,\boldsymbol{z}} = 0 \qquad \text{in} \quad \Theta_z \tag{33}$$

where $\boldsymbol{I}$ is identity matrix and

$$G_z(\boldsymbol{z}) = H_{,\boldsymbol{z}}(\boldsymbol{z}) \equiv H_{(i,z_j)kl}(\boldsymbol{z}) \tag{34}$$

is a micro-scale polarization function. The integrals of the polarization function in Θ_z vanish due to periodicity conditions. Since equation (33) should be valid for arbitrary mesoscopic fields, we may first consider the case of $\boldsymbol{d}_{meso} = 0$ (and $\omega^0 = 0$) but $\varepsilon_{meso} \neq 0$, which yields:

$$\{\boldsymbol{L}_{mic}(\boldsymbol{I} + \boldsymbol{G}_z)\}_{,\boldsymbol{z}} = 0 \tag{35}$$

Equation (35) together with the Z-periodic boundary conditions comprise a linear boundary value problem for H in Θ_z. The weak form of (35) is solved for three right hand side vectors in 2-D and six in 3-D (see for example [21][26]). In absence of damage, the strain asymptotic expansions (13) and (20) can be expressed in terms of the mesoscopic strain ε_{meso} as

$$\varepsilon_{mic} = \boldsymbol{A}_z\, \varepsilon_{meso} + O(\varsigma) \tag{36}$$

where A_z is elastic strain concentration function in Θ_z defined as

$$\boldsymbol{A}_z = \boldsymbol{I} + \boldsymbol{G}_z \tag{37}$$

After solving (35) for H, we proceed to finding $\boldsymbol{d}_{meso}$ from (33). Premultiplying it by H and integrating it by parts in Θ_z with consideration of Z-periodic boundary conditions yields

$$\int_{\Theta_z} \boldsymbol{G}_z^T \boldsymbol{L}_{mic}(\boldsymbol{A}_z \varepsilon_{meso} + \boldsymbol{G}_z \boldsymbol{d}_{meso})\, d\Theta = 0 \tag{38}$$

and the expression for the mesoscopic damage induced strain becomes

$$\boldsymbol{d}_{meso} = -\left\{\int_{\Theta_z} \boldsymbol{G}_z^T \mathrm{L}_{mic} \boldsymbol{G}_z\, d\Theta\right\}^{-1} \left\{\int_{\Theta_z} \boldsymbol{G}_z^T \mathrm{L}_{mic} \boldsymbol{A}_z\, d\Theta\right\} \varepsilon_{meso} \tag{39}$$

Let $\boldsymbol{f} \equiv \{f^{(\alpha)}(\boldsymbol{y});\quad \alpha = 1, 2, \ldots\}$ and $\boldsymbol{g} \equiv \{g^{(\alpha\beta)}(\boldsymbol{z});\quad \alpha, \beta = 1, 2, \ldots\}$ be the two sets of C^{-1} continuous functions, then the damage variable $\omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is assumed to have the following decomposition

$$\omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_\alpha \sum_\beta \omega(\boldsymbol{x}) f^{(\alpha)}(\boldsymbol{y})\, g^{(\alpha\beta)}(\boldsymbol{z}) \tag{40}$$

where $\omega(\boldsymbol{x})$ is the macroscopic damage variable; $f^{(\alpha)}(\boldsymbol{y})$ is a damage distribution function on the meso-scale RVE; $g^{(\alpha\beta)}(\boldsymbol{z})$ represents the damage distribution function in the micro-scale RVE β corresponding to phase α in the meso-scale RVE. Rewriting (39) in terms of $\boldsymbol{A}_{mic}$ and $\omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ yields

$$\boldsymbol{d}_{meso} = \boldsymbol{D}_{meso} \varepsilon_{meso} \tag{41}$$

where

$$\boldsymbol{D}_{meso} = \left(\boldsymbol{I} - \sum_\alpha \sum_\beta \omega(\boldsymbol{x}) f^{(\alpha)}(\boldsymbol{y})\, \boldsymbol{B}^{(\alpha\beta)}\right)^{-1} \left(\sum_\alpha \sum_\beta \omega(\boldsymbol{x}) f^{(\alpha)}(\boldsymbol{y})\, \boldsymbol{C}^{(\alpha\beta)}\right) \tag{42}$$

$$\boldsymbol{B}^{(\alpha\beta)} = \frac{1}{|\Theta_z|}(\tilde{\boldsymbol{L}}_{meso} - \bar{\boldsymbol{L}}_{meso})^{-1} \int_{\Theta_z} g^{(\alpha\beta)}(\boldsymbol{z}) \boldsymbol{G}_z^T \boldsymbol{L}_{mic} \boldsymbol{G}_z\, d\Theta \tag{43}$$

$$\boldsymbol{C}^{(\alpha\beta)} = \frac{1}{|\Theta_z|}(\tilde{\boldsymbol{L}}_{meso} - \bar{\boldsymbol{L}}_{meso})^{-1} \int_{\Theta_z} g^{(\alpha\beta)}(\boldsymbol{z}) \boldsymbol{G}_z^T \boldsymbol{L}_{mic} \boldsymbol{A}_z\, d\Theta \tag{44}$$

$$\tilde{\boldsymbol{L}}_{meso} = \frac{1}{|\Theta_z|} \int_{\Theta_z} \boldsymbol{L}_{mic}\, d\Theta \tag{45}$$

$$\bar{\boldsymbol{L}}_{meso} = \frac{1}{|\Theta_z|}\int_{\Theta_z} \boldsymbol{L}_{mic}\boldsymbol{A}_z \, d\Theta = \frac{1}{|\Theta_z|}\int_{\Theta_z} \boldsymbol{A}_z^T \boldsymbol{L}_{mic}\boldsymbol{A}_z \, d\Theta \tag{46}$$

where $\tilde{\boldsymbol{L}}_{meso}$ is the overall stiffness on Θ_z; $\bar{\boldsymbol{L}}_{meso}$ is the elastic homogenized stiffness [19][22]; $|\Theta_z|$ denotes the volume of the micro-scale RVE. Note that the integrals in $\boldsymbol{B}^{(\alpha\beta)}$ and $\boldsymbol{C}^{(\alpha\beta)}$ are history-independent and thus can be precomputed. This provides one of the main motivations for the decomposition given in (40).

Based on (32) and (41), the asymptotic expansion of the strain field (13) can be finally cast as

$$\varepsilon_{mic} = (\boldsymbol{A}_z + \boldsymbol{G}_z\boldsymbol{D}_{meso})\varepsilon_{meso} + O(\varsigma) \tag{47}$$

where $\boldsymbol{G}_z$ is a local damage strain distribution function in Θ_z. Note that the asymptotic expansion of the strain field is given as a sum of mechanical fields induced by the mesoscopic strain via elastic strain concentration function $\boldsymbol{A}_z$ and thermodynamical fields governed by the damage-induced strain $\boldsymbol{d}_{meso} = \boldsymbol{D}_{meso}\varepsilon_{meso}$ through the distribution function $\boldsymbol{G}_z$.

We now consider the $O(\varsigma^{-1})$ equilibrium equation (28). By integrating (28) over Θ_z, making use of the Z-periodicity condition and the definition of the mesoscopic stress in (25), we get

$$(\sigma_{meso})_{,\boldsymbol{y}} = \left(\frac{1}{|\Theta_z|}\int_{\Theta_z} \sigma^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \, d\Theta\right)_{,\boldsymbol{y}} = 0 \qquad \text{in} \quad \Theta_y \tag{48}$$

which represents the mesomechanical equilibrium equation in Θ_y. Based on the asymptotic expansion of the strain field in (13) and (19), the constitutive equation in (21) and the decomposition (47) we can rewrite (48) as

$$\{\mathrm{L}_{meso}\varepsilon_{meso}\}_{,\boldsymbol{y}} = \{\mathrm{L}_{meso}(\varepsilon_{mac} + \boldsymbol{u}^1_{,\boldsymbol{y}})\}_{,\boldsymbol{y}} = 0 \qquad \text{in} \quad \Theta_y \tag{49}$$

$$\mathrm{L}_{meso} = \frac{1}{|\Theta_z|}\int_{\Theta_z} \mathrm{L}_{mic}(\boldsymbol{A}_z + \boldsymbol{G}_z\boldsymbol{D}_{meso}) \, d\Theta \tag{50}$$

where L_{meso} is a history-dependent mesoscopic stiffness matrix.

To solve for the mesomechanical equilibrium equation, we first note that (49) is similar to its microscopic counterpart in (30). Thus, a similar procedure can be employed by introducing the decomposition:

$$\boldsymbol{u}^1(\boldsymbol{x}, \boldsymbol{y}) = \hat{H}(\boldsymbol{y})\{\varepsilon_{mac}(\boldsymbol{x}) + \boldsymbol{d}_{mac}(\boldsymbol{x})\} \tag{51}$$

where $\hat{H}(\boldsymbol{y}) \equiv \hat{H}_{ikl}(\boldsymbol{y})$ is a Y-periodic third order tensor on the meso-scale, symmetric with respect to indices k and l; $\boldsymbol{d}_{mac}(\boldsymbol{x})$ is the macroscopic damage-induced strain driven by the macroscopic strain $\varepsilon_{mac}(\boldsymbol{x})$. Based on this decomposition, (49) becomes:

$$\{\mathrm{L}_{meso}\{(\boldsymbol{I} + \boldsymbol{G}_y)\varepsilon_{mac} + \boldsymbol{G}_y\boldsymbol{d}_{mac}\}\}_{,\boldsymbol{y}} = 0 \qquad \text{in} \quad \Theta_y \tag{52}$$

where

$$\boldsymbol{G}_y(\boldsymbol{y}) = \hat{H}_{,\boldsymbol{y}}(\boldsymbol{y}) \equiv \hat{H}_{(i,y_j)kl}(\boldsymbol{y}) \tag{53}$$

is a polarization function on the meso-scale whose integral in Θ_y vanishes due to Y-periodicity conditions. Once again, since (53) is valid for arbitrary macroscopic fields, we first consider the case of damage-free, i.e. $\boldsymbol{d}_{mac} = 0$ and $\omega^0 = 0$ but $\varepsilon_{mac} \neq 0$, which yields:

$$\{\boldsymbol{L}_{meso}(\boldsymbol{I} + \boldsymbol{G}_y)\}_{,\boldsymbol{y}} = 0 \tag{54}$$

Equation (54) comprise a linear boundary value problem for $\hat{H}$ in the meso-scale domain Θ_y subjected to Y-periodic boundary conditions. Based on the decomposition in (51) and in absence of damage, the mesoscopic strain in (19) can be expressed in terms of the macroscopic strain ε_{mac} as:

$$\varepsilon_{meso} = \boldsymbol{A}_y \, \varepsilon_{mac} \tag{55}$$

where A_y is the mesoscopic elastic strain concentration function defined as

$$\boldsymbol{A}_y = \boldsymbol{I} + \boldsymbol{G}_y \tag{56}$$

After obtaining $\hat{H}$, $\boldsymbol{d}_{mac}$ can be obtained from (52) by premultiplying it with $\hat{H}$ and integrating it by parts in Θ_y with consideration of Y-periodic boundary conditions, which yields

$$\boldsymbol{d}_{mac} = \boldsymbol{D}_{mac}\varepsilon_{mac} \tag{57}$$

$$\boldsymbol{D}_{mac} = -\left\{\int_{\Theta_y} \boldsymbol{G}_y^T \mathrm{L}_{meso} \boldsymbol{G}_y d\Theta\right\}^{-1} \left\{\int_{\Theta_y} \boldsymbol{G}_y^T \mathrm{L}_{meso} \boldsymbol{A}_y d\Theta\right\} \tag{58}$$

where L_{meso} in (50) is a history-dependent stiffness matrix. Based on the decomposition of the damage variable in (40), L_{meso} is given as

$$\mathrm{L}_{meso} = \left(\bar{\boldsymbol{L}}_{meso} - \sum_{\alpha}\sum_{\beta}\omega(\boldsymbol{x})f^{(\alpha)}(\boldsymbol{y})\ \bar{\boldsymbol{L}}_{meso}^{(\alpha\beta)}\right)(\boldsymbol{I} + \boldsymbol{D}_{meso}) - \left(\tilde{\boldsymbol{L}}_{meso} - \sum_{\alpha}\sum_{\beta}\omega(\boldsymbol{x})f^{(\alpha)}(\boldsymbol{y})\ \tilde{\boldsymbol{L}}_{meso}^{(\alpha\beta)}\right)\boldsymbol{D}_{meso} \tag{59}$$

where

$$\tilde{\boldsymbol{L}}_{meso}^{(\alpha\beta)} = \frac{1}{|\Theta_z|}\int_{\Theta_z} g^{(\alpha\beta)}(\boldsymbol{z})\boldsymbol{L}_{mic}\ d\Theta \tag{60}$$

$$\bar{\boldsymbol{L}}_{meso}^{(\alpha\beta)} = \frac{1}{|\Theta_z|}\int_{\Theta_z} g^{(\alpha\beta)}(\boldsymbol{z})\boldsymbol{L}_{mic}\boldsymbol{A}_z\ d\Theta \tag{61}$$

To this end, the mesoscopic strain (19) and the asymptotic expansion of the microscopic strain field (13) can be directly linked to the macroscopic strain as

$$\varepsilon_{meso} = (\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac})\varepsilon_{mac} \tag{62}$$

$$\varepsilon_{mic} = (\boldsymbol{A}_z + \boldsymbol{G}_z\boldsymbol{D}_{meso})(\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac})\varepsilon_{mac} + O(\varsigma) \tag{63}$$

where $\boldsymbol{G}_y$, the counterpart of $\boldsymbol{G}_z$ in Θ_z, is a local distribution function of the damage-induced strain in Θ_y.

