# Uniformly Valid Multiple Spatial-Temporal Scale Modeling for Wave Propagation in Heterogeneous Media

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**Abstract:** A novel dispersive model for wave propagation in heterogeneous media is developed. The method is based on a higher-order mathematical homogenization theory with multiple spatial and temporal scales. By this approach a fast spatial scale and a series of slow temporal scales are introduced to account for rapid spatial fluctuations of material properties as well as for the long-term behavior of the homogenized solution. The problem of secularity arising from the classical multiple spatial scale homogenization theory for wave propagation problems is resolved, giving rise to uniformly valid dispersive model. The proposed dispersive model is solved analytically and its solution has be found to be in good agreement with the numerical solution of the source problem in a heterogeneous medium.

## 1. Introduction

When a wavelength of a traveling signal in a heterogeneous medium is comparable to the characteristic length of the microstructure, successive reflection and refraction of the waves between the interfaces of the material lead to significant dispersion effects (see for example [1][2][3]). The interest on the subject matter stems from the fact that the phenomenon of dispersion cannot be captured by the classical homogenization theory.

The use of multiple-scale expansions as a systematic tool of homogenization for problems other than elastodynamics can be traced to Sanchez-Palencia [4], Benssousan, Lions and Papanicoulau [5], and Bakhvalov and Panasenko [6]. The role of higher order terms in the asymptotic expansion has been investigated in statics by Gambin and Kroner [7], and Boutin [8]. In elastodynamics, Boutin and Auriault [9] demonstrated that the terms of a higher order successively introduce effects of polarization, dispersion and attenuation.

For wave propagation in heterogeneous media, a single-frequency time dependence is typically assumed [10]. Notable exceptions are the recent articles of Fish and Chen [11][12], which investigated the initial-boundary value problem with rapidly varying coefficients. In [11] it has been shown that while higher-order multiple scale expansion in space is capable of capturing the dispersion effect when the temporal scale of observation is small, it introduces secular terms which grow unbounded with time. In [12], a slow temporal scale was introduced to eliminated the secular terms up to the second order and to capture the long-term behavior of the homogenized solution.

In an attempt to develop a uniformly valid dispersive model up to an arbitrary order, we extend the theory developed in [12] to fast spatial and a series of slow temporal scales. The fast spatial scale is designated to account for rapid spatial fluctuations of material properties and a series of slow temporal scales are aimed at capturing the long-term evolution of the homogenization solutions. This results in a dispersive uniformly valid model, solution of which is obtained analytically and subsequently validated against the numerical solution of the source problem in a heterogeneous medium.

## 2. Problem Statement

We consider wave propagation normal to layers of an array of elastic bilaminates in a periodic arrangement with $\Omega$ as a characteristic length as shown in Figure 1. The governing elastodynamics equation of is given by

$$\rho(x/\varepsilon)u_{;tt} - \{E(x/\varepsilon)u_{;x}\}_{;x} = 0 \tag{1}$$

with appropriate boundary and initial conditions

$$u(x, 0) = f(x), \quad u_{;t}(x, 0) = q(x) \tag{2}$$

where $u(x, t)$ denotes the displacement field; $\rho(x/\varepsilon)$ and $E(x/\varepsilon)$ the mass density and elastic modulus, respectively; $(\ )_{;x}$ and $(\ )_{;t}$ denote the differentiation with respect to $x$ and time respectively; and $0 < \varepsilon \ll 1$ in (1) denotes a rapid spatial variation of material properties.

The goal is to establish an effective homogeneous model in which local fluctuations introduced by material heterogeneity do not appear explicitly and the response of a heterogeneous medium can be approximated by the response of the effective homogeneous medium. This is facilitated by the method of multiple scale asymptotic expansion in space and time.

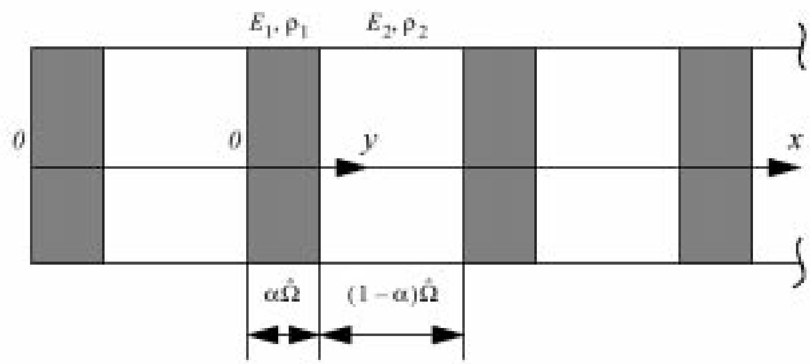


Figure 1: A bilaminate with a periodic microstructure

## 3. Asymptotic Analysis with Multiple Spatial and Temporal Scales

Under the premise that the macro domain $L = \lambda/(2\pi)$ is much larger than the unit cell domain $\Omega$, i.e. $\Omega/L = (\omega\Omega)/c = k\Omega \ll 1$, it is convenient to introduce a microscopic spatial length variable $y$ [9][16] such that

$$y = x/\varepsilon \tag{3}$$

where $\lambda$, $\omega$, $k$ and $c$ are the macroscopic wavelength, the circular frequency, the wave number and the phase velocity of the macroscopic wave, respectively. In addition to this fast spatial variable, we introduce multiple time scales

$$t_k = \varepsilon^{2k} t, \quad (k = 0, 1, 2, \ldots, m) \tag{4}$$

where $t_0$ is the usual time coordinate and $t_k$, $k > 0$ are various slow time scales. Since the response quantities $u$ and $\sigma$ depend on $x$, $y = x/\varepsilon$, $t_0$, $t_1$, $\ldots$ and $t_m$, a multiple-scale asymptotic expansion is employed to approximate the displacement and stress fields

$$u(x, y, t_0, t_1, \ldots, t_m) = \sum_{i=0}^{m} \varepsilon^i u_i(x, y, t_0, t_1, \ldots, t_m),$$

$$\sigma(x, y, t_0, t_1, \ldots, t_m) = \sum_{i=-1}^{m} \varepsilon^i \sigma_i(x, y, t_0, t_1, \ldots, t_m) \tag{5}$$

Homogenization process consists of inserting the asymptotic expansions (5) into the governing equation (1), identifying the terms with the equal power of $\varepsilon$, and then solving the resulting problems.

Following the aforementioned procedure and expressing the spatial and temporal derivatives in terms of the fast and slow space-time coordinates

$$u_{;x} = u_{,x} + \varepsilon^{-1} u_{,y} \tag{6}$$

$$u_{;t} = u_{,t_0} + \varepsilon^2 u_{,t_1} + \varepsilon^4 u_{,t_2} + \ldots + \varepsilon^{2m} u_{,t_m} \tag{7}$$

we obtain a series of equations in ascending power of $\varepsilon$ starting with $\varepsilon^{-2}$.