Finally, we integrate the $O(\varsigma^0)$ equilibrium equation (29) over $\Theta_y \times \Theta_z$. The $\int_{\Theta_y}\sigma^1_{,\boldsymbol{y}}d\Theta$ and $\int_{\Theta_z}\sigma^2_{,\boldsymbol{z}}d\Theta$ terms in the integral vanish due to periodicity and we obtain:

$$\left(\frac{1}{|\Theta_y|}\frac{1}{|\Theta_z|}\int_{\Theta_y}\int_{\Theta_z}\sigma^0(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\ d\Theta\ d\Theta\right)_{,\boldsymbol{x}} + \boldsymbol{b} = 0 \qquad \text{in} \quad \Omega \tag{64}$$

Substituting the constitutive relation (21), the asymptotic expansion of the strain field (63), the microscopic and mesoscopic instantaneous stiffnesses in (31) and (50) into (64) yields the macroscopic equilibrium equation

$$(\sigma_{mac})_{,\boldsymbol{x}} + \boldsymbol{b} = 0 \qquad \text{and} \qquad (\mathrm{L}_{mac}\varepsilon_{mac})_{,\boldsymbol{x}} + \boldsymbol{b} = 0 \tag{65}$$

where

$$\mathrm{L}_{mac} = \frac{1}{|\Theta_y|}\int_{\Theta_y}\mathrm{L}_{meso}(\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac})\ d\Theta \tag{66}$$

is a macroscopic instantaneous secant stiffness.

3.0 Nonlocal Piecewise Constant Damage Model for Multi-Phase Materials

Accumulation of damage leads to strain softening and loss of ellipticity in quasi-static problems. The local approach, stating that in absence of thermal effects, stresses at a material point are completely determined by the deformation and the deformation history at that point, may result in a physically unacceptable localization of the deformation [4]-[7], [13], [24]. A number of regularization techniques have been developed to limit strain localization and to alleviate mesh sensitivity associated with strain softening [4]-[10], [13], [14], [15]. One of these approaches is based on smearing solution variables causing strain softening over the characteristic volume of the material [4][6]. Following [6] and [22], the nonlocal damage variable $\bar{\omega}(\boldsymbol{x})$ is defined as:

$$\bar{\omega}(\boldsymbol{x}) = \frac{1}{|\Theta_{C_y}|}\frac{1}{|\Theta_{C_z}|}\int_{\Theta_{Cy}}\int_{\Theta_{Cz}} \varphi_y(\boldsymbol{y})\varphi_z(\boldsymbol{z})\omega^0(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\, d\Theta\, d\Theta \tag{67}$$

where $\varphi_z(\boldsymbol{y})$ and $\varphi_y(\boldsymbol{y})$ are weight functions on micro- and meso-scale, respectively; Θ_{C_z} and Θ_{C_y} are the characteristic volumes on the micro- and meso- scales with characteristic length l_{C_z} and l_{C_y}, respectively. The characteristic length is defined (for example) as a radius of the largest inscribed sphere in a characteristic volume, which is related to the size of the material inhomogeneties [6]. l_{H_z} and l_{H_y} are the radii of the largest inscribed spheres in the Statistically Homogeneous Volumes (SHV), which is the smallest volume for which the corresponding local periodicity assumptions are valid. Several guidelines for determining the value of characteristic length have been provided in [5] and [24]. The characteristic lengths, l_{C_z} and l_{C_y}, as indicated in [6], are usually smaller than the corresponding l_{H_z} and l_{H_y} in particular for random local structures. Following the two-scale nonlocal damage model in [22], we redefine the Representative Volume Element (RVE) as the maximum between Statistically Homogeneous Volume and the characteristic volume. Schematically, this can be expressed as

$$l_{RVE_{mic}} = max\{l_{H_z}, l_{C_z}\} \tag{68}$$

$$l_{RVE_{meso}} = max\{l_{H_y}, l_{C_y}\} \tag{69}$$

where $l_{RVE_{mic}}$ and $l_{RVE_{meso}}$ denote the radii of the largest inscribed spheres in the mesoscopic and macroscopic RVEs, respectively. Figure 2 illustrates the two possibilities for construction of the RVE on the mesoscale in a two-phase medium: one for random microstructure where RVE typically coincides with SHV, and another one for periodic microstructure, where RVE and SHV are of the same order of magnitude. Figure 2 is also applicable to the definition of the microscopic RVE.

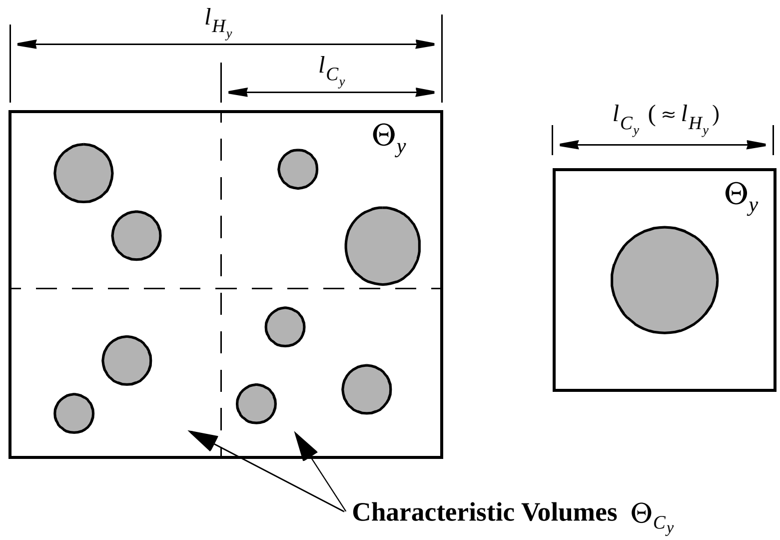

FIGURE 2. Selection of the Representative Volume Element

In particular, we assume that the microscopic and mesoscopic damage distribution functions, $g^{(\alpha\beta)}(\boldsymbol{z})$ and $f^{(\alpha)}(\boldsymbol{y})$ in (40), are both piecewise constant; $f^{(\alpha)}(\boldsymbol{y})$ is assumed to be unity within the domain of the mesoscopic phase $\Theta_y^{(\alpha)}$ which satisfies $\Theta_y^{(\alpha)} \subset \Theta_{C_y} \subset \Theta_y$, and to vanish elsewhere, i.e.

$$f^{(\alpha)}(\boldsymbol{y}) = \begin{cases} 1 & \text{if } \boldsymbol{y} \in \Theta_y^{(\alpha)} \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

where $\bigcup_{\alpha=1}^{k_\alpha} \Theta_y^{(\alpha)} = \Theta_y$ and $\Theta_y^{(i)} \cap \Theta_y^{(j)} = \varnothing$ for $i \neq j$ and $i, j = 1, 2, \ldots, k_\alpha$; k_α is a product of the number of different mesoscopic phases and the number of mesoscopic characteristic volumes in the mesoscopic RVE Θ_y; $g^{(\alpha\beta)}(\boldsymbol{z})$ is unity within the microphase $\Theta_z^{(\alpha\beta)}$, such that $\Theta_z^{(\alpha\beta)} \subset \Theta_{C_z}^{(\alpha)} \subset \Theta_z^{(\alpha)}$, but vanish elsewhere, i.e.

$$g^{(\alpha\beta)}(\boldsymbol{z}) = \begin{cases} 1 & \text{if } \boldsymbol{z} \in \Theta_z^{(\alpha\beta)} \\ 0 & \text{otherwise} \end{cases} \tag{71}$$

where $\bigcup_{\beta=1}^{k_{\alpha\beta}} \Theta_z^{(\alpha\beta)} = \Theta_z^{(\alpha)}$ and $\Theta_y^{(i)} \cap \Theta_y^{(j)} = \varnothing$ for $i \neq j$ and $i, j = 1, 2, \ldots, k_{\alpha\beta}$; $k_{\alpha\beta}$ is the product of the number of different microphases and the number of microscopic characteristic volumes in $\Theta_z^{(\alpha)}$.

We further define the weight function in (67) as

$$\varphi_y(\boldsymbol{y}) = \mu_y^{(\alpha)} f^{(\alpha)}(\boldsymbol{y}) \quad \text{and} \quad \varphi_z(\boldsymbol{z}) = \mu_z^{(\alpha\beta)} g^{(\alpha\beta)}(\boldsymbol{z}) \tag{72}$$

where the constants $\mu_y^{(\alpha)}$ and $\mu_z^{(\alpha\beta)}$ are determined by the orthogonality conditions

$$\frac{\mu_y^{(\alpha)}}{|\Theta_{C_y}|}\int_{\Theta_{Cy}} f^{(\alpha)}(\boldsymbol{y}) f^{(\hat{\alpha})}(\boldsymbol{y})\, d\Theta = \delta_{\alpha\hat{\alpha}}\,, \qquad \alpha, \hat{\alpha} = 1, 2, \ldots, k_\alpha \tag{73}$$

$$\frac{\mu_z^{(\alpha\beta)}}{|\Theta_{C_z}^{(\alpha)}|}\int_{\Theta_{Cz}^{(\alpha)}} g^{(\alpha\beta)}(\boldsymbol{z}) g^{(\alpha\hat{\beta})}(\boldsymbol{z})\, d\Theta = \delta_{\beta\hat{\beta}}\,, \qquad \beta, \hat{\beta} = 1, 2, \ldots, k_{\alpha\beta} \tag{74}$$

$\delta_{\alpha\hat{\alpha}}$ and $\delta_{\beta\hat{\beta}}$ are the Kronecker deltas. Substituting (40) and (70)-(74) into (67) shows that $\omega^{(\alpha\beta)}$ coincides with the nonlocal phase average damage variable:

$$\bar{\omega}(\boldsymbol{x}) = \frac{\mu_y^{(\alpha)}}{|\Theta_{C_y}|}\frac{\mu_z^{(\alpha\beta)}}{|\Theta_{C_z}^{(\alpha)}|}\int_{\Theta_{Cy}}\int_{\Theta_{Cz}^{(\alpha)}} \omega(\boldsymbol{x})\{f^{(\alpha)}(\boldsymbol{y}) g^{(\alpha\beta)}(\boldsymbol{z})\}^2\, d\Theta\, d\Theta \equiv \omega^{(\alpha\beta)}(\boldsymbol{x}) \tag{75}$$

The average strains in each microphase, termed as microscopic phase average strain, are obtained by integrating (63) over $\Theta_z^{(\alpha\beta)}$

$$\varepsilon_{meso}^{(\alpha\beta)} = \frac{1}{|\Theta_z^{(\alpha\beta)}|}\int_{\Theta_z^{(\alpha\beta)}} \varepsilon_{mic}\, d\Theta = (\bar{\boldsymbol{A}}_z^{(\alpha\beta)} + \bar{\boldsymbol{G}}_z^{(\alpha\beta)} \boldsymbol{D}_{meso}^{(\alpha)})(\boldsymbol{A}_y^{(\alpha)} + \boldsymbol{G}_y^{(\alpha)} \boldsymbol{D}_{mac})\varepsilon_{mac} \tag{76}$$

where $\Theta_z^{(\alpha\beta)}$ represents the domain of phase β in $\Theta_z^{(\alpha)}$

$$\boldsymbol{G}_y^{(\alpha)} \equiv \boldsymbol{G}_y(\boldsymbol{y} \in \Theta_y^{(\alpha)}) \quad \text{and} \quad \boldsymbol{A}_y^{(\alpha)} = \boldsymbol{I} + \boldsymbol{G}_y^{(\alpha)} \tag{77}$$

$$\bar{\boldsymbol{G}}_z^{(\alpha\beta)} = \frac{1}{|\Theta_z^{(\alpha\beta)}|}\int_{\Theta_z^{(\alpha\beta)}} \boldsymbol{G}_z(\boldsymbol{z})\, d\Theta \quad \text{and} \quad \bar{\boldsymbol{A}}_z^{(\alpha\beta)} = \boldsymbol{I} + \bar{\boldsymbol{G}}_z^{(\alpha\beta)} \tag{78}$$

$$\boldsymbol{D}_{meso}^{(\alpha)} \equiv \boldsymbol{D}_{meso}(\boldsymbol{x}, \boldsymbol{y} \in \Theta_y^{(\alpha)}) = \left(\boldsymbol{I} - \sum_\alpha \omega^{(\alpha\beta)} \boldsymbol{B}^{(\alpha\beta)}\right)^{-1} \left(\sum_\beta \omega^{(\alpha\beta)} \boldsymbol{C}^{(\alpha\beta)}\right) \tag{79}$$

and following (43)-(46) and (71) yields

$$\boldsymbol{B}^{(\alpha\beta)} = \frac{1}{|\Theta_z^{(\alpha)}|}(\tilde{\boldsymbol{L}}_{meso}^{(\alpha)} - \bar{\boldsymbol{L}}_{meso}^{(\alpha)})^{-1}\int_{\Theta_z^{(\alpha\beta)}} \boldsymbol{G}_z^T \boldsymbol{L}_{mic} \boldsymbol{G}_z\, d\Theta \tag{80}$$

$$\boldsymbol{C}^{(\alpha\beta)} = \frac{1}{|\Theta_z^{(\alpha)}|}(\tilde{\boldsymbol{L}}_{meso}^{(\alpha)} - \bar{\boldsymbol{L}}_{meso}^{(\alpha)})^{-1}\int_{\Theta_z^{(\alpha\beta)}} \boldsymbol{G}_z^T \boldsymbol{L}_{mic} \boldsymbol{A}_z\, d\Theta \tag{81}$$

$$\tilde{\boldsymbol{L}}_{meso}^{(\alpha)} = \frac{1}{\left|\Theta_z^{(\alpha)}\right|}\int_{\Theta_z^{(\alpha)}} \boldsymbol{L}_{mic}\, d\Theta \tag{82}$$

$$\bar{\boldsymbol{L}}_{meso}^{(\alpha)} = \frac{1}{\left|\Theta_z^{(\alpha)}\right|}\int_{\Theta_z^{(\alpha)}} \boldsymbol{L}_{mic}\boldsymbol{A}_z\, d\Theta = \frac{1}{\left|\Theta_z^{(\alpha)}\right|}\int_{\Theta_z^{(\alpha)}} \boldsymbol{A}_z^T\boldsymbol{L}_{mic}\boldsymbol{A}_z\, d\Theta \tag{83}$$