### 3.1 $O(1)$ Homogenization

At $O(\varepsilon^{-2})$, we have

$$\{E(y) u_{0,y}\}_{,y} = 0 \tag{8}$$

The general solution of (8) is

$$u_0 = a_1(x, t_0, t_1, \ldots, t_m) \int_{y_0}^{y_0 + y} \frac{1}{E(y)} dy + a_2(x, t_0, t_1, \ldots, t_m) \tag{9}$$

where $a_1(x, t_0, t_1, \ldots, t_m)$ and $a_2(x, t_0, t_1, \ldots, t_m)$ are functions of macro coordinates and multiple temporal scales. To ensure periodicity of $u_0$ over the unit cell domain $\hat{\Omega} = \Omega/\varepsilon$ in the stretched coordinate system $y$, $a_1$ must vanish, implying that the leading-order displacement depends only on the macroscale

$$u_0 = u_0(x, t_0, t_1, \ldots, t_m) \tag{10}$$

At order $O(\varepsilon^{-1})$, the perturbation equation is

$$\{E(y)(u_{0,x}+u_{1,y})\}_{,y} = 0 \tag{11}$$

Due to linearity of the above equation, the general solution of $u_1$ is

$$u_1(x, y, t_0, t_1, \ldots, t_m) = U_1(x, t_0, t_1, \ldots, t_m) + L(y)u_{0,x} \tag{12}$$

Substituting (12) into (11) yields

$$\{E(y)(1+L_{,y})\}_{,y} = 0 \tag{13}$$

For a $\hat{\Omega}$-periodic function $g(x, y, t_0, t_1, \ldots, t_m)$, we define an averaging operator

$$\langle g\rangle = \frac{1}{|\hat{\Omega}|}\int_{\hat{\Omega}} g(x, y, t_0, t_1, \ldots, t_m)dy \tag{14}$$

The boundary conditions for the unit cell problem described by (13) are

(a) Periodicity: $u_1(y=0) = u_1(y=\hat{\Omega})$, $\quad \sigma_0(y=0) = \sigma_0(y=\hat{\Omega})$

(b) Continuity: $\lceil u_1(y=\alpha\hat{\Omega})\rceil = 0$, $\quad \lceil \sigma_0(y=\alpha\hat{\Omega})\rceil = 0$

(c) Normalization: $\langle u_1(x, y, t_0, t_1, \ldots, t_m)\rangle = U_1(x, t_0, t_1, \ldots, t_m) \quad \Rightarrow \quad \langle L(y)\rangle = 0$ (15)

where $0 \le \alpha \le 1$ is the volume fraction of the unit cell; $\lceil\ \ \rceil$ is the jump operator; and

$$\sigma_i = E(y)(u_{i,x}+u_{i+1,y}), \qquad i = 0, 1, \ldots, nn \tag{16}$$

Equation (13) together with the boundary conditions (15) define the unit cell boundary value problem from which $L(y)$ can be uniquely determined as

$$L_1(y) = \frac{(1-\alpha)(E_2-E_1)}{(1-\alpha)E_1+\alpha E_2}\left[y-\frac{\alpha\hat{\Omega}}{2}\right], \quad L_2(y) = \frac{\alpha(E_1-E_2)}{(1-\alpha)E_1+\alpha E_2}\left[y-\frac{(1+\alpha)\hat{\Omega}}{2}\right] \tag{17}$$

At $O(1)$, the perturbation equation is

$$\rho(y)u_{0,t_0t_0} - \{E(y)(u_{0,x}+u_{1,y})\}_{,x} - \{E(y)(u_{1,x}+u_{2,y})\}_{,y} = 0 \tag{18}$$

Applying the averaging operator defined in (14) to the above equation and taking into account periodicity of $\sigma_1$, we get the macroscopic equation of motion at $O(1)$:

$$\rho_0 u_{0,t_0t_0} - E_0 u_{0,xx} = 0 \tag{19}$$

where

$$\rho_0 = \langle\rho\rangle = \alpha\rho_1 + (1-\alpha)\rho_2, \quad E_0 = \langle E(y)(1+L_{,y})\rangle = \frac{E_1E_2}{(1-\alpha)E_1 + \alpha E_2} \tag{20}$$

The above macroscopic equation of motion is non-dispersive. In order to capture dispersion effects, we next consider higher-order equilibrium equations.

### 3.2 $O(\varepsilon)$ Homogenization

$u_2$ is determined from $O(1)$ perturbation equation (18). Substituting (12) and (19) into (18), yields

$$\{E(y)u_{2,y}\}_{,y} = \{E_0(\theta(y)-1) - (E(y)L)_{,y}\}u_{0,xx} - \{E(y)U_{1,x}\}_{,y} \tag{21}$$

where

$$\theta(y) = \rho(y)/\rho_0 \tag{22}$$

We seek for the solution of $u_2$ in the form of

$$u_2(x, y, t_0, t_1, \ldots, t_m) = U_2(x, t_0, t_1, \ldots, t_m) + L(y)U_{1,x} + M(y)u_{0,xx} \tag{23}$$

Substituting (23) into (21) yields

$$\{E(y)(L + M_{,y})\}_{,y} = E_0(\theta(y)-1) \tag{24}$$

The boundary conditions for the above equation are: periodicity and continuity of $u_2$ and $\sigma_1$ as well as the normalization condition $\langle M(y)\rangle = 0$. Once the solution of $M(y)$ is obtained it can be easily verified that $M(y)$ satisfies

$$\langle\rho L\rangle = 0, \quad \langle E(L + M_{,y})\rangle = 0 \tag{25}$$

Consider the $O(\varepsilon)$ equilibrium equation:

$$\rho(y)u_{1,t_0t_0} - \{E(y)(u_{1,x} + u_{2,y})\}_{,x} - \{E(y)(u_{2,x} + u_{3,y})\}_{,y} = 0 \tag{26}$$

Applying the averaging operator to the above equation, and exploiting (25) together with periodicity of $\sigma_2$ yields

$$\rho_0 U_{1,t_0t_0} - E_0 U_{1,xx} = 0 \tag{27}$$

### 3.3 $O(\varepsilon^2)$ Homogenization

$u_3$ is determined from the $O(\varepsilon)$ perturbation equation (26). Inserting (12) and (23) into (26) and making use of the macroscopic equations of motion (19) and (27), gives

$$\{E(y)u_{3,y}\}_{,y} = \{E_0\theta(y)L - E(y)(L + M_{,y}) - (E(y)M)_{,y}\}u_{0,xxx} +$$

$$\{E_0(\theta(y) - 1) - (E(y)L)_{,y}\}U_{1,xx} - \{E(y)U_{2,x}\}_{,y} \tag{28}$$

Due to linearity of (28) the general solution of $u_3$ is as follows:

$$u_3(x, y, t_0, t_1, \ldots, t_m) = U_3(x, t_0, t_1, \ldots, t_m) + L(y)U_{2,x} +$$

$$M(y)U_{1,xx} + N(y)u_{0,xxx} \tag{29}$$

Substituting (29) into (28) gives

$$\{E(y)(M + N_{,y})\}_{,y} = E_0L\theta(y) - E(y)(L + M_{,y}) \tag{30}$$

The above equation, together with the periodicity and continuity of $u_3$ and $\sigma_2$ as well as the normalization condition $\langle N(y)\rangle = 0$, fully determine $N(y)$. After $N(y)$ is solved for, we can calculate

$$\langle\rho M\rangle = \frac{[\alpha(1-\alpha)]^2(\rho_2 - \rho_1)(E_1\rho_1 - E_2\rho_2)E_0\hat{\Omega}^2}{12\rho_0E_1E_2} \tag{31}$$

$$\langle E(M + N_{,y})\rangle = -\frac{\alpha(1-\alpha)E_0\hat{\Omega}^2}{12\rho_0}\left\{\frac{(E_2 - E_1)[\alpha^2\rho_1 - (1-\alpha)^2\rho_2] + E_0\rho_0}{(1-\alpha)E_1 + \alpha E_2} - \rho_0\right\} \tag{32}$$

Consider the equilibrium equation of $O(\varepsilon^2)$:

$$\rho(y)(u_{2,t_0t_0} + 2u_{0,t_0t_1}) - \{E(y)(u_{2,x} + u_{3,y})\}_{,x} - \{E(y)(u_{3,x} + u_{4,y})\}_{,y} = 0 \tag{33}$$

Applying the averaging operator to the above equation, and exploiting periodicity of $\sigma_3$ and making use of (31) and (32) leads to

$$\rho_0U_{2,t_0t_0} - E_0U_{2,xx} = \frac{1}{\varepsilon^2}E_du_{0,xxxx} - 2\rho_0u_{0,t_0t_1} \tag{34}$$

where

$$E_d = \frac{[\alpha(1-\alpha)]^2(E_1\rho_1 - E_2\rho_2)^2E_0\Omega^2}{12\rho_0^2[(1-\alpha)E_1 + \alpha E_2]^2} \tag{35}$$

$E_d$ characterizes the effect of the microstructure on the macroscopic behavior. It is proportional to the square of the dimension of the unit cell $\Omega$. Note that for homogeneous materials (i.e., $\alpha = 0$ or $\alpha = 1$) and in the case of impedance ratio $r = z_2/z_1$ ($z = \sqrt{E\rho}$) equal to unity, $E_d$ vanishes.