We denote the volume fractions for phase β in $\Theta_z^{(\alpha)}$ by $v^{(\alpha\beta)} \equiv \left|\Theta_z^{(\alpha\beta)}\right| / \left|\Theta_z^{(\alpha)}\right|$ such that $\sum_\beta v^{(\alpha\beta)} = 1$. It can be readily seen that

$$\varepsilon_{meso}^{(\alpha)} \equiv \varepsilon_{meso}(\boldsymbol{x}, \boldsymbol{y} \in \Theta_y^{(\alpha)}) = \sum_{\beta} v^{(\alpha\beta)}\varepsilon_{meso}^{(\alpha\beta)} \tag{84}$$

Similarly, the phase average strain in the mesoscopic RVE Θ_y can be obtained by integrating (62) over $\Theta_y^{(\alpha)}$, which yields

$$\varepsilon_{mac}^{(\alpha)} = \frac{1}{\left|\Theta_y^{(\alpha)}\right|}\int_{\Theta_y^{(\alpha)}} \varepsilon_{meso}\, d\Theta = (\bar{\boldsymbol{A}}_y^{(\alpha)} + \bar{\boldsymbol{G}}_y^{(\alpha)}\boldsymbol{D}_{mac})\varepsilon_{mac} \tag{85}$$

where the mesoscopic phase average strain concentration function $\bar{\boldsymbol{A}}_y^{(\alpha)}$ is defined as

$$\bar{\boldsymbol{G}}_y^{(\alpha)} = \frac{1}{\left|\Theta_y^{(\alpha)}\right|}\int_{\Theta_y^{(\alpha)}} \boldsymbol{G}_y(\boldsymbol{y})\, d\Theta \quad \text{and} \quad \bar{\boldsymbol{A}}_y^{(\alpha)} = \boldsymbol{I} + \bar{\boldsymbol{G}}_y^{(\alpha)} \tag{86}$$

Also, we have the relation

$$\varepsilon_{mac} = \sum_{\alpha} v^{(\alpha)}\varepsilon_{mac}^{(\alpha)} \tag{87}$$

where $v^{(\alpha)} \equiv \left|\Theta_y^{(\alpha)}\right| / \left|\Theta_y\right|$ is the volume fraction of phase α in Θ_y satisfying $\sum_\alpha v^{(\alpha)} = 1$

To construct the nonlocal constitutive relation between phase averages, we define the local average stress in $\Theta_z^{(\alpha)}$ as:

$$\sigma_{meso}^{(\alpha\beta)} \equiv \frac{1}{\left|\Theta_z^{(\alpha\beta)}\right|}\int_{\Theta_z^{(\alpha\beta)}} \sigma^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\, d\Theta \tag{88}$$

By combining (88) with microscopic constitutive equation (21), the asymptotic expansion of strain in (13) and (47) and the piecewise constant damage variable defined in (40), (70) and (71), we get

$$\sigma_{meso}^{(\alpha\beta)} = (1 - \omega^{(\alpha\beta)})\boldsymbol{L}_{mseo}^{(\alpha\beta)}\varepsilon_{meso}^{(\alpha\beta)} \tag{89}$$

where $\boldsymbol{L}_{meso}^{(\alpha\beta)} \equiv \boldsymbol{L}_{mic}(\boldsymbol{x}, \boldsymbol{y} \in \Theta_y^{(\alpha)}, \boldsymbol{z} \in \Theta_z^{(\alpha\beta)})$. Here, we assume that each phase in the micro-scale RVE is homogeneous, i.e. $\boldsymbol{L}_{meso}^{(\alpha\beta)}$ is assumed to be independent of the position vector $\boldsymbol{z}$ within each phase $\Theta_z^{(\alpha\beta)}$.

The phase average stress in the mesoscopic RVE Θ_y is defined in a similar fashion to (88) as:

$$\sigma_{mac}^{(\alpha)} \equiv \frac{1}{\left|\Theta_y^{(\alpha)}\right|} \int_{\Theta_y^{(\alpha)}} \sigma_{meso} \, d\Theta \tag{90}$$

With the definition of σ_{meso} in (25), the microscopic constitutive equation (21), equations (31) and (50), the definition in (90) can be restated as

$$\sigma_{mac}^{(\alpha)} = \mathrm{L}_{mac}^{(\alpha)} \varepsilon_{mac} \tag{91}$$

where the macroscopic phase average instantaneous secant stiffness $\mathrm{L}_{mac}^{(\alpha)}$ in $\Theta_y^{(\alpha)}$ is given as

$$\mathrm{L}_{mac}^{(\alpha)} = \frac{1}{\left|\Theta_y^{(\alpha)}\right|} \int_{\Theta_y^{(\alpha)}} \mathrm{L}_{meso}^{(\alpha)} (\boldsymbol{A}_y + \boldsymbol{G}_y \boldsymbol{D}_{mac}) d\Theta \tag{92}$$

and

$$\sigma_{mac} = \sum_{\alpha} v^{(\alpha)} \sigma_{mac}^{(\alpha)} \quad \text{and} \quad \mathrm{L}_{mac} = \sum_{\alpha} v^{(\alpha)} \mathrm{L}_{mac}^{(\alpha)} \tag{93}$$

which follows from (26) and (91). The mesoscopic instantaneous stiffness $\mathrm{L}_{meso}^{(\alpha)}$ can be obtained from (59) in conjunction with the definition of the piecewise constant damage so that

$$\begin{aligned} \mathrm{L}_{meso}^{(\alpha)} &\equiv \mathrm{L}_{meso}(\boldsymbol{x}, \boldsymbol{y} \in \Theta_y^{(\alpha)}) \\ &= \left(\bar{\boldsymbol{L}}_{meso} - \sum_{\beta} v^{(\alpha\beta)} \omega^{(\alpha\beta)} \bar{\boldsymbol{L}}_{meso}^{(\alpha\beta)}\right)(\boldsymbol{I} + \boldsymbol{D}_{meso}^{(\alpha)}) - \left(\tilde{\boldsymbol{L}}_{meso} - \sum_{\beta} v^{(\alpha\beta)} \omega^{(\alpha\beta)} \tilde{\boldsymbol{L}}_{meso}^{(\alpha\beta)}\right) \boldsymbol{D}_{meso}^{(\alpha)} \end{aligned} \tag{94}$$

where the phase average stiffness $\tilde{\boldsymbol{L}}_{meso}^{(\alpha\beta)}$ and the homogenized stiffness $\bar{\boldsymbol{L}}_{meso}^{(\alpha\beta)}$ are obtained from (60) and (61):

$$\tilde{\boldsymbol{L}}_{meso}^{(\alpha\beta)} = \frac{1}{\left|\Theta_z^{(\alpha\beta)}\right|} \int_{\Theta_z^{(\alpha\beta)}} \boldsymbol{L}_{mic} \, d\Theta \quad \text{and} \quad \bar{\boldsymbol{L}}_{meso}^{(\alpha\beta)} = \frac{1}{\left|\Theta_z^{(\alpha\beta)}\right|} \int_{\Theta_z^{(\alpha\beta)}} \boldsymbol{L}_{mic} \boldsymbol{A}_z \, d\Theta \tag{95}$$

The constitutive equations (89) and (91) have a nonlocal character in the sense that they relate between phase averages in the microscopic and mesoscopic RVEs, respectively. The response characteristics between the phases are not smeared as the damage evolution law and thermomechanical properties of phases might be significantly different, in particular when damage occurs in a single phase. In the next section we focus on the multiscale damage modeling in woven composites.

4.0 Damage Evolution for Two-Phase Composites

As a special case depicted in Figure 1, we consider a three-scale composite whose mesoscopic structure is composed of reinforcement phase $(\alpha = F)$ and matrix phase $(\alpha = M)$ such that $\Theta_y = \Theta_y^{(F)} \cup \Theta_y^{(M)}$ and the volume fractions satisfy $v^{(F)} + v^{(M)} = 1$. We assume that the matrix phase $\Theta_y^{(M)}$ in the mesoscopic RVE Θ_y is homogeneous and isotropic, i.e. the stiffness of matrix is independent of any position vectors such that

$$\bar{\boldsymbol{L}}_{meso}^{(M)} \equiv \tilde{\boldsymbol{L}}_{meso}^{(M)} \equiv \boldsymbol{L}_{mac}^{(M)} = \text{constant} \tag{96}$$

Since the matrix phase in Θ_y is homogeneous, there are no microscopic structures for $\Theta_y^{(M)}$, and therefore $\boldsymbol{G}_z \equiv 0$ and $\boldsymbol{A}_z \equiv \boldsymbol{I}$ in $\Theta_y^{(M)}$. From (79)-(83), follows that

$$\boldsymbol{D}_{meso}^{(M)} \equiv 0 \tag{97}$$

For the reinforcement phase $\Theta_y^{(F)}$, we assume that it consists of a two-phase composite (reinforcement $\beta = f$ and matrix $\beta = m$) characterized by the microscopic RVE $\Theta_z^{(F)}$, such that $\Theta_z^{(F)} = \Theta_z^{(Ff)} \cup \Theta_z^{(Fm)}$ and $v^{(Ff)} + v^{(Fm)} = 1$.

For simplicity, we further assume that damage occurs in the matrix only. The expression of the damage variable (40), (70) and (71) can be further simplified for this special case as

$$\omega^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \begin{cases} \omega^{(M)}(\boldsymbol{x}) & \boldsymbol{y} \in \Theta_y^{(M)} \\ \omega^{(Fm)}(\boldsymbol{x}) & \boldsymbol{y} \in \Theta_y^{(F)} \text{ and } \boldsymbol{z} \in \Theta_z^{(Fm)} \\ 0 & \text{otherwise} \end{cases} \tag{98}$$

Accordingly, the mesoscopic instantaneous stiffness for $\Theta_y^{(F)}$ and $\Theta_y^{(M)}$ in (94) becomes

$$\mathrm{L}_{meso}^{(F)} = \{\bar{\boldsymbol{L}}_{meso}^{(F)} - v^{(Fm)}\omega^{(Fm)}\bar{\boldsymbol{L}}_{meso}^{(Fm)}\}(\boldsymbol{I} + \boldsymbol{D}_{meso}^{(F)}) - \{\tilde{\boldsymbol{L}}_{meso}^{(F)} - v^{(Fm)}\omega^{(Fm)}\tilde{\boldsymbol{L}}_{meso}^{(Fm)}\}\boldsymbol{D}_{meso}^{(F)} \tag{99}$$

$$\mathrm{L}_{meso}^{(M)} = (1 - \omega^{(M)})\boldsymbol{L}_{mac}^{(M)} \tag{100}$$

where

$$\boldsymbol{D}_{meso}^{(F)} = (\boldsymbol{I} - \omega^{(Fm)}\boldsymbol{B}^{(Fm)})^{-1}\omega^{(Fm)}\boldsymbol{C}^{(Fm)} \tag{101}$$

The macroscopic phase average instantaneous stiffness $\mathrm{L}_{mac}^{(\alpha)}$ $(\alpha = F, M)$ given in (92) takes the following form:

$$\mathrm{L}_{mac}^{(F)} = \frac{1}{\left|\Theta_y^{(F)}\right|}\int_{\Theta_y^{(F)}} \mathrm{L}_{meso}^{(F)}(\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac})d\Theta \tag{102}$$

$$\mathrm{L}_{mac}^{(M)} = \mathrm{L}_{meso}^{(M)}(\bar{\boldsymbol{A}}_y^{(M)} + \bar{\boldsymbol{G}}_y^{(M)}\boldsymbol{D}_{mac}) \tag{103}$$

The expression of $\boldsymbol{D}_{mac}$ can be further modified by substituting (99) and (100)

$$\boldsymbol{D}_{mac} = -\left\{\int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T \mathrm{L}_{meso}^{(F)} \boldsymbol{G}_y d\Theta + \int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T \mathrm{L}_{meso}^{(M)} \boldsymbol{G}_y d\Theta\right\}^{-1} \left\{\int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T \mathrm{L}_{meso}^{(F)} \boldsymbol{A}_y d\Theta + \int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T \mathrm{L}_{meso}^{(M)} \boldsymbol{A}_y d\Theta\right\} \tag{104}$$

The damage variable $\omega^{(\eta)}$ $(\eta = M, Fm)$ is assumed to be a monotonically increasing function of nonlocal phase deformation history parameter $\kappa^{(\eta)}$ (see, for example, [12], [22], [24], [27], [28] and [38]) which characterizes the maximum deformation experienced throughout the loading history. In general, the evolution of phase damage at time t can be expressed as

$$\omega^{(\eta)}(\boldsymbol{x}, t) = f(< \kappa^{(\eta)}(\boldsymbol{x}, t) - \bar{\vartheta}_{ini}^{(\eta)} >_+) \qquad \text{and} \qquad \frac{\partial f(\kappa^{(\eta)}(\boldsymbol{x}, t))}{\partial \kappa^{(\eta)}} \geq 0 \tag{105}$$

where $\eta = M, Fm$; the operator $< >_+$ denotes the positive part, i.e. $< \bullet >_+ = sup\{ 0 , \bullet \}$; the phase deformation history parameter $\kappa^{(\eta)}$ is determined by the evolution of the nonlocal phase damage equivalent strain, denoted by $\bar{\vartheta}^{(\eta)}$

$$\kappa^{(\eta)}(\boldsymbol{x}, t) = max\{\bar{\vartheta}^{(\eta)}(\boldsymbol{x}, \tau) | (\tau \leq t) , \bar{\vartheta}_{ini}^{(\eta)}\} \tag{106}$$

where $\bar{\vartheta}_{ini}^{(\eta)}$ represents the threshold value of the damage equivalent strain prior to the initiation of phase damage. Based on the strain-based damage theory [38] we defined the damage equivalent strain $\bar{\vartheta}^{(\eta)}$ as

$$\bar{\vartheta}^{(\eta)} = \sqrt{\frac{1}{2}(\boldsymbol{F}^{(\eta)}\hat{\boldsymbol{\varepsilon}}_{mac}^{(\eta)})^T \hat{\boldsymbol{L}}_{mac}^{(M)} (\boldsymbol{F}^{(\eta)}\hat{\boldsymbol{\varepsilon}}_{mac}^{(\eta)})}, \qquad \eta = M, Fm \tag{107}$$

where $\hat{\varepsilon}^{(\eta)}_{mac}$ represents the principal nonlocal phase strain vector, i.e. $\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac} = [\varepsilon_1^{(\eta)}, \varepsilon_2^{(\eta)}, \varepsilon_3^{(\eta)}]^T_{mac}$ and $\hat{\boldsymbol{L}}^{(M)}_{mac}$ is the mapping of $\boldsymbol{L}^{(M)}_{mac}$ in the corresponding principal strain directions; $\boldsymbol{F}^{(\eta)}$ denotes the weighting matrix aimed at accounting for different damage accumulation in tension and compression

$$\boldsymbol{F}^{(\eta)} = \begin{bmatrix} h_1^{(\eta)} & 0 & 0 \\ 0 & h_2^{(\eta)} & 0 \\ 0 & 0 & h_3^{(\eta)} \end{bmatrix}, \qquad \eta = M, Fm \tag{108}$$

$$h_\xi^{(\eta)} \equiv h(\varepsilon_\xi^{(\eta)}) = \frac{1}{2} + \frac{1}{\pi}\mathrm{atan}[c_1^{(\eta)}(\varepsilon_\xi^{(\eta)} - c_2^{(\eta)})], \qquad \xi = 1, 2, 3 \tag{109}$$

where $c_1^{(\eta)}$ and $c_2^{(\eta)}$ are constants selected to represent the contributions of each component of the nonlocal principal phase strain to phase damage equivalent strain $\vartheta^{(\eta)}$. Figure 3 illustrates the influences of both constants. As an extreme case, when $c_1^{(\eta)} \to \infty$ and $c_2^{(\eta)} = 0$, the weight function reduces to $h(\varepsilon_\xi^{(\eta)}) = \langle \varepsilon_\xi^{(\eta)} \rangle_+ / \varepsilon_\xi^{(\eta)}$ so that the compressive principal phase strain components have no contribution to the phase damage equivalent strain.

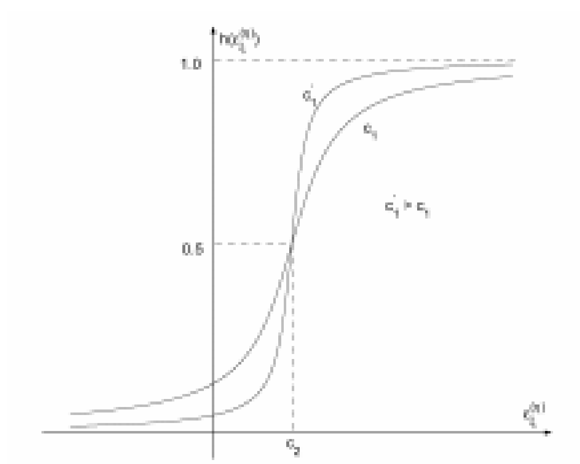

FIGURE 3. Weight Functions for Principal Phase Average Strains

The phase average strain in (107) is defined as

$$\varepsilon^{(\alpha\beta)}_{mac} = \frac{1}{|\Theta_y^{(\alpha)}|}\frac{1}{|\Theta_z^{(\alpha\beta)}|}\int_{\Theta_y^{(\alpha)}}\int_{\Theta_z^{(\alpha\beta)}} \varepsilon_{mic}\, d\Theta\, d\Theta \tag{110}$$

When damage occurs in the matrix phase only, $\alpha\beta = \eta = M, Fm$, equation (110) is applied to matrix phases on the two scales. Exploiting (76) and (85), we get

$$\varepsilon_{mac}^{(Fm)} = \frac{1}{\left|\Theta_y^{(F)}\right|}\int_{\Theta_y^{(F)}} (\bar{\boldsymbol{A}}_z^{(Fm)} + \bar{\boldsymbol{G}}_z^{(Fm)}\boldsymbol{D}_{meso}^{(F)})(\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac})\, d\Theta\ \varepsilon_{mac} \tag{111}$$

$$\varepsilon_{mac}^{(M)} = (\bar{\boldsymbol{A}}_y^{(M)} + \bar{\boldsymbol{G}}_y^{(M)}\boldsymbol{D}_{mac})\varepsilon_{mac} \tag{112}$$

The arctangent form evolution law [22]

$$\Phi^{(\eta)} \equiv \frac{\mathrm{atan}\left\{a^{(\eta)}\left(\frac{\langle \bar{\vartheta}^{(\eta)} - \bar{\vartheta}_{ini}^{(\eta)} \rangle_+}{\bar{\vartheta}_0^{(\eta)}}\right) - b^{(\eta)}\right\} + \mathrm{atan}(b^{(\eta)})}{\frac{\pi}{2} + \mathrm{atan}(b^{(\eta)})} - \omega^{(\eta)} = 0, \qquad \eta = M, Fm \tag{113}$$

is adopted, in which $a^{(\eta)}, b^{(\eta)}$ are material parameters; $\bar{\vartheta}_0^{(\eta)}$ denotes the critical value of the strain history parameter beyond which the damage will develop very quickly.

Based on the definition of the phase damage equivalent strain $\bar{\vartheta}^{(\eta)}$ in (107), the damage evolution conditions can be expressed as

$$if\ \bar{\vartheta}^{(\eta)} - \kappa^{(\eta)} = 0,\ \dot{\kappa}^{(\eta)} > 0 \quad \Rightarrow \quad \text{damage process:}\ \dot{\omega}^{(\eta)} > 0 \tag{114}$$

$$\left.\begin{array}{c} if\ \bar{\vartheta}^{(\eta)} - \kappa^{(\eta)} < 0 \\ \text{or} \\ if\ \bar{\vartheta}^{(\eta)} - \kappa^{(\eta)} = 0,\ \dot{\kappa}^{(\eta)} = 0 \end{array}\right\} \quad \Rightarrow \quad \text{elastic process:}\ \dot{\omega}^{(\eta)} = 0 \tag{115}$$

where $\dot{\omega}^{(\eta)}$ denotes the rate of damage.

5.0 Computational issues

In this section, we describe the computational aspects of the nonlocal piecewise constant damage model for the two-phase material developed in Section 4.0. Due to the nonlinear character of the problem an incremental analysis is employed. Prior to the nonlinear analysis elastic strain concentration factors, $\boldsymbol{G}_y(\boldsymbol{y})$ and $\boldsymbol{G}_z(\boldsymbol{z})$, are computed in the mesoscopic and microscopic RVEs, respectively, using the finite element method. Subsequently, the phase average elastic strain concentration factor $\bar{\boldsymbol{A}}_y^{(M)}$ and the damage distribution factor $\bar{\boldsymbol{G}}_y^{(M)}$ are precomputed using (86), (78) and subsequently $\bar{\boldsymbol{A}}_z^{(FM)}$ and $\bar{\boldsymbol{G}}_z^{(Fm)}$ are evaluated.

5.1 Stress update (integration) procedure

Given: displacement vector ${}_t\boldsymbol{u}_{mac}$; overall strain ${}_t\varepsilon_{mac}$; strain history parameters ${}_t\kappa^{(M)}$ and ${}_t\kappa^{(Fm)}$; phase damage variables ${}_t\omega^{(M)}$ and ${}_t\omega^{(Fm)}$; and displacement increment $\Delta\boldsymbol{u}_{mac}$ calculated from the finite element analysis for the global problem. The left subscript denotes the increment step, i.e., ${}_{t+\Delta t}\square$ is the variables in the current increment, whereas ${}_t\square$ is a converged variable from the last increment. For simplicity, we will omit the left subscript for the current increment, i.e., $\square \equiv {}_{t+\Delta t}\square$, and use superscript η $(= M, Fm)$ to denote the matrix phases in both mesoscopic and microscopic RVEs.

Find: overall strain $\boldsymbol{\varepsilon}_{mac}$; nonlocal strain history parameters $\kappa^{(M)}$ and $\kappa^{(Fm)}$; nonlocal phase damage variables $\omega^{(M)}$ and $\omega^{(Fm)}$; overall stress σ_{mac} and nonlocal phase stresses $\sigma_{mac}^{(F)}$ and $\sigma_{mac}^{(M)}$.

The stress update procedure consists of the following steps:

step i.) Calculate the macroscopic strain increment, $\Delta\boldsymbol{\varepsilon}_{mac} = \Delta\boldsymbol{u}_{,x}$, and update macroscopic strains through $\varepsilon_{mac} = {}_t\varepsilon_{mac} + \Delta\varepsilon_{mac}$.

step ii.) Compute the damage equivalent strain $\bar{\vartheta}^{(\eta)}$ defined by (111) and (112) in terms of ${}_t\omega^{(\eta)}$ and ε_{mac}.

step iii.) Check the damage evolution conditions (114) and (115). Note that $\kappa^{(\eta)}$ is defined by (106) and $\dot{\kappa}^{(\eta)}$ is integrated as $\Delta\kappa^{(\eta)} = \kappa^{(\eta)} - {}_t\kappa^{(\eta)}$. The procedure for strain and damage variable updates is given as:

IF: elastic process, $\bar{\vartheta}^{(M)} \le {}_t\kappa^{(M)}$ *and* $\bar{\vartheta}^{(Fm)} \le {}_t\kappa^{(Fm)}$*, THEN*

  set $\omega^{(\eta)} = {}_t\omega^{(\eta)}$ *and* $\kappa^{(\eta)} = {}_t\kappa^{(\eta)}$ *with* $\eta \equiv M, Fm$

ELSE: damage process,

  update for $\bar{\vartheta}^{(\eta)}$ *and* $\omega^{(\eta)}$ *by solving the system of nonlinear equations (113);*

  set $\kappa^{(\eta)} = \bar{\vartheta}^{(\eta)}$

  IF: elastic process in mesoscopic matrix phase, i.e. $\bar{\vartheta}^{(M)} < {}_t\kappa^{(M)}$*, THEN*

    set $\omega^{(M)} = {}_t\omega^{(M)}$ *and* $\kappa^{(M)} = {}_t\kappa^{(M)}$*;*

    update for $\bar{\vartheta}^{(Fm)}$ *and* $\omega^{(Fm)}$ *by solving (113) with* $\eta = Fm$ *only;*

    set $\kappa^{(Fm)} = \bar{\vartheta}^{(Fm)}$

  ENDIF

  IF: elastic process in microscopic matrix phase, i.e. $\bar{\vartheta}^{(Fm)} < {}_t\kappa^{(Fm)}$*, THEN* *set* $\omega^{(Fm)} = {}_t\omega^{(Fm)}$ *and* $\kappa^{(Fm)} = {}_t\kappa^{(Fm)}$*;*

update for $\bar{\vartheta}^{(M)}$ *and* $\omega^{(M)}$ *by solving (113) with* $\eta = M$ *only;*

set $\kappa^{(M)} = \bar{\vartheta}^{(M)}$

ENDIF

ENDIF

Since $\bar{\vartheta}^{(\eta)}$ is determined by the current phase average strains in the meso- and micro-scale matrix phases, which in turn depend on the current damage variable, it follows that the damage evolution laws in (113) comprise a set of nonlinear equations for $\omega^{(\eta)}$. Using the Newton's method we construct an iterative process for the damage variables on both scales:

$$\begin{bmatrix} {}^{k+1}\omega^{(M)} \\ {}^{k+1}\omega^{(Fm)} \end{bmatrix} = \begin{bmatrix} {}^{k}\omega^{(M)} \\ {}^{k}\omega^{(Fm)} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial\Phi^{(M)}}{\partial\omega^{(M)}} & \dfrac{\partial\Phi^{(M)}}{\partial\omega^{(Fm)}} \\ \dfrac{\partial\Phi^{(Fm)}}{\partial\omega^{(M)}} & \dfrac{\partial\Phi^{(Fm)}}{\partial\omega^{(Fm)}} \end{bmatrix}^{-1}_{{}^k\omega^{(\eta)}} \begin{bmatrix} \Phi^{(M)} \\ \Phi^{(Fm)} \end{bmatrix}_{{}^k\omega^{(\eta)}} \tag{116}$$