### 3.4 $O(\varepsilon^3)$ Homogenization

$u_4$ is determined from $O(\varepsilon^2)$ perturbation equation (33). Substituting (12), (23) and (29) into (33) and making use of (19), (27) and (34) yields

$$\begin{aligned}\{E(y)u_{4,y}\}_{,y} = {} & \{\theta(y)(E_0M + E_d/\varepsilon^2) - E(y)(M + N_{,y}) - (E(y)N)_{,y}\}u_{0,xxxx} + \\ & \{E_0\theta(y)L - E(y)(L + M_{,y}) - (E(y)M)_{,y}\}U_{1,xxx} + \\ & \{E_0(\theta(y) - 1) - (E(y)L)_{,y}\}U_{2,xx} - \{E(y)U_{3,x}\}_{,y}\end{aligned} \tag{36}$$

Due to linearity, the general solution of $u_4$ can be sought in the form

$$\begin{aligned}u_4(x, y, t_0, t_1, \ldots, t_m) = {} & U_4(x, t_0, t_1, \ldots, t_m) + L(y)U_{3,x} + \\ & M(y)U_{2,xx} + N(y)U_{1,xxx} + P(y)u_{0,xxxx}\end{aligned} \tag{37}$$

Substituting (37) into (36) yields

$$\{E(y)(N + P_{,y})\}_{,y} = \theta(y)(E_0M + E_d/\varepsilon^2) - E(y)(M + N_{,y}) \tag{38}$$

The above equation, together with the periodicity and continuity of $u_4$ and $\sigma_3$ as well as the normalization condition $\langle P(y)\rangle = 0$, uniquely determines $P(y)$. The solution of $P(y)$ satisfies

$$\langle\rho N\rangle = 0, \quad \langle E(N + P_{,y})\rangle = 0 \tag{39}$$

The equilibrium equation at $O(\varepsilon^3)$ is:

$$\rho(y)(u_{3,t_0t_0} + 2u_{1,t_0t_1}) - \{E(y)(u_{3,x} + u_{4,y})\}_{,x} - \{E(y)(u_{4,x} + u_{5,y})\}_{,y} = 0 \tag{40}$$

Applying the averaging operator to (40), exploiting periodicity of $\sigma_4$ and making use of (39) yields

$$\rho_0U_{3,t_0t_0} - E_0U_{3,xx} = \frac{1}{\varepsilon^2}E_dU_{1,xxxx} - 2\rho_0U_{1,t_0t_1} \tag{41}$$

### 3.5 $O(\varepsilon^4)$ Homogenization

$u_5$ is determined from $O(\varepsilon^3)$ perturbation equation (40). Substituting (12), (23), (29) and (37) into (40) and making use of (19), (27), (34) and (41) yields

$$\begin{aligned}\{E(y)u_{5,y}\}_{,y} \;=\; & \{\theta(y)(E_0N + LE_d/\varepsilon^2) - E(y)(N + P_{,y}) - (E(y)P)_{,y}\}u_{0,xxxxx} + \\ & \{\theta(y)(E_0M + E_d/\varepsilon^2) - E(y)(M + N_{,y}) - (E(y)N)_{,y}\}U_{1,xxxx} + \\ & \{E_0\theta(y)L - E(y)(L + M_{,y}) - (E(y)M)_{,y}\}U_{2,xxx} + \\ & \{E_0(\theta(y) - 1) - (E(y)L)_{,y}\}U_{3,xx} - \{E(y)U_{4,x}\}_{,y}\end{aligned} \tag{42}$$

Due to linearity, the general solution of $u_5$ can be sought in the form

$$\begin{aligned}u_5(x, y, t_0, t_1, \ldots, t_m) \;=\; & U_5(x, t_0, t_1, \ldots, t_m) + L(y)U_{4,x} + \\ & M(y)U_{3,xx} + N(y)U_{2,xxx} + P(y)U_{1,xxxx} + Q(y)u_{0,xxxxx}\end{aligned} \tag{43}$$

Substituting (43) into (42) gives

$$\{E(y)(P + Q_{,y})\}_{,y} \;=\; \theta(y)(E_0N + LE_d/\varepsilon^2) - E(y)(N + P_{,y}) \tag{44}$$

The above equation, together with the periodicity and continuity of $u_5$ and $\sigma_4$ as well as the normalization condition $\langle Q(y)\rangle = 0$, uniquely determines $Q(y)$. After $Q(y)$ is solved for, expressions for $\langle\rho P\rangle$ and $\langle E(P + Q_{,y})\rangle$ can be derived.

The $O(\varepsilon^4)$ equilibrium equation is:

$$\begin{aligned}\rho(y)(u_{4,t_0t_0} + 2u_{2,t_0t_1} + 2u_{0,t_0t_2} + u_{0,t_1t_1}) - \{E(y)(u_{4,x} + u_{5,y})\}_{,x} - \\ \{E(y)(u_{5,x} + u_{6,y})\}_{,y} \;=\; 0\end{aligned} \tag{45}$$

Applying the averaging operator to the above equation and taking into account periodicity of $\sigma_5$, gives

$$\rho_0\frac{\partial^2 U_4}{\partial t_0^2} - E_0\frac{\partial^2 U_4}{\partial x^2} \;=\; \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_2}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 u_0}{\partial x^6} - 2\rho_0\frac{\partial^2 U_2}{\partial t_0\partial t_1} - 2\rho_0\frac{\partial^2 u_0}{\partial t_0\partial t_2} - \rho_0\frac{\partial^2 u_0}{\partial t_1^2} \tag{46}$$

where

$$E_g = \frac{[\alpha(1-\alpha)]^2(E_1\rho_1 - E_2\rho_2)^2 E_0\Omega^4}{360\rho_0^4[(1-\alpha)E_1 + \alpha E_2]^4}\{\alpha^2 E_2^2[2\alpha^2\rho_1^2 - (1-\alpha)^2\rho_2^2 +$$

$$6\alpha(1-\alpha)\rho_1\rho_2] + 2\alpha(1-\alpha)E_1E_2[3\alpha^2\rho_1^2 + 3(1-\alpha)^2\rho_2^2 +$$

$$11\alpha(1-\alpha)\rho_1\rho_2] - (1-\alpha)^2E_1^2[\alpha^2\rho_1^2 - 2(1-\alpha)^2\rho_2^2 - 6\alpha(1-\alpha)\rho_1\rho_2]\} \quad (47)$$