The derivatives with respect to $\omega^{(M)}$ and $\omega^{(Fm)}$ in (116) can be evaluated by differentiating (113) with $\kappa^{(\eta)} = \bar{\vartheta}^{(\eta)}$ $(\eta = M, Fm)$ provided that the damage processes occur on both scales, such that

$$\frac{\partial\Phi^{(M)}}{\partial\omega^{(M)}} = \gamma^{(M)}\frac{\partial\bar{\vartheta}^{(M)}}{\partial\omega^{(M)}} - 1\,, \qquad \frac{\partial\Phi^{(M)}}{\partial\omega^{(Fm)}} = \gamma^{(M)}\frac{\partial\bar{\vartheta}^{(M)}}{\partial\omega^{(Fm)}} \tag{117}$$

$$\frac{\partial\Phi^{(Fm)}}{\partial\omega^{(M)}} = \gamma^{(Fm)}\frac{\partial\bar{\vartheta}^{(Fm)}}{\partial\omega^{(M)}}\,, \qquad \frac{\partial\Phi^{(Fm)}}{\partial\omega^{(Fm)}} = \gamma^{(Fm)}\frac{\partial\bar{\vartheta}^{(Fm)}}{\partial\omega^{(Fm)}} - 1 \tag{118}$$

where $\gamma^{(M)}$ and $\gamma^{(Fm)}$ are given as

$$\gamma^{(\eta)} = \frac{\{a^{(\eta)}\kappa_0^{(\eta)}\}/\{\pi/2 + \operatorname{atan}(b^{(\eta)})\}}{(\kappa_0^{(\eta)})^2 + \{a^{(\eta)}(\bar{\vartheta}^{(\eta)} - \bar{\vartheta}_{ini}^{(\eta)}) - b^{(\eta)}\kappa_0^{(\eta)}\}^2}\,, \qquad \eta = M, Fm \tag{119}$$

The derivatives of the nonlocal damage equivalent strain $\bar{\vartheta}^{(M)}$ and $\bar{\vartheta}^{(Fm)}$ with respect to the damage variables are presented in Appendix.

When damage process occurs only on one scale, either micro- or meso-scale, the set of nonlinear equations (113) reduces to a single nonlinear function and the Newton's iterative method gives

$$ {}^{k+1}\omega^{(\eta)} = {}^{k}\omega^{(\eta)} - \left(\frac{\partial\Phi^{(\eta)}}{\partial\omega^{(\eta)}}\right)^{-1}\Phi^{(\eta)}\Big|_{{}^{k}\omega^{(\eta)}}, \qquad \eta = M \text{ or } Fm \tag{120} $$

where $\partial\Phi^{(\eta)}/\partial\omega^{(\eta)}$ is given in (117) for $\eta = M$ and in (118) for $\eta = Fm$.

step vi.) Update the nonlocal phase stresses, $\sigma_{mac}^{(F)}$ and $\sigma_{mac}^{(M)}$, using (91) with macroscopic instantaneous stiffness $\mathrm{L}_{mac}^{(F)}$ and $\mathrm{L}_{mac}^{(M)}$ defined in (102) and (103), respectively. The macroscopic stress σ_{mac} can be finally obtained from (93).

5.2 Consistent tangent stiffness

To this end we focus on the computation of the consistent tangent stiffness matrix on the macro level. We start by taking material derivative of the incremental form of the constitutive equation (91)

$$ \dot{\sigma}_{mac}^{(F)} = \dot{\mathrm{L}}_{mac}^{(F)}\varepsilon_{mac} + \mathrm{L}_{mac}^{(F)}\dot{\varepsilon}_{mac}, \qquad \dot{\sigma}_{mac}^{(M)} = \dot{\mathrm{L}}_{mac}^{(M)}\varepsilon_{mac} + L_{mac}^{(M)}\dot{\varepsilon}_{mac} \tag{121} $$

where $\dot{\mathrm{L}}_{mac}^{(\alpha)}$ can be obtained by taking time derivative of (102) and (103), which yields

$$ \dot{\mathrm{L}}_{mac}^{(F)} = \frac{1}{\left|\Theta_y^{(F)}\right|}\int_{\Theta_y^{(F)}}\{\dot{\mathrm{L}}_{meso}^{(F)}(\boldsymbol{A}_y + \boldsymbol{G}_y\boldsymbol{D}_{mac}) + L_{meso}^{(F)}\boldsymbol{G}_y\dot{\boldsymbol{D}}_{mac}\}d\Theta \tag{122} $$

$$ \dot{\mathrm{L}}_{mac}^{(M)} = \dot{\mathrm{L}}_{meso}^{(M)}(\bar{\boldsymbol{A}}_y^{(M)} + \bar{\boldsymbol{G}}_y^{(M)}\boldsymbol{D}_{mac}) + L_{meso}^{(M)}\bar{\boldsymbol{G}}_y^{(M)}\dot{\boldsymbol{D}}_{mac} \tag{123} $$

Following (99) and (100), the rate form of the mesoscopic stiffness is given as

$$ \dot{\mathrm{L}}_{meso}^{(F)} = \vec{L}_{meso}^{(F)}\dot{\omega}^{(Fm)}, \qquad \dot{\mathrm{L}}_{meso}^{(M)} = \vec{L}_{meso}^{(M)}\dot{\omega}^{(M)} \tag{124} $$

where

$$ \begin{aligned} \vec{L}_{meso}^{(F)} = {} & \{(\bar{\boldsymbol{L}}_{meso}^{(F)} - \tilde{\boldsymbol{L}}_{meso}^{(F)}) - v^{(Fm)}\omega^{(Fm)}(\bar{\boldsymbol{L}}_{meso}^{(Fm)} - \tilde{\boldsymbol{L}}_{meso}^{(Fm)})\}\vec{\boldsymbol{D}}_{meso}^{(F)} \\ & - v^{(Fm)}\{\bar{\boldsymbol{L}}_{meso}^{(Fm)} + (\bar{\boldsymbol{L}}_{meso}^{(Fm)} - \tilde{\boldsymbol{L}}_{meso}^{(Fm)})\boldsymbol{D}_{meso}^{(F)}\} \end{aligned} \tag{125} $$

$$ \vec{L}_{meso}^{(M)} = -\boldsymbol{L}_{mac}^{(M)} \tag{126} $$

and $\vec{\boldsymbol{D}}_{meso}^{(F)}$ in (125) is obtained by taking time derivative of $\boldsymbol{D}_{meso}^{(F)}$ (101)

$$ \boldsymbol{D}_{meso}^{(F)} \equiv \vec{\boldsymbol{D}}_{meso}^{(F)}\dot{\omega}^{(Fm)} = (\boldsymbol{I} - \omega^{(Fm)}\boldsymbol{B}^{(Fm)})^{-2}\boldsymbol{C}^{(Fm)}\dot{\omega}^{(Fm)} \tag{127} $$

From (124)-(127), the time derivative of $\boldsymbol{D}_{mac}$ defined in (104) can be expressed as

$$\dot{\boldsymbol{D}}_{mac} = \vec{\boldsymbol{D}}_{mac}^{(M)}\dot{\omega}^{(M)} + \vec{\boldsymbol{D}}_{mac}^{(Fm)}\dot{\omega}^{(Fm)} \tag{128}$$

where

$$\vec{\boldsymbol{D}}_{mac}^{(M)} = -\left\{\int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T L_{meso}^{(F)} \boldsymbol{G}_y d\Theta + \int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T L_{meso}^{(M)} \boldsymbol{G}_y d\Theta\right\}^{-1} \left\{\int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T \vec{L}_{meso}^{(M)} \boldsymbol{G}_y d\Theta \boldsymbol{D}_{mac} + \int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T \vec{L}_{meso}^{(M)} \boldsymbol{A}_y d\Theta\right\} \tag{129}$$

$$\vec{\boldsymbol{D}}_{mac}^{(Fm)} = -\left\{\int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T L_{meso}^{(F)} \boldsymbol{G}_y d\Theta + \int_{\Theta_y^{(M)}} \boldsymbol{G}_y^T L_{meso}^{(M)} \boldsymbol{G}_y d\Theta\right\}^{-1} \left\{\int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T \vec{L}_{meso}^{(F)} \boldsymbol{G}_y d\Theta \boldsymbol{D}_{mac} + \int_{\Theta_y^{(F)}} \boldsymbol{G}_y^T \vec{L}_{meso}^{(F)} \boldsymbol{A}_y d\Theta\right\} \tag{130}$$

Substituting (124) and (128) into (122) and (123), the rate form of the macroscopic phase stiffness becomes

$$\dot{L}_{mac}^{(F)} = \vec{L}_{mac}^{(FM)}\dot{\omega}^{(M)} + \vec{L}_{mac}^{(FF)}\dot{\omega}^{(Fm)} \tag{131}$$

$$\dot{L}_{mac}^{(M)} = \vec{L}_{mac}^{(MM)}\dot{\omega}^{(M)} + \vec{L}_{mac}^{(MF)}\dot{\omega}^{(Fm)} \tag{132}$$

where

$$\vec{L}_{mac}^{(FM)} = \frac{1}{\left|\Theta_y^{(F)}\right|}\int_{\Theta_y^{(F)}} L_{meso}^{(F)} \boldsymbol{G}_y d\Theta \vec{\boldsymbol{D}}_{mac}^{(M)} \tag{133}$$

$$\vec{L}_{mac}^{(FF)} = \frac{1}{\left|\Theta_y^{(F)}\right|}\left\{\int_{\Theta_y^{(F)}} \vec{L}_{meso}^{(F)} (\boldsymbol{A}_y + \boldsymbol{G}_y \boldsymbol{D}_{mac}) d\Theta + \int_{\Theta_y^{(F)}} L_{meso}^{(F)} \boldsymbol{G}_y d\Theta \vec{\boldsymbol{D}}_{mac}^{(B)}\right\} \tag{134}$$

$$\vec{L}_{mac}^{(MM)} = \vec{L}_{meso}^{(M)}(\overline{\boldsymbol{A}}_y^{(M)} + \overline{\boldsymbol{G}}_y^{(M)} \boldsymbol{D}_{mac}) + L_{meso}^{(M)} \overline{\boldsymbol{G}}_y^{(M)} \vec{\boldsymbol{D}}_{mac}^{(M)} \tag{135}$$

$$\vec{L}_{mac}^{(MF)} = L_{meso}^{(M)} \overline{\boldsymbol{G}}_y^{(M)} \vec{\boldsymbol{D}}_{mac}^{(B)} \tag{136}$$

To this end, the rate form of the constitutive equation (121) takes the following form

$$\dot{\sigma}_{mac}^{(F)} = \vec{L}_{mac}^{(FM)} \varepsilon_{mac} \dot{\omega}^{(M)} + \vec{L}_{mac}^{(FF)} \varepsilon_{mac} \dot{\omega}^{(Fm)} + L_{mac}^{(F)} \dot{\varepsilon}_{mac} \tag{137}$$

$$\dot{\sigma}_{mac}^{(M)} = \vec{L}_{mac}^{(MM)}\varepsilon_{mac}\dot{\omega}^{(M)} + \vec{L}_{mac}^{(MF)}\varepsilon_{mac}\dot{\omega}^{(Fm)} + L_{mac}^{(M)}\dot{\varepsilon}_{mac} \tag{138}$$

In order to obtain $\dot{\omega}^{(M)}$ and $\dot{\omega}^{(Fm)}$, we make use of damage cumulative law (113) with consideration of damage/elastic processes as described in Section 5.1. In the case of elastic process, $\dot{\omega}^{(M)} = 0$ and/or $\dot{\omega}^{(Fm)} = 0$. For damage process, $\dot{\omega}^{(M)}$ and $\dot{\omega}^{(Fm)}$ are obtained by taking time derivatives of (113). The derivation is detailed in Appendix and the final expressions can be summarized as

$$\dot{\omega}^{(M)} = (\boldsymbol{w}^{(M)})^T\dot{\varepsilon}_{mac}, \quad \dot{\omega}^{(Fm)} = (\boldsymbol{w}^{(Fm)})^T\dot{\varepsilon}_{mac} \tag{139}$$

Substituting (139) into (137) and (138), we get the following relations between the rate of the overall strain and the nonlocal phase average stresses in the matrix phases:

$$\dot{\sigma}_{mac}^{(F)} = \wp_{mac}^{(F)}\dot{\varepsilon}_{mac}, \quad \dot{\sigma}_{mac}^{(M)} = \wp_{mac}^{(M)}\dot{\varepsilon}_{mac} \tag{140}$$

where

$$\wp_{mac}^{(F)} = \vec{L}_{mac}^{(FM)}\varepsilon_{mac}(\boldsymbol{w}^{(M)})^T + \vec{L}_{mac}^{(FF)}\varepsilon_{mac}(\boldsymbol{w}^{(Fm)})^T + L_{mac}^{(F)} \tag{141}$$

$$\wp_{mac}^{(M)} = \vec{L}_{mac}^{(MM)}\varepsilon_{mac}(\boldsymbol{w}^{(M)})^T + \vec{L}_{mac}^{(MF)}\varepsilon_{mac}(\boldsymbol{w}^{(Fm)})^T + L_{mac}^{(M)} \tag{142}$$

The overall consistent tangent stiffness is constructed by substituting (140) into the rate form of the overall stress-strain relation (93)

$$\dot{\sigma}_{mac} = \wp_{mac}\dot{\varepsilon}_{mac} \tag{143}$$

$$\wp_{mac} = v^{(F)}\wp_{mac}^{(F)} + v^{(M)}\wp_{mac}^{(M)} \tag{144}$$

Finally, we remark that the integrals in the mesoscopic RVE, (129) and (130), have to be evaluated at each Gaussian point in the global finite element mesh and for every load increment. This may lead to enormous computational complexity, especially when the finer mesh is used for mesoscopic RVE. A close look at the above formulations reveals that the source of the computational complexity stems from the history-dependent form of $\boldsymbol{D}_{meso}^{(F)}$. To remedy the situation, we make use of Taylor expansion in (101) with respect to $\omega^{(Fm)}$ so that the history-dependent variable can be moved out of the integrals.

6.0 Numerical Examples

We consider a woven composite material made of Blackglas/Nextel 2D 5-harness satin weave as shown in Figure 4 . The fabric is made of 600 denier bundles of Nextel 312 fibers, spaced at

46 threads per inch, and surrounded by Blackglas 493C matrix material. We will refer to this material system as AF10. The micrograph in Figure 4 was produced at Northrop-Gruman.

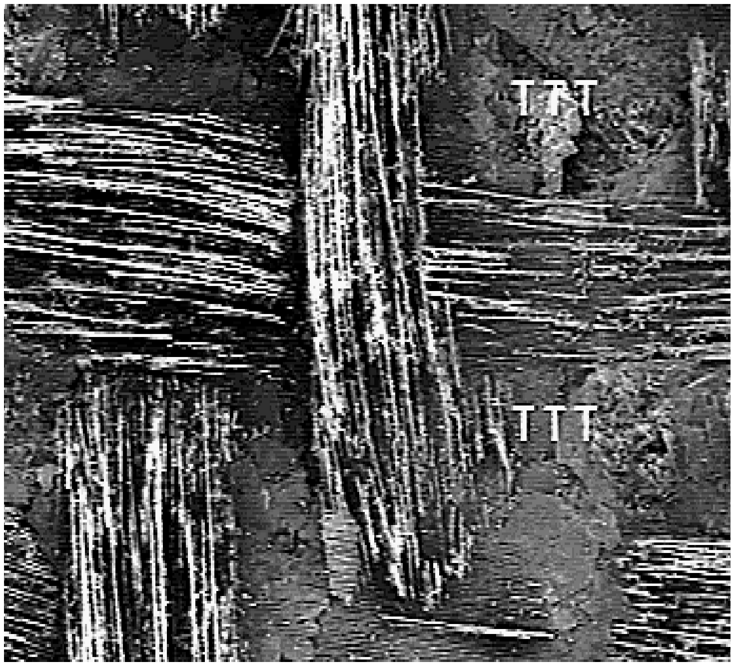

FIGURE 4. Blackglas™/Nextel 5-harness Satin Weave

In this set of numerical examples, the mesoscopic RVE is defined as a two-phase material (bundle/Blackglas 493C), while bundles consist of unidirectional fibrous composite (microscopic RVE). The phase properties of RVEs on both scales are summarized below:

Microscopic (bundle) RVE:

Blackglas 493C Matrix: volume fraction $\nu^{(Fm)} = 0.733$; Young's modulus $E^{(Fm)} = 82.7\text{GPa}$; Poisson's ratio 0.26.
Nextel™ 312 Fiber: volume fraction $\nu^{(Ff)} = 0.267$; Young's modulus $E^{(Ff)} = 151.7\text{GPa}$; Poisson's ratio 0.24.

Mesoscopic RVE (AF10 woven composite architecture, Figure 4):

Blackglas 493C Matrix (with reduce stiffness): volume fraction $\nu^{(M)} = 0.548$; Young's modulus $E^{(M)} = 26.2\text{GPa}$; Poisson's ratio = 0.26.
Bundle: volume fraction $\nu^{(F)} = 0.452$; properties determined by the homogenized stiffness of the microscopic RVE.

Note that the matrix phase in both RVEs is made of the same material. The stiffness reduction of matrix phase in the mesoscopic RVE is due to initial inter-bundle cracks. The parameters of the damage evolution laws are chosen as $a^{(\eta)} = 7.2$, $b^{(\eta)} = 16.3$ and $\bar{\vartheta}_0^{(\eta)} = 0.22$ with $\eta = M, Fm$. For the matrix phase in the mesoscopic RVE, we choose $\bar{\vartheta}_{ini}^{(Fm)} = 0$. For the matrix phase in the microscopic RVE we define $\bar{\vartheta}_{ini}^{(Fm)} = (\sqrt{E^{(Fm)}/E^{(M)}} - 1)\bar{\vartheta}_0^{(M)} = 0.17$. We assume that the two matrix phases reach the critical value $\bar{\vartheta}_0^{(\eta)}$ at the same time under the equal uniaxial strains, i.e. $\bar{\vartheta}^{(Fm)} - \bar{\vartheta}_{ini}^{(Fm)} = \bar{\vartheta}^{(M)} - \bar{\vartheta}_{ini}^{(M)} = \bar{\vartheta}_0^{(M)}$ with $h(\varepsilon_\xi^{(Fm)})\varepsilon_\xi^{(Fm)} = h(\varepsilon_\xi^{(M)})\varepsilon_\xi^{(M)}$ where $\xi = 1, 2, \text{or } 3$. The damage evolution laws are depicted in Figure 5 .

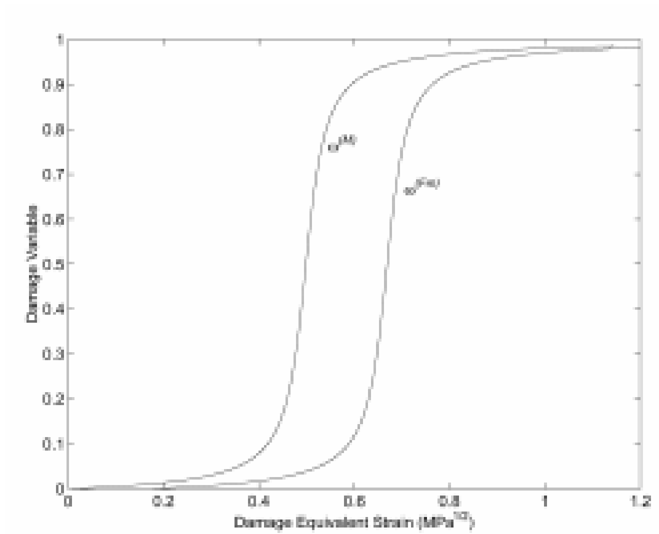

FIGURE 5. Damage Evolution Law on Micro- and Meso- Scales

We further assume that the compressive principal strain components do not contribute to the damage evolution. Thus, the parameters in (109) are chosen as $c_1^{(\eta)} = 10^5$ and $c_2^{(\eta)} = 0$ $(\eta = M, Fm)$. The microscopic RVE is discretized with 351 tetrahedral elements as shown in Figure 6 .

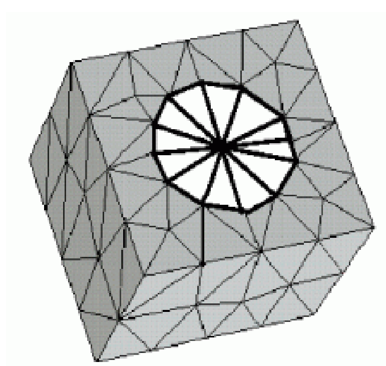

FIGURE 6. Microscopic RVE for the Bundle

The finite element mesh of the bundle phase (mesoscopic RVE) is depicted in Figure 7 . The finite element mesh of the mesoscopic RVE contains 6857 tetrahedral elements. We utilize our model to predict the ultimate strength under uniaxial tension and the 4-point bending tests.

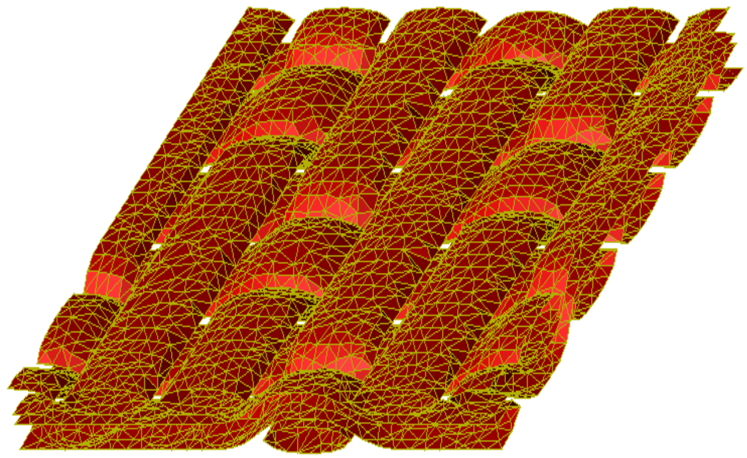

FIGURE 7. Mesoscopic RVE of AF10 Woven Composites (only bundle displayed)

The stress-strain curve for the uniaxial tension in the weave's plane is illustrated in Figure 8 .

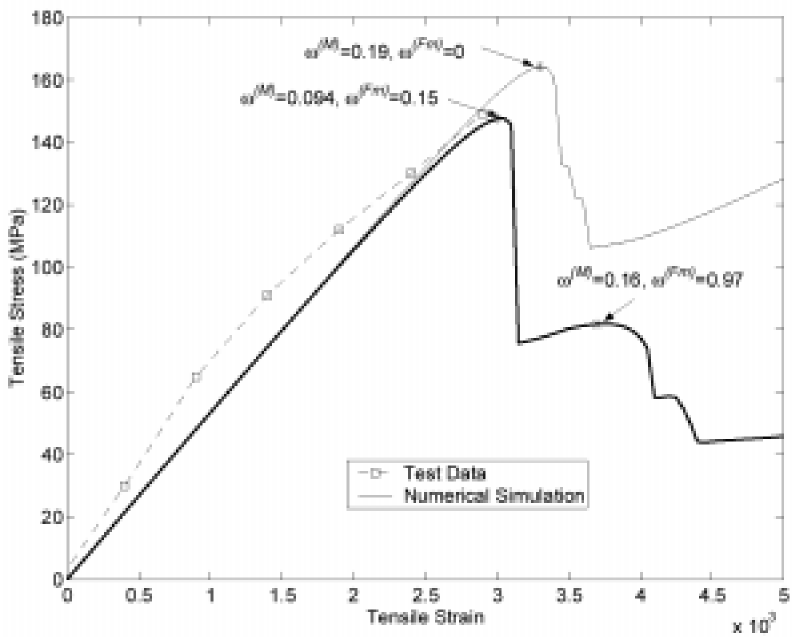

FIGURE 8. Strain-Stress Curves for Uniaxial Tension in Weave Plane

To illustrate the importance of the three-scale model, we compare the results with the two-scale model developed in [22], where the damage occurs in the matrix phase while the bundle is assumed to remain elastic throughout the loading. It can be seen from Figure 8 that the post-ultimate loading curve obtained with the two-scale method has a significant loading capacity due to bending of elastic bundles. On the other hand, the stress-strain curve of the three-scale model shows two sharp drops due to successive failures of the matrix phases in the two scales. The numerical simulation using the three-scale model gives $\sigma_u = 148\text{MPa}$ at $\varepsilon_u = 3.1\times10^{-3}$ compared with the ultimate experimental stress/strain values in the uniaxial tension test of $\sigma_u = 150 \pm 7\text{MPa}$ at $\varepsilon_u = 2.5\times10^{-3} \pm 0.3\times10^{-3}$.