### 3.6 $O(\varepsilon^5)$ Homogenization

Next we determine the value of $u_6$ from $O(\varepsilon^4)$ perturbation equation (45). Substituting (12), (23), (29), (37) and (43) into (45) and making use of (19), (27), (34), (41) and (46), yields

$$\{E(y)u_{6,y}\}_{,y} = \{\theta(y)(E_0P + ME_d/\varepsilon^2 - E_g/\varepsilon^4) - E(y)(P + Q_{,y}) -$$

$$(E(y)Q)_{,y}\}u_{0,xxxxxx} + \{\theta(y)(E_0N + LE_d/\varepsilon^2) - E(y)(N + P_{,y}) -$$

$$(E(y)P)_{,y}\}U_{1,xxxxx} + \{\theta(y)(E_0M + E_d/\varepsilon^2) - E(y)(M + N_{,y}) -$$

$$(E(y)N)_{,y}\}U_{2,xxxx} + \{E_0\theta(y)L - E(y)(L + M_{,y}) - (E(y)M)_{,y}\}U_{3,xxx} +$$

$$\{E_0(\theta(y) - 1) - (E(y)L)_{,y}\}U_{4,xx} - \{E(y)U_{5,x}\}_{,y} \quad (48)$$

Due to linearity of (48) the general solution of $u_6$ can be sought in the form

$$u_6(x, y, t_0, t_1, \ldots, t_m) = U_6(x, t_0, t_1, \ldots, t_m) + L(y)U_{5,x} +$$

$$M(y)U_{4,xx} + N(y)U_{3,xxx} + P(y)U_{2,xxxx} + Q(y)U_{1,xxxxx} + R(y)u_{0,xxxxxx} \quad (49)$$

Substituting the above expression into (48) yields

$$\{E(y)(Q + R_{,y})\}_{,y} = \theta(y)(E_0P + ME_d/\varepsilon^2 - E_g/\varepsilon^4) - E(y)(P + Q_{,y}) \quad (50)$$

The above equation, together with the periodicity and continuity of $u_6$ and $\sigma_5$ as well as the normalization condition $\langle R(y)\rangle = 0$, uniquely determines $R(y)$. After $R(y)$ is solved for, it can be easily shown that

$$\langle\rho Q\rangle = 0, \quad \langle E(Q + R_{,y})\rangle = 0 \quad (51)$$

The equilibrium equation at $O(\varepsilon^5)$ is:

$$\rho(y)(u_{5,t_0t_0} + 2u_{3,t_0t_1} + 2u_{1,t_0t_2} + u_{1,t_1t_1}) - \{E(y)(u_{5,x} + u_{6,y})\}_{,x} -$$

$$\{E(y)(u_{6,x} + u_{7,y})\}_{,y} = 0 \tag{52}$$

Applying the averaging operator to the above equation, exploiting periodicity of $\sigma_6$ and making use of (51) yields

$$\rho_0\frac{\partial^2 U_5}{\partial t_0^2} - E_0\frac{\partial^2 U_5}{\partial x^2} = \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_3}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 U_1}{\partial x^6} - 2\rho_0\frac{\partial^2 U_3}{\partial t_0 \partial t_1} - 2\rho_0\frac{\partial^2 U_1}{\partial t_0 \partial t_2} - \rho_0\frac{\partial^2 U_1}{\partial t_1^2} \tag{53}$$

### 3.7 Higher Order Homogenization and Summary of Macroscopic Equations

The homogenization process described in the previous section can be systematically generalized to an arbitrary order. In this section we summarize various order macroscopic equations of motion and state the initial and boundary conditions.

The macroscopic equations of motion are:

$$O(1): \quad \rho_0 u_{0,t_0t_0} - E_0 u_{0,xx} = 0 \tag{19}$$

$$O(\varepsilon): \quad \rho_0 U_{1,t_0t_0} - E_0 U_{1,xx} = 0 \tag{27}$$

$$O(\varepsilon^2): \quad \rho_0 U_{2,t_0t_0} - E_0 U_{2,xx} = \frac{1}{\varepsilon^2}E_d u_{0,xxxx} - 2\rho_0 u_{0,t_0t_1} \tag{34}$$

$$O(\varepsilon^3): \quad \rho_0 U_{3,t_0t_0} - E_0 U_{3,xx} = \frac{1}{\varepsilon^2}E_d U_{1,xxxx} - 2\rho_0 U_{1,t_0t_1} \tag{41}$$

$$O(\varepsilon^4): \quad \rho_0\frac{\partial^2 U_4}{\partial t_0^2} - E_0\frac{\partial^2 U_4}{\partial x^2} = \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_2}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 u_0}{\partial x^6} - 2\rho_0\frac{\partial^2 U_2}{\partial t_0 \partial t_1} - 2\rho_0\frac{\partial^2 u_0}{\partial t_0 \partial t_2} - \rho_0\frac{\partial^2 u_0}{\partial t_1^2} \tag{46}$$

$$O(\varepsilon^5): \quad \rho_0\frac{\partial^2 U_5}{\partial t_0^2} - E_0\frac{\partial^2 U_5}{\partial x^2} = \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_3}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 U_1}{\partial x^6} - 2\rho_0\frac{\partial^2 U_3}{\partial t_0 \partial t_1} - 2\rho_0\frac{\partial^2 U_1}{\partial t_0 \partial t_2} - \rho_0\frac{\partial^2 U_1}{\partial t_1^2} \tag{53}$$

$O(\varepsilon^6)$ and higher:

$$\rho_0\frac{\partial^2 U_{2m}}{\partial t_0^2} - E_0\frac{\partial^2 U_{2m}}{\partial x^2} = \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_{2(m-1)}}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 U_{2(m-2)}}{\partial x^6} +$$

$$\frac{E_{s1}}{\varepsilon^6}\frac{\partial^8 U_{2(m-3)}}{\partial x^8} - \frac{E_{s2}}{\varepsilon^8}\frac{\partial^{10} U_{2(m-4)}}{\partial x^{10}} + \ldots + (-1)^{m+1}\frac{E_{s(m-2)}}{\varepsilon^{2m}}\frac{\partial^{2(m+1)} u_0}{\partial x^{2(m+1)}} -$$

$$2\rho_0\left(\frac{\partial^2 u_0}{\partial t_0 \partial t_m} + \frac{\partial^2 u_0}{\partial t_1 \partial t_{m-1}} + \ldots + \frac{1}{2}\frac{\partial^2 u_0}{\partial t_{m/2}^2}\right) -$$

$$2\rho_0\left(\frac{\partial^2 U_2}{\partial t_0 \partial t_{m-1}} + \frac{\partial^2 U_2}{\partial t_1 \partial t_{m-2}} + \ldots + \frac{1}{2}\frac{\partial^2 U_2}{\partial t_{(m-1)/2}^2}\right) -$$

$$2\rho_0\left(\frac{\partial^2 U_4}{\partial t_0 \partial t_{m-2}} + \frac{\partial^2 U_4}{\partial t_1 \partial t_{m-3}} + \ldots + \frac{1}{2}\frac{\partial^2 U_4}{\partial t_{(m-2)/2}^2}\right) - \ldots - 2\rho_0\frac{\partial^2 U_{2(m-1)}}{\partial t_0 \partial t_1} \tag{54}$$

$$\rho_0\frac{\partial^2 U_{2m+1}}{\partial t_0^2} - E_0\frac{\partial^2 U_{2m+1}}{\partial x^2} = \frac{E_d}{\varepsilon^2}\frac{\partial^4 U_{2m-1}}{\partial x^4} - \frac{E_g}{\varepsilon^4}\frac{\partial^6 U_{2m-3}}{\partial x^6} +$$

$$\frac{E_{s1}}{\varepsilon^6}\frac{\partial^8 U_{2m-5}}{\partial x^8} - \frac{E_{s2}}{\varepsilon^8}\frac{\partial^{10} U_{2m-7}}{\partial x^{10}} + \ldots + (-1)^{m+1}\frac{E_{s(m-2)}}{\varepsilon^{2m}}\frac{\partial^{2(m+1)} U_1}{\partial x^{2(m+1)}} -$$

$$2\rho_0\left(\frac{\partial^2 U_1}{\partial t_0 \partial t_m} + \frac{\partial^2 U_1}{\partial t_1 \partial t_{m-1}} + \ldots + \frac{1}{2}\frac{\partial^2 U_1}{\partial t_{m/2}^2}\right) -$$

$$2\rho_0\left(\frac{\partial^2 U_3}{\partial t_0 \partial t_{m-1}} + \frac{\partial^2 U_3}{\partial t_1 \partial t_{m-2}} + \ldots + \frac{1}{2}\frac{\partial^2 U_3}{\partial t_{(m-1)/2}^2}\right) -$$

$$2\rho_0\left(\frac{\partial^2 U_5}{\partial t_0 \partial t_{m-2}} + \frac{\partial^2 U_5}{\partial t_1 \partial t_{m-3}} + \ldots + \frac{1}{2}\frac{\partial^2 U_5}{\partial t_{(m-2)/2}^2}\right) - \ldots - 2\rho_0\frac{\partial^2 U_{2m-1}}{\partial t_0 \partial t_1} \tag{55}$$

where $m = 3, 4, 5, \ldots$; $t_k$ $(k = 0, 1, 2, \ldots, m)$; $E_{s1}$, $E_{s2}$, $\ldots$, $E_{s(m-2)}$ can be evaluated using higher-order homogenization process.

Subsequently, we consider the following model problem: a domain composed of an array of bilaminates with fixed boundary at $x = 0$ and free boundary at $x = l$ subjected to an initial disturbance $f(x)$ in the displacement field. At $O(1)$, the displacement field is determined by the equation of motion (19) and the following initial and boundary conditions

$$\text{ICs:} \quad u_0(x, 0, 0, \dots, 0) = f(x), \quad u_{0;t}(x, 0, 0, \dots, 0) = q(x) = 0 \tag{56}$$

$$\text{BCs:} \quad u_0(0, t_0, t_1, \dots, t_m) = 0, \quad u_{0,x}(l, t_0, t_1, \dots, t_m) = 0 \tag{57}$$

The calculation of $\varepsilon U_1(x, t_0, t_1, \dots, t_m)$ is obtained by solving the equation of motion (27). The initial and boundary conditions applied to $\varepsilon U_1$ must be such that the global field $u_0 + \varepsilon U_1$ meets macroscopic initial conditions and conditions imposed on the boundary:

$$u_0(x, 0, 0, \dots, 0) + \varepsilon U_1(x, 0, 0, \dots, 0) = f(x)$$

$$u_{0;t}(x, 0, 0, \dots, 0) + \varepsilon U_{1;t}(x, 0, 0, \dots, 0) = 0$$

$$u_0(0, t_0, t_1, \dots, t_m) + \varepsilon U_1(0, t_0, t_1, \dots, t_m) = 0$$

$$u_{0,x}(l, t_0, t_1, \dots, t_m) + \varepsilon U_{1,x}(l, t_0, t_1, \dots, t_m) = 0$$

Taking into account (56) and (57), the initial and boundary conditions for $\varepsilon U_1$ become

$$\text{ICs:} \quad \varepsilon U_1(x, 0, 0, \dots, 0) = 0, \quad \varepsilon U_{1;t}(x, 0, 0, \dots, 0) = 0$$

$$\text{BCs:} \quad \varepsilon U_1(0, t_0, t_1, \dots, t_m) = 0, \quad \varepsilon U_{1,x}(l, t_0, t_1, \dots, t_m) = 0$$

Similarly, the macroscopic field $\varepsilon^2 U_2(x, t_0, t_1, \dots, t_m)$ is determined from the equation of motion (34), with the initial and boundary conditions for $\varepsilon^2 U_2$ such that the global field $u_0 + \varepsilon U_1 + \varepsilon^2 U_2$ should satisfy the macroscopic initial and boundary conditions.

With this in mind, we obtain the initial and boundary conditions for different order equations of motion:

$$\text{ICs:} \quad u_0(x, 0, 0, \dots, 0) = f(x), \quad u_{0;t}(x, 0, 0, \dots, 0) = q(x) = 0$$

$$U_i(x, 0, 0, \dots, 0) = 0, \quad U_{i;t}(x, 0, 0, \dots, 0) = 0 \quad (i = 1, 2, 3, \dots) \tag{58}$$

$$\text{BCs:} \quad U_i(0, t_0, t_1, \dots, t_m) = 0, \quad U_{i,x}(l, t_0, t_1, \dots, t_m) = 0 \quad (i = 0, 1, 2, \dots) \tag{59}$$

From the above equations of motion and the initial-boundary conditions, we can readily deduce that

$$U_{2m+1}(x, t_0, t_1, \ldots, t_m) \equiv 0, \quad (m = 0, 1, 2, \ldots) \tag{60}$$

# 4. Solution of Macroscopic Equations

We start with the zero-order equation of motion (19), the solution of which can be sought by means of separation of variables in the form

$$u_0(x, t_0, t_1, \ldots, t_m) = X(x)T(t_0, t_1, \ldots, t_m) \tag{61}$$

Substituting the above equation into (19) and dividing by the product $X \cdot T$ yields

$$\frac{1}{T}\frac{\partial^2 T}{\partial t_0^2} = c^2\frac{X''}{X} = -\lambda^2 \tag{62}$$

where $\lambda$ is the separation constant and

$$c = \sqrt{E_0 / \rho_0} \tag{63}$$

The resulting differential equations and corresponding solutions are

$$X'' + \frac{\lambda^2}{c^2}X = 0, \qquad \frac{\partial^2 T}{\partial t_0^2} + \lambda^2 T = 0 \tag{64}$$

$$X(x) = A_1 \sin\frac{\lambda x}{c} + A_2 \cos\frac{\lambda x}{c} \tag{65}$$

$$T(t_0, t_1, \ldots, t_m) = D(t_1, t_2, \ldots, t_m)\sin(\lambda t_0) + F(t_1, t_2, \ldots, t_m)\cos(\lambda t_0) \tag{66}$$

where $A_1$ and $A_2$ are constants of integration; $D(t_1, t_2, \ldots, t_m)$ and $F(t_1, t_2, \ldots, t_m)$ are undetermined functions.