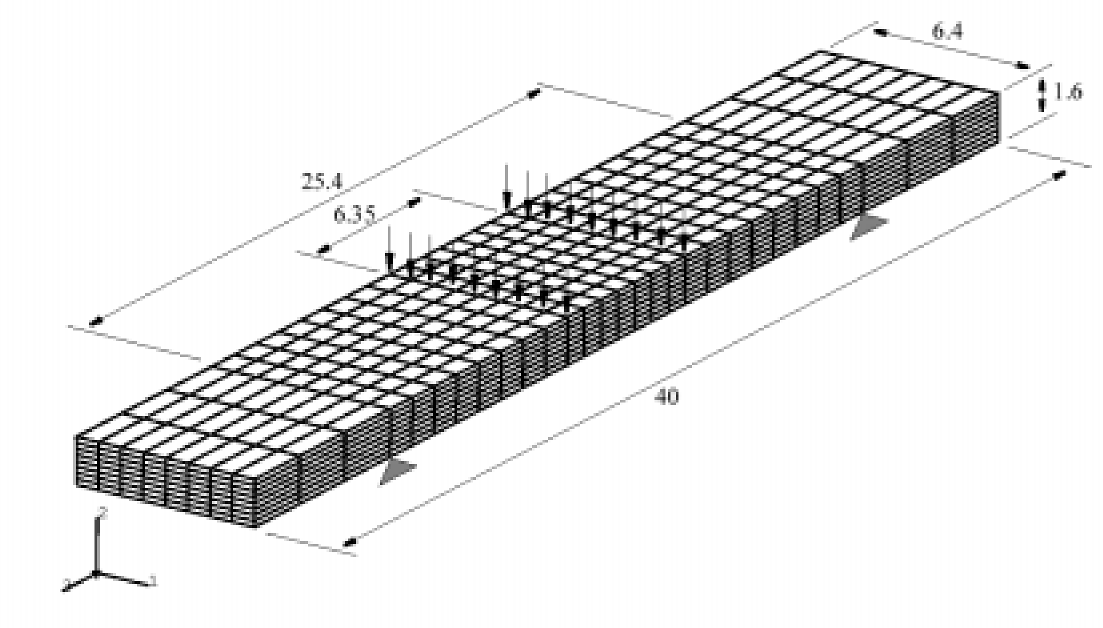

FIGURE 9. Configuration and FE Mesh of Composite Beam

The configuration of the composite beam used for 4-point bending test is shown in Figure 9 , where the loading direction (normal to the plane of the weave) is aligned along the Y axis. The finite element model of the composite beam (global structure) is composed of 1856 brick elements. Numerical simulation results as well as the test data for 4-point bending problem are shown in Figure 10 . Experiments have been conducted on five identical beams [11] and the scattered experimental data of the applied load versus the displacement at the point of load application in the beam are shown by the gray area in Figure 10 . It can be seen that the numerical simulation results are in good agreement with the experimental data in terms of predicting the overall behavior and the dominant failure mode. Both the numerical simulation and the experimental data predict that the dominant failure mode is in bending. Figure 11 and

Figure 12 illustrate the distribution of the phase damage in the composite beam corresponding to the ultimate point in the load-displacement curve in Figure 10 .

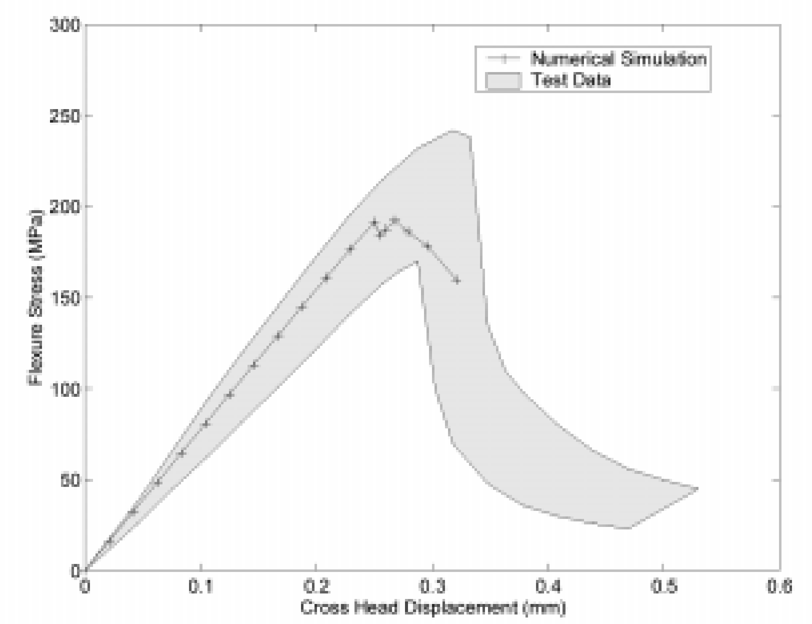

FIGURE 10. 4-Point Bending Flexure Strain-Stress Curves

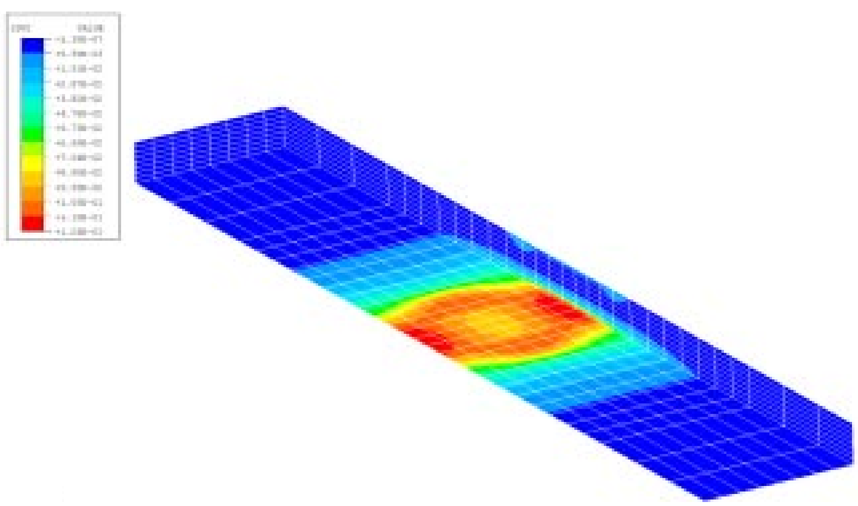

FIGURE 11. Distribution of $\omega^{(M)}$ at Ultimate Point

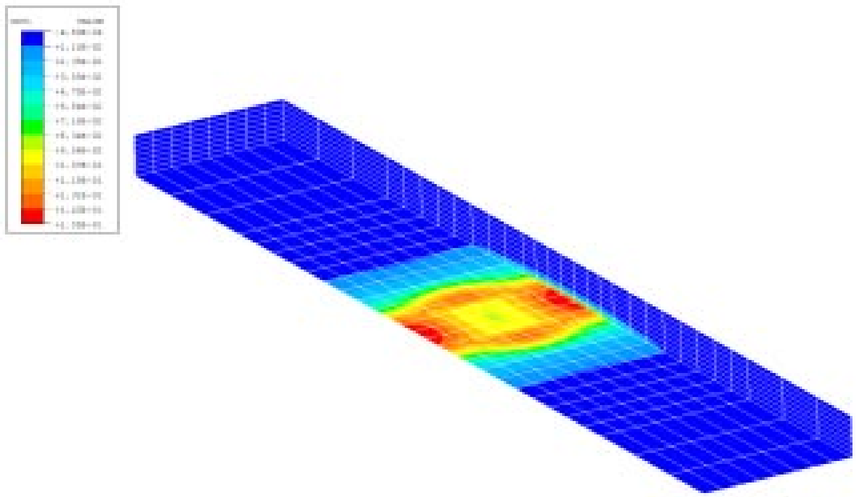

FIGURE 12. Distribution of $\omega^{(Fm)}$ at Ultimate Point

7.0 Summary and Future Research Directions

A multiscale nonlocal damage theory for brittle composite materials has been developed based on the triple-scale asymptotic expansions of damage and displacement fields. The closed form expressions relating microscopic, mesoscopic and overall strains and damage have been derived. The damage evolution is stated on the smallest scale of interest and the nonlocality is taken into account to alleviate the spurious mesh dependence by introducing the weighted phase average fields over the micro- and meso-phases. Numerical results revealed the superior performance of the three-scale method over the two-scale damage model [22] for woven composites.

The present work represents only the first step towards developing a robust simulation framework for prediction of complex damage processes in composite materials. Several important issues, such as interfacial debonding, coupled plasticity-damage effects, have not been accounted for in our model. Moreover, the assumptions of local periodicity and uniformity of macroscopic fields, which are embedded in our formulation, may yield inaccurate solutions in the vicinity of free edges or in the case of the nonperiodic microstructures. These issues will be studied in our future work.

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Appendix

In this section we present detailed derivations for $\partial\bar{\vartheta}^{(\lambda)}/\partial\omega^{(\eta)}$ $\lambda, \eta = M, Fm$ in (117) and (118), and $\dot{\omega}^{(\eta)}$ in (139). We start with the first derivative by differentiating (107) with respect to $\omega^{(\lambda)}$

$$\frac{\partial\bar{\vartheta}^{(\eta)}}{\partial\omega^{(\lambda)}} = (\boldsymbol{e}^{(\eta)})^T \frac{\partial(\boldsymbol{F}^{(\eta)}\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac})}{\partial\omega^{(\lambda)}}, \qquad \lambda, \eta = M, Fm \tag{A1}$$

where the vector $\boldsymbol{e}^{(\eta)} \equiv [e_1^{(\eta)}, e_2^{(\eta)}, e_3^{(\eta)}]^T$ takes following form

$$(\boldsymbol{e}^{(\eta)})^T = \frac{1}{2\bar{\vartheta}^{(m)}}(\boldsymbol{F}^{(\eta)}\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac})^T\hat{\boldsymbol{L}}^{(\eta)}_{mac} \tag{A2}$$

With the definition of $\boldsymbol{F}^{(\eta)}$ in (108), the derivative in (A1) can be expressed as

$$\frac{\partial(\boldsymbol{F}^{(\eta)}\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac})}{\partial\omega^{(\lambda)}} = \left[\frac{\partial(h_1^{(\eta)}\hat{\varepsilon}_1^{(\eta)})}{\partial\omega^{(\lambda)}} \quad \frac{\partial(h_2^{(\eta)}\hat{\varepsilon}_2^{(\eta)})}{\partial\omega^{(\lambda)}} \quad \frac{\partial(h_3^{(\eta)}\hat{\varepsilon}_3^{(\eta)})}{\partial\omega^{(\lambda)}}\right]^T \tag{A3}$$

Since the three components of the vector in (A3) have a similar form, we simply denote them by $\partial(h_\xi^{(\eta)}\hat{\varepsilon}_\xi^{(\eta)})/(\partial\omega^{(\lambda)})$ with $\xi = 1, 2, 3$ and then by using (109) we have

$$\frac{\partial(h_\xi^{(\eta)}\hat{\varepsilon}_\xi^{(\eta)})}{\partial\omega^{(\lambda)}} = \left\{\frac{\partial h_\xi^{(\eta)}}{\partial\hat{\varepsilon}_\xi^{(\eta)}}\hat{\varepsilon}_\xi^{(\eta)} + h_\xi^{(\eta)}\right\}\frac{\partial\hat{\varepsilon}_\xi^{(\eta)}}{\partial\omega^{(\lambda)}} \; ; \qquad \xi = 1, 2, 3 \tag{A4}$$

where

$$\frac{\partial h_\xi^{(\eta)}}{\partial\hat{\varepsilon}_\xi^{(\eta)}} = \frac{c_1^{(\eta)}/\pi}{1 + \{c_1^{(\eta)}(\hat{\varepsilon}_\xi^{(\eta)} - c_2^{(\eta)})\}^2} \tag{A5}$$

To this end we need to compute the derivative of each component of principal strain $\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac}$ with respect to the damage variable $\omega^{(\lambda)}$. The principal components of a second order tensor satisfy Hamilton’s Theorem, i.e.

$$(\hat{\varepsilon}_\xi^{(\eta)})^3 - I_1(\hat{\varepsilon}_\xi^{(\eta)})^2 + I_2\hat{\varepsilon}_\xi^{(\eta)} - I_3 = 0 \tag{A6}$$

where I_1, I_2 and I_3 are three invariants of $\varepsilon^{(\eta)}_{mac}$ (or $\hat{\boldsymbol{\varepsilon}}^{(\eta)}_{mac}$) which can be expressed as

$$I_1 = \varepsilon_{ii}^{(\eta)} = \hat{\varepsilon}_1^{(\eta)} + \hat{\varepsilon}_2^{(\eta)} + \hat{\varepsilon}_3^{(\eta)} \tag{A7}$$

$$I_2 = \frac{1}{2}\cdot(\varepsilon_{ii}^{(\eta)}\varepsilon_{jj}^{(\eta)} - \varepsilon_{ij}^{(\eta)}\varepsilon_{ji}^{(\eta)}) = \hat{\varepsilon}_1^{(\eta)}\hat{\varepsilon}_2^{(\eta)} + \hat{\varepsilon}_2^{(\eta)}\hat{\varepsilon}_3^{(\eta)} + \hat{\varepsilon}_3^{(\eta)}\hat{\varepsilon}_1^{(\eta)} \tag{A8}$$

$$I_3 = \frac{1}{6}\cdot(2\varepsilon_{ij}^{(\eta)}\varepsilon_{jk}^{(\eta)}\varepsilon_{ki}^{(\eta)} - 3\varepsilon_{ij}^{(\eta)}\varepsilon_{ji}^{(\eta)}\varepsilon_{kk}^{(\eta)} + \varepsilon_{ii}^{(\eta)}\varepsilon_{jj}^{(\eta)}\varepsilon_{kk}^{(\eta)}) = \hat{\varepsilon}_1^{(\eta)}\hat{\varepsilon}_2^{(\eta)}\hat{\varepsilon}_3^{(\eta)} \tag{A9}$$

where the tensorial notations are adopted for $\varepsilon^{(\eta)}_{mac}$, i.e $\varepsilon^{(\eta)}_{mac} \equiv \varepsilon^{(\eta)}_{ij}$. Differentiating (A6) with respect to $\omega^{(\lambda)}$ gives