Substituting (61), (65) and (66) into (59) gives

$$A_2 = 0, \qquad A_1 \cos\frac{\lambda l}{c} = 0 \tag{67}$$

The second equation in (67) yields

$$\lambda_n = (2n-1)\frac{\pi c}{2l}, \quad (n = 1, 2, 3, \ldots) \tag{68}$$

Due to linearity of the differential equation, the total solution consists of the sum of individual solutions. Hence, we may write

$$u_0(x, t_0, t_1, \ldots, t_m) = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}[D_n(t_1, t_2, \ldots, t_m)\sin(\lambda_n t_0) +$$
$$F_n(t_1, t_2, \ldots, t_m)\cos(\lambda_n t_0)] \tag{69}$$

which can be shown to satisfy the boundary conditions. Inserting (69) into the second order macroscopic equation of motion (34) yields

$$\frac{\partial^2 U_2}{\partial t_0^2} - c^2\frac{\partial^2 U_2}{\partial x^2} = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}\left\{\left[\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{\lambda_n}{c}\right)^4 D_n + 2\lambda_n\frac{\partial F_n}{\partial t_1}\right]\sin(\lambda_n t_0) +\right.$$
$$\left.\left[\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{\lambda_n}{c}\right)^4 F_n - 2\lambda_n\frac{\partial D_n}{\partial t_1}\right]\cos(\lambda_n t_0)\right\} \tag{70}$$

The right-hand-side in (70) is also a solution to the corresponding homogeneous equation, and therefore, it generates secular terms. In order to eliminate secular terms and to avoid unbounded resonance of $U_2(x, t_0, t_1, \ldots, t_m)$, the source term must vanish, i.e.

$$\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{\lambda_n}{c}\right)^4 D_n + 2\lambda_n\frac{\partial F_n}{\partial t_1} = 0, \quad \frac{E_d}{\varepsilon^2\rho_0}\left(\frac{\lambda_n}{c}\right)^4 F_n - 2\lambda_n\frac{\partial D_n}{\partial t_1} = 0 \tag{71}$$

Let

$$\omega_n = \frac{E_d}{2c\rho_0}\left(\frac{\lambda_n}{c}\right)^3 \tag{72}$$

Then (71) can be written as

$$\varepsilon^2\frac{\partial F_n}{\partial t_1} + \omega_n D_n = 0, \quad \varepsilon^2\frac{\partial D_n}{\partial t_1} - \omega_n F_n = 0 \tag{73}$$

Differentiating the first equation in (73) and inserting the second equation into the resulting equation leads to

$$\varepsilon^4\frac{\partial^2 F_n}{\partial t_1^2} + \omega_n^2 F_n = 0 \tag{74}$$

Likewise, differentiating the second equation in (73) and inserting the first equation into the resulting equation yields

$$\varepsilon^4 \frac{\partial^2 D_n}{\partial t_1^2} + \omega_n^2 D_n = 0 \tag{75}$$

The general solutions to (74) and (75) are

$$D_n(t_1, t_2, \ldots, t_m) = G_n(t_2, t_3, \ldots, t_m) \sin\left(\frac{\omega_n t_1}{\varepsilon^2}\right) + J_n(t_2, t_3, \ldots, t_m) \cos\left(\frac{\omega_n t_1}{\varepsilon^2}\right) \tag{76}$$

$$F_n(t_1, t_2, \ldots, t_m) = K_n(t_2, t_3, \ldots, t_m) \sin\left(\frac{\omega_n t_1}{\varepsilon^2}\right) + S_n(t_2, t_3, \ldots, t_m) \cos\left(\frac{\omega_n t_1}{\varepsilon^2}\right) \tag{77}$$

where $G_n(t_2, t_3, \ldots, t_m)$, $J_n(t_2, t_3, \ldots, t_m)$, $K_n(t_2, t_3, \ldots, t_m)$ and $S_n(t_2, t_3, \ldots, t_m)$ are undetermined functions.

Solutions (76) and (77) must satisfy (73). Inserting (76) and (77) into (73) gives

$$J_n = -K_n, \quad G_n = S_n \tag{78}$$

Substituting (76), (77) and (78) into (69) yields

$$u_0(x, t_0, t_1, \ldots, t_m) = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}\left[K_n(t_2, t_3, \ldots, t_m)\sin\left(\frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right) + \right.$$

$$\left. S_n(t_2, t_3, \ldots, t_m)\cos\left(\frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right)\right] \tag{79}$$

Since the source term of $O(\varepsilon^2)$ macroscopic equation of motion has been set to zero and considering the initial and boundary conditions described in Section 3.7, we deduce

$$U_2(x, t_0, t_1, \ldots, t_m) = 0 \tag{80}$$

Inserting (79) and (80) into the fourth order macroscopic equation of motion (46) yields

$$\frac{\partial^2 U_4}{\partial t_0^2} - c^2 \frac{\partial^2 U_4}{\partial x^2} = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}\left\{\left[\frac{1}{\varepsilon^4}\left[\frac{E_g}{\rho_0}\left(\frac{\lambda_n}{c}\right)^6 + \omega_n^2\right]K_n - 2\lambda_n \frac{\partial S_n}{\partial t_2}\right]\sin\left(\frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right) + \right.$$

$$\left.\left[\frac{1}{\varepsilon^4}\left[\frac{E_g}{\rho_0}\left(\frac{\lambda_n}{c}\right)^6 + \omega_n^2\right]S_n + 2\lambda_n \frac{\partial K_n}{\partial t_2}\right]\cos\left(\frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right)\right\} \tag{81}$$

Again, the right-hand-side term in the above equation is the solution of the corresponding homogeneous equation and thus will generate secular terms. In order to eliminate secularity arising from $U_4$, we set

$$\frac{1}{\varepsilon^4}\left[\frac{E_g}{\rho_0}\left(\frac{\lambda_n}{c}\right)^6 + \omega_n^2\right]K_n - 2\lambda_n\frac{\partial S_n}{\partial t_2} = 0, \quad \frac{1}{\varepsilon^4}\left[\frac{E_g}{\rho_0}\left(\frac{\lambda_n}{c}\right)^6 + \omega_n^2\right]S_n + 2\lambda_n\frac{\partial K_n}{\partial t_2} = 0 \tag{82}$$

Let

$$\gamma_n = \frac{1}{\lambda_n}\left[\frac{E_g}{2\rho_0}\left(\frac{\lambda_n}{c}\right)^6 + \frac{1}{2}\omega_n^2\right] = \frac{1}{2c\rho_0}\left(E_g + \frac{E_d^2}{4c^2\rho_0}\right)\left(\frac{\lambda_n}{c}\right)^5 \tag{83}$$

Equation (82) can be written as

$$\varepsilon^4\frac{\partial S_n}{\partial t_2} - \gamma_n K_n = 0, \qquad \varepsilon^4\frac{\partial K_n}{\partial t_2} + \gamma_n S_n = 0 \tag{84}$$

The general solutions to the above equations are

$$K_n(t_2, t_3, \ldots, t_m) = V_n(t_3, t_4, \ldots, t_m)\sin\left(\frac{\gamma_n t_2}{\varepsilon^4}\right) + W_n(t_3, t_4, \ldots, t_m)\cos\left(\frac{\gamma_n t_2}{\varepsilon^4}\right) \tag{85}$$

$$S_n(t_2, t_3, \ldots, t_m) = W_n(t_3, t_4, \ldots, t_m)\sin\left(\frac{\gamma_n t_2}{\varepsilon^4}\right) - V_n(t_3, t_4, \ldots, t_m)\cos\left(\frac{\gamma_n t_2}{\varepsilon^4}\right) \tag{86}$$

Substituting (85) and (86) into (79) gives

$$u_0(x, t_0, t_1, \ldots, t_m) = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}\left[W_n(t_3, t_4, \ldots, t_m)\sin\left(\frac{\gamma_n t_2}{\varepsilon^4} + \frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right) - \right.$$

$$\left. V_n(t_3, t_4, \ldots, t_m)\cos\left(\frac{\gamma_n t_2}{\varepsilon^4} + \frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right)\right] \tag{87}$$

Since the source term of $O(\varepsilon^4)$ macroscopic equation of motion vanishes and considering the initial and boundary conditions prescribed in Section 3.7, we conclude that

$$U_4(x, t_0, t_1, \ldots, t_m) = 0 \tag{88}$$

The above procedure can be systematically extended to higher order equations, which yields

$$u_0(x, t_0, t_1, \ldots, t_m) = \sum_{n=1}^{\infty} \sin\frac{\lambda_n x}{c}\left[A_n \sin\left(\frac{\beta_{(m-2)n}t_m}{\varepsilon^{2m}} + \ldots + \frac{\beta_{2n}t_4}{\varepsilon^8} + \frac{\beta_{1n}t_3}{\varepsilon^6} + \frac{\gamma_n t_2}{\varepsilon^4} + \right.\right.$$

$$\left.\left.\frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right) + B_n \cos\left(\frac{\beta_{(m-2)n}t_m}{\varepsilon^{2m}} + \ldots + \frac{\beta_{2n}t_4}{\varepsilon^8} + \frac{\beta_{1n}t_3}{\varepsilon^6} + \frac{\gamma_n t_2}{\varepsilon^4} + \frac{\omega_n t_1}{\varepsilon^2} - \lambda_n t_0\right)\right] \quad (89)$$

where $A_n$ and $B_n$ are constants of integration and

$$\beta_{1n} = \frac{1}{\lambda_n}\left[\frac{E_{s1}}{2\rho_0}\left(\frac{\lambda_n}{c}\right)^8 + \omega_n \gamma_n\right] \quad (90)$$

$$\beta_{2n} = \frac{1}{\lambda_n}\left[\frac{E_{s2}}{2\rho_0}\left(\frac{\lambda_n}{c}\right)^{10} + \omega_n \beta_{1n} + \frac{1}{2}\gamma_n^2\right] \quad (91)$$

$$\beta_{3n} = \frac{1}{\lambda_n}\left[\frac{E_{s3}}{2\rho_0}\left(\frac{\lambda_n}{c}\right)^{12} + \omega_n \beta_{2n} + \gamma_n \beta_{1n}\right] \quad (92)$$

$$\ldots\ldots$$

$$\beta_{(m-2)n} = \frac{1}{\lambda_n}\left[\frac{E_{s(m-2)}}{2\rho_0}\left(\frac{\lambda_n}{c}\right)^{2(m+1)} + \omega_n \beta_{(m-3)n} + \gamma_n \beta_{(m-4)n} + \right.$$

$$\left.\beta_{1n}\beta_{(m-5)n} + \beta_{2n}\beta_{(m-6)n} + \ldots + \frac{1}{2}\beta^2_{(m/2-2)n}\right] \quad (93)$$

where $\beta_{in}$ $(i = 1, 2, \ldots, (m-2))$ and $i$ is an integer.

Inserting $t_k = \varepsilon^{2k}t$, $(k = 0, 1, 2, \ldots, m)$ into (89) and using the initial conditions (58) we can determine $A_n$ and $B_n$ as

$$A_n = 0, \qquad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{(2n-1)\pi x}{2l}dx \quad (94)$$

and thus a uniformly valid dispersive solution, denoted here as $u_d$, is given as

$$u_d(x, t_0, t_1, \ldots, t_m) = \sum_{n=1}^{\infty} B_n \sin\frac{\lambda_n x}{c}\cos\left[\lambda_n t_0 - \left(\frac{\omega_n t_1}{\varepsilon^2} + \frac{\gamma_n t_2}{\varepsilon^4} + \frac{\beta_{1n} t_3}{\varepsilon^6} + \right.\right.$$

$$\left.\left.\frac{\beta_{2n}t_4}{\varepsilon^8} + \ldots + \frac{\beta_{(m-2)n}t_m}{\varepsilon^{2m}}\right)\right] \tag{95}$$

For function evaluation, we insert $t_k = \varepsilon^{2k}t$, $(k = 0, 1, 2, \ldots, m)$, which yields

$$u_d(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{\lambda_n x}{c}\cos\{[\lambda_n - (\omega_n + \gamma_n + \beta_{1n} + \beta_{2n} + \ldots + \beta_{(m-2)n})]t\} \tag{96}$$

## 5. Numerical Results

To assess the accuracy of the proposed formulation, we construct a reference solution by utilizing a very fine finite element mesh to discretize the problem domain. We consider the following initial disturbance in displacements:

$$f(x) = f_0 a_0[x - (x_0 - \delta)]^2[x - (x_0 + \delta)]^2\{1 - H[x - (x_0 + \delta)]\} \cdot$$
$$[1 - H(x_0 - \delta - x)]$$

where $a_0 = 1/\delta^4$ and $H(x)$ is the Heaviside step function; $f_0$, $x_0$ and $\delta$ are the magnitude, the location of the maximum value and the half width of the initial pulse. Substituting the initial disturbance $f(x)$ into (94) and integrating analytically, we get

$$B_n = \frac{2}{l}\int_{x_0-\delta}^{x_0+\delta} f_0 a_0[x - (x_0 - \delta)]^2[x - (x_0 + \delta)]^2 \sin\frac{(2n-1)\pi x}{2l}dx$$

$$= \frac{256 l^2 f_0}{\delta^4[(2n-1)\pi]^5}\left\{\left[12l^2 - ((2n-1)\delta\pi)^2\right]\sin\frac{(2n-1)\pi x_0}{2l}\sin\frac{(2n-1)\pi\delta}{2l} - \right.$$

$$\left. 6(2n-1)\delta\pi l \sin\frac{(2n-1)\pi x_0}{2l}\cos\frac{(2n-1)\pi\delta}{2l}\right\}$$

The material properties considered are: $E_1 = 120$ *GPa*, $E_2 = 6$ *GPa*, $\rho_1 = 8000$ *Kg/m*$^3$, $\rho_2 = 3000$ *Kg/m*$^3$, and volume fraction $\alpha = 0.5$. The dimension of the macro-domain and that of the unit cell are set as $l = 40$ *m* and $\Omega = 0.2$ *m*, respectively. The homogenized material properties are calculated as $E_0 = 11.43$ *GPa*, $\rho_0 = 5500$ *Kg/m*$^3$ and $E_d = 1.76 \times 10^7$ *N*. In this case, $E_1/E_2 = 20$ and the ratio of the impedance of the two material constituents is $r = 7.30$. The initial pulse is centered at the midpoint of the domain, i.e. $x_0 = 20\ m$, with the magnitude $f_0 = 1.0\ m$.