$$\frac{\partial\hat{\varepsilon}_\xi^{(\eta)}}{\partial\omega^{(\lambda)}} = \{3(\hat{\varepsilon}_\xi^{(\eta)})^2 - 2I_1\hat{\varepsilon}_\xi^{(\eta)} + I_2\}^{-1}\cdot\left\{\frac{\partial I_1}{\partial\omega^{(\lambda)}}(\hat{\varepsilon}_\xi^{(\eta)})^2 - \frac{\partial I_2}{\partial\omega^{(\lambda)}}\hat{\varepsilon}_\xi^{(\eta)} + \frac{\partial I_3}{\partial\omega^{(\lambda)}}\right\} \tag{A10}$$

where the derivative of the invariants with respect to $\omega^{(\lambda)}$ can be obtained by using (A7)-(A9) such that

$$\frac{\partial I_1}{\partial \omega^{(\lambda)}} \equiv (\boldsymbol{p}_1^{(\eta)})^T \frac{\partial \varepsilon_{mac}^{(\eta)}}{\partial \omega^{(\lambda)}} = \delta_{jk}\delta_{ik}\frac{\partial \varepsilon_{ij}^{(\eta)}}{\partial \omega^{(\lambda)}} \tag{A11}$$

$$\frac{\partial I_2}{\partial \omega^{(\lambda)}} \equiv (\boldsymbol{p}_2^{(\eta)})^T \frac{\partial \varepsilon_{mac}^{(\eta)}}{\partial \omega^{(\lambda)}} = (\varepsilon_{mm}^{(\eta)}\delta_{jk}\delta_{ik} - \varepsilon_{ji}^{(\eta)})\frac{\partial \varepsilon_{ij}^{(\eta)}}{\partial \omega^{(\lambda)}} \tag{A12}$$

$$\frac{\partial I_3}{\partial \omega^{(\lambda)}} \equiv (\boldsymbol{p}_3^{(\eta)})^T \frac{\partial \varepsilon_{mac}^{(\eta)}}{\partial \omega^{(\lambda)}} \tag{A13}$$
$$= \left(\varepsilon_{jk}^{(\eta)}\varepsilon_{ki}^{(\eta)} - \varepsilon_{mm}^{(\eta)}\varepsilon_{ji}^{(\eta)} - \frac{1}{2}\varepsilon_{mn}^{(\eta)}\varepsilon_{nm}^{(\eta)}\delta_{jk}\delta_{ik} + \frac{1}{2}\varepsilon_{mm}^{(\eta)}\varepsilon_{nn}^{(\eta)}\delta_{jk}\delta_{ik}\right)\frac{\partial \varepsilon_{ij}^{(\eta)}}{\partial \omega^{(\lambda)}}$$

Substituting (A10)-(A13) into (A4), gives

$$\frac{\partial (h_\xi^{(\eta)}\hat{\varepsilon}_\xi^{(\eta)})}{\partial \omega^{(\lambda)}} = (\boldsymbol{q}_\xi^{(\eta)})^T \frac{\partial \varepsilon_{mac}^{(\eta)}}{\partial \omega^{(\lambda)}}, \qquad \xi = 1, 2, 3 \tag{A14}$$

where $\boldsymbol{q}_\xi^{(\eta)}$ is given as

$$\boldsymbol{q}_\xi^{(\eta)} = \left\{\frac{\partial h_\xi^{(\eta)}}{\partial \hat{\varepsilon}_\xi^{(\eta)}}\hat{\varepsilon}_\xi^{(\eta)} + h_\xi^{(\eta)}\right\}\{3(\hat{\varepsilon}_\xi^{(\eta)})^2 - 2I_1\hat{\varepsilon}_\xi^{(\eta)} + I_2\}^{-1}\{\boldsymbol{p}_1^{(\eta)}(\hat{\varepsilon}_\xi^{(\eta)})^2 - \boldsymbol{p}_2^{(\eta)}\hat{\varepsilon}_\xi^{(\eta)} + \boldsymbol{p}_3^{(\eta)}\} \tag{A15}$$

Finally, $\partial\bar{\vartheta}^{(\eta)}/\partial\omega^{(\lambda)}$ in (A1) can be written in a concise form by using (A3) and (A14), which yields

$$\frac{\partial \vartheta^{(\eta)}}{\partial \omega^{(\lambda)}} = (\boldsymbol{\aleph}^{(\eta)})^T \left(\frac{\partial \varepsilon_{mac}^{(\eta)}}{\partial \omega^{(\lambda)}}\right), \qquad \lambda, \eta = M, Fm \tag{A16}$$

where

$$\boldsymbol{\aleph}^{(\eta)} = \sum_{\xi=1}^{3} e_\xi^{(\eta)}\boldsymbol{q}_\xi^{(\eta)}, \qquad \eta = M, Fm \tag{A17}$$

and the derivative on right hand side of (A16) can be evaluated by differentiating (111) and (112) such that

$$d\varepsilon_{mac}^{(M)} = \boldsymbol{s}_1^{(M)}d\omega^{(M)} + \boldsymbol{s}_2^{(M)}d\omega^{(Fm)} + \boldsymbol{S}^{(M)}d\varepsilon_{mac} \tag{A18}$$

$$d\varepsilon_{mac}^{(Fn)} = \boldsymbol{s}_1^{(F)} d\omega^{(M)} + \boldsymbol{s}_2^{(F)} d\omega^{(Fm)} + \boldsymbol{S}^{(F)} d\varepsilon_{mac} \tag{A19}$$

where for (A18)

$$\frac{\partial\varepsilon_{mac}^{(M)}}{\partial\omega^{(M)}} \equiv \boldsymbol{s}_1^{(M)} = \overline{\boldsymbol{G}}_y^{(M)} \overrightarrow{\boldsymbol{D}}_{mac}^{(M)} \varepsilon_{mac} \tag{A20}$$

$$\frac{\partial\varepsilon_{mac}^{(M)}}{\partial\omega^{(Fm)}} \equiv \boldsymbol{s}_2^{(M)} = \overline{\boldsymbol{G}}_y^{(M)} \overrightarrow{\boldsymbol{D}}_{mac}^{(F)} \varepsilon_{mac} \tag{A21}$$

$$\frac{\partial\varepsilon_{mac}^{(M)}}{\partial\varepsilon_{mac}} \equiv \boldsymbol{S}^{(M)} = \overline{\boldsymbol{A}}_y^{(M)} + \overline{\boldsymbol{G}}_y^{(M)} \boldsymbol{D}_{mac} \tag{A22}$$

and for (A19)

$$\frac{\partial\varepsilon_{mac}^{(Fm)}}{\partial\omega^{(M)}} \equiv \boldsymbol{s}_1^{(Fm)} = \frac{1}{\left|\Theta_y^{(F)}\right|} \int_{\Theta_y^{(F)}} (\overline{\boldsymbol{A}}_z^{(Fm)} + \overline{\boldsymbol{G}}_z^{(Fm)} \boldsymbol{D}_{meso}^{(F)}) \boldsymbol{G}_y \overrightarrow{\boldsymbol{D}}_{mac}^{(M)} \, d\Theta \ \varepsilon_{mac} \tag{A23}$$

$$\begin{aligned} \frac{\partial\varepsilon_{mac}^{(Fm)}}{\partial\omega^{(Fm)}} \equiv \boldsymbol{s}_2^{(Fm)} = \frac{1}{\left|\Theta_y^{(F)}\right|} \int_{\Theta_y^{(F)}} \{ & \overline{\boldsymbol{G}}_z^{(Fm)} \overrightarrow{\boldsymbol{D}}_{meso}^{(F)} (\boldsymbol{A}_y + \boldsymbol{G}_y \boldsymbol{D}_{mac}) \\ & + (\overline{\boldsymbol{A}}_z^{(Fm)} + \overline{\boldsymbol{G}}_z^{(Fm)} \boldsymbol{D}_{meso}^{(F)}) \boldsymbol{G}_y \overrightarrow{\boldsymbol{D}}_{mac}^{(F)} \} \, d\Theta \ \varepsilon_{mac} \end{aligned} \tag{A24}$$

$$\frac{\partial\varepsilon_{mac}^{(Fm)}}{\partial\varepsilon_{mac}} \equiv \boldsymbol{S}^{(Fm)} = \frac{1}{\left|\Theta_y^{(F)}\right|} \int_{\Theta_y^{(F)}} (\overline{\boldsymbol{A}}_z^{(Fm)} + \overline{\boldsymbol{G}}_z^{(Fm)} \boldsymbol{D}_{meso}^{(F)}) (\boldsymbol{A}_y + \boldsymbol{G}_y \boldsymbol{D}_{mac}) \, d\Theta \tag{A25}$$

The time derivatives of the phase damage variable, $\dot{\omega}^{(M)}$ and $\dot{\omega}^{(Fm)}$, can be obtained by taking time derivatives of damage evolution law (113) and making use of (A18) and (A19). From (113) and assuming that damage processes occur on both scales, i.e. $\kappa^{(\eta)} = \overline{\vartheta}^{(\eta)}$ $(\eta = M, Fm)$, we have

$$\dot{\omega}^{(M)} = \gamma^{(M)} \dot{\overline{\vartheta}}^{(M)} \tag{A26}$$

$$\dot{\omega}^{(Fm)} = \gamma^{(Fm)} \dot{\overline{\vartheta}}^{(Fm)} \tag{A27}$$

where $\gamma^{(M)}$ and $\gamma^{(Fm)}$ are given in (119); $\dot{\overline{\vartheta}}^{(M)}$ and $\dot{\overline{\vartheta}}^{(Fm)}$ can be derived in a similar way as for $\partial\overline{\vartheta}^{(\eta)} / \partial\omega^{(\lambda)}$, which yields:

$$\dot{\overline{\vartheta}}^{(\eta)} = (\boldsymbol{\aleph}^{(\eta)})^T \dot{\varepsilon}_{mac}^{(\eta)} , \qquad \eta = M, Fm \tag{A28}$$

where $\dot{\varepsilon}_{mac}^{(\eta)}$ can be evaluated by using (A18) and (A19). From (A26) and (A18), we can get

$$n_1^{(M)}\dot{\omega}^{(M)} + n_2^{(M)}\dot{\omega}^{(Fm)} + (\boldsymbol{r}^{(M)})^T\dot{\varepsilon}_{mac} = 0 \tag{A29}$$

where $n_1^{(M)}$, $n_2^{(M)}$ and $\boldsymbol{r}^{(M)}$ are given as

$$n_1^{(M)} = (\boldsymbol{\aleph}^{(M)})^T\boldsymbol{s}_1^{(M)} - 1/\gamma^{(M)} \tag{A30}$$

$$n_2^{(M)} = (\boldsymbol{\aleph}^{(M)})^T\boldsymbol{s}_2^{(M)} \tag{A31}$$

$$(\boldsymbol{r}^{(M)})^T = (\boldsymbol{\aleph}^{(M)})^T\boldsymbol{S}^{(M)} \tag{A32}$$

Similarly, from (A28) and (A19), we have

$$n_1^{(Fm)}\dot{\omega}^{(M)} + n_2^{(Fm)}\dot{\omega}^{(Fm)} + (\boldsymbol{r}^{(Fm)})^T\dot{\varepsilon}_{mac} = 0 \tag{A33}$$

where $n_1^{(Fm)}$, $n_2^{(Fm)}$ and $\boldsymbol{r}^{(Fm)}$ are given as

$$n_1^{(Fm)} = (\boldsymbol{\aleph}^{(Fm)})^T\boldsymbol{s}_1^{(Fm)} \tag{A34}$$

$$n_2^{(Fm)} = (\boldsymbol{\aleph}^{(Fm)})^T\boldsymbol{s}_2^{(Fm)} - 1/\gamma^{(Fm)} \tag{A35}$$

$$(\boldsymbol{s}^{(Fm)})^T = (\boldsymbol{\aleph}^{(Fm)})^T\boldsymbol{S}^{(Fm)} \tag{A36}$$

Finally, solving for (A29) and (A33) yields

$$\dot{\omega}^{(M)} = (\boldsymbol{w}^{(M)})^T\dot{\varepsilon}_{mac}, \qquad \dot{\omega}^{(Fm)} = (\boldsymbol{w}^{(Fm)})^T\dot{\varepsilon}_{mac} \tag{A37}$$

where

$$\boldsymbol{w}^{(M)} = \frac{n_2^{(M)}\boldsymbol{r}^{(Fm)} - n_2^{(Fm)}\boldsymbol{r}^{(M)}}{n_1^{(M)}n_2^{(Fm)} - n_2^{(M)}n_1^{(Fm)}} \tag{A38}$$

$$\boldsymbol{w}^{(Fm)} = \frac{n_1^{(Fm)}\boldsymbol{r}^{(M)} - n_1^{(M)}\boldsymbol{r}^{(Fm)}}{n_1^{(M)}n_2^{(Fm)} - n_2^{(M)}n_1^{(Fm)}} \tag{A39}$$

The Jacobian matrix (116)-(118) follows from (A16), (A20), (A21), (A23), (A24), (A30), (A31), (A34) and (A35):

$$\boldsymbol{J} = \begin{bmatrix} \gamma^{(M)}n_1^{(M)} & \gamma^{(M)}n_2^{(M)} \\ \gamma^{(Fm)}n_1^{(Fm)} & \gamma^{(Fm)}n_2^{(Fm)} \end{bmatrix} \tag{A40}$$