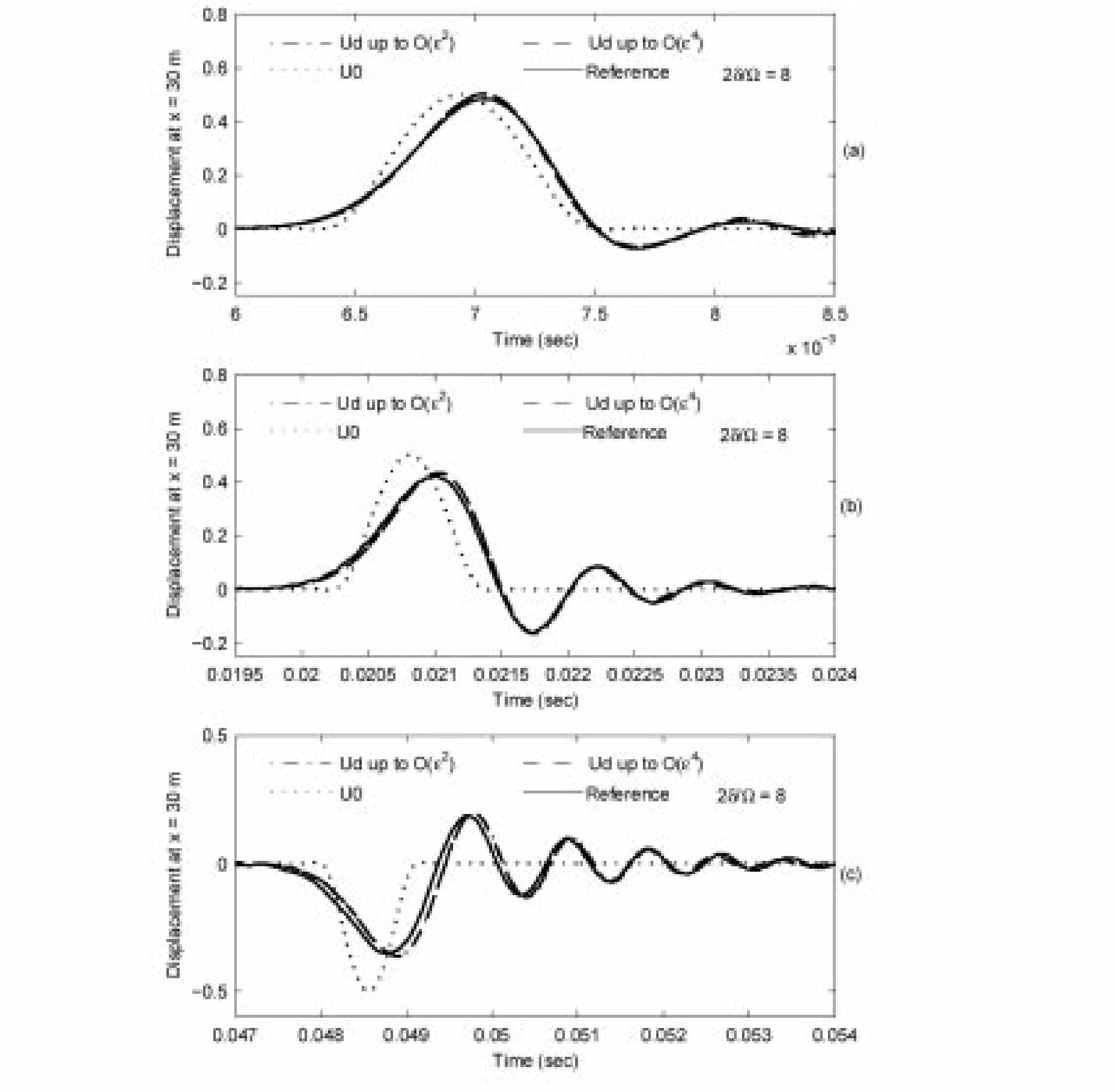


Figure 2: Displacements at $x = 30m$ for the initial half pulse width $\delta = 0.8m$.

Evolution of displacements of the point *(x = 30 m)* is plotted in figures 2-4 for three cases corresponding to $\delta = 0.8\ m$, $\delta = 0.5\ m$ and $\delta = 0.3\ m$, respectively. In other words, the ratios between the pulse width and the unit cell dimension $2\delta/\Omega$ are 8, 5 and 3, respectively. Each of the figures 2-4 depicts four graphs corresponding to the finite element solution of the source problem, the analytical nondispersive solution $u_0(x, t)$, the dispersive solution $u_d(x, t)$ up to the second order and the dispersive solution up to the fourth order.

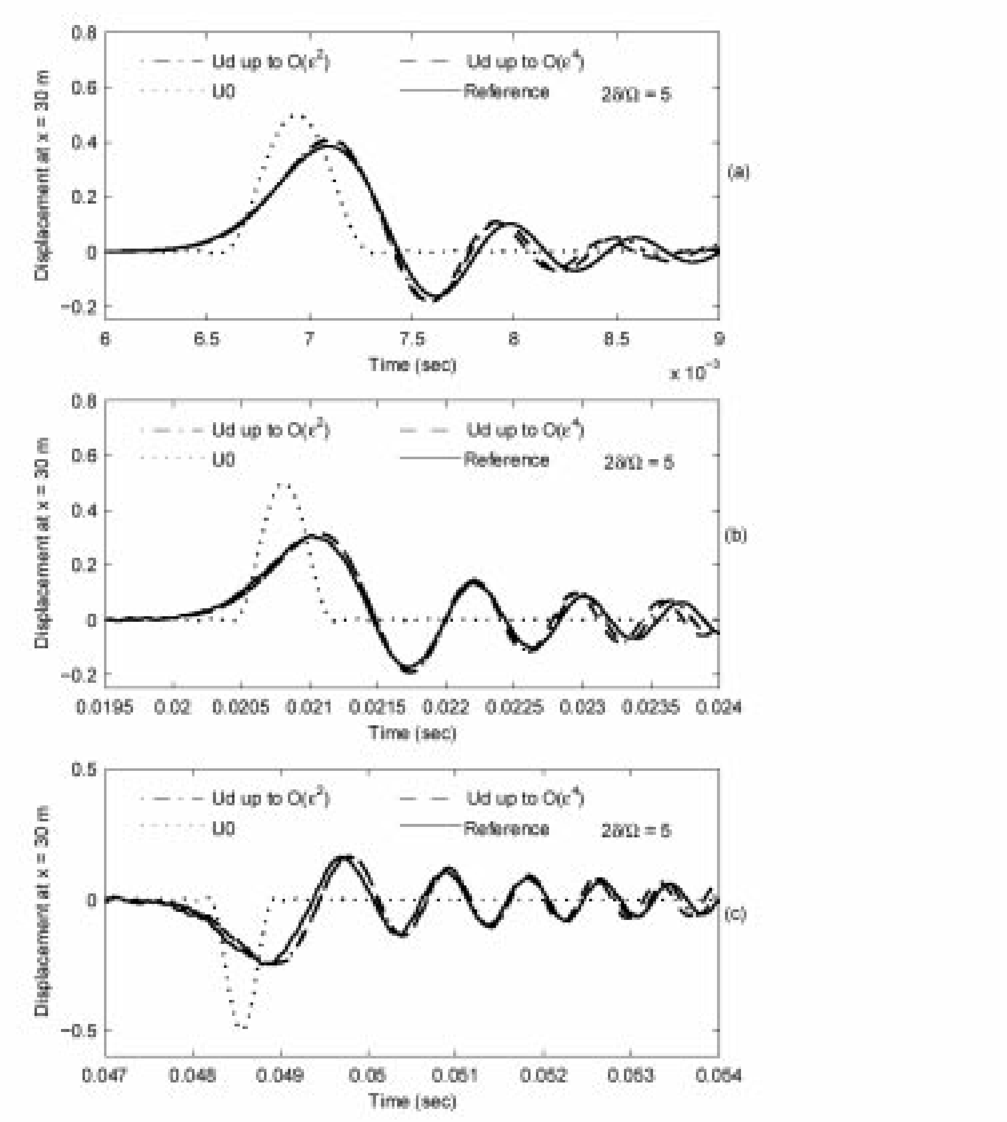


Figure 3: Displacements at $x = 30m$ for the initial half pulse width $\delta = 0.5m$

The phenomenon of dispersion can be clearly observed in Figures 2-4. Figure 2 corresponds to rather low frequency case, where the pulse almost maintains its initial shape except for some minor wiggles at the wavefront. In this case, the leading-order homogenization can give a reasonable approximation to the response of a heterogeneous medium. However, when the pulse width of the initial disturbance is comparable to the dimension of the unit cell and the observation time is large, which are the cases shown in Figures 3 and 4, the wave becomes strongly dispersive and the leading-order homogenization errs badly. It can be readily observed that the dispersive solution $u_d(x, t)$ provides a good approximation to the response of the heterogeneous media even as the initial pulse width is only 3 times of the unit cell dimension.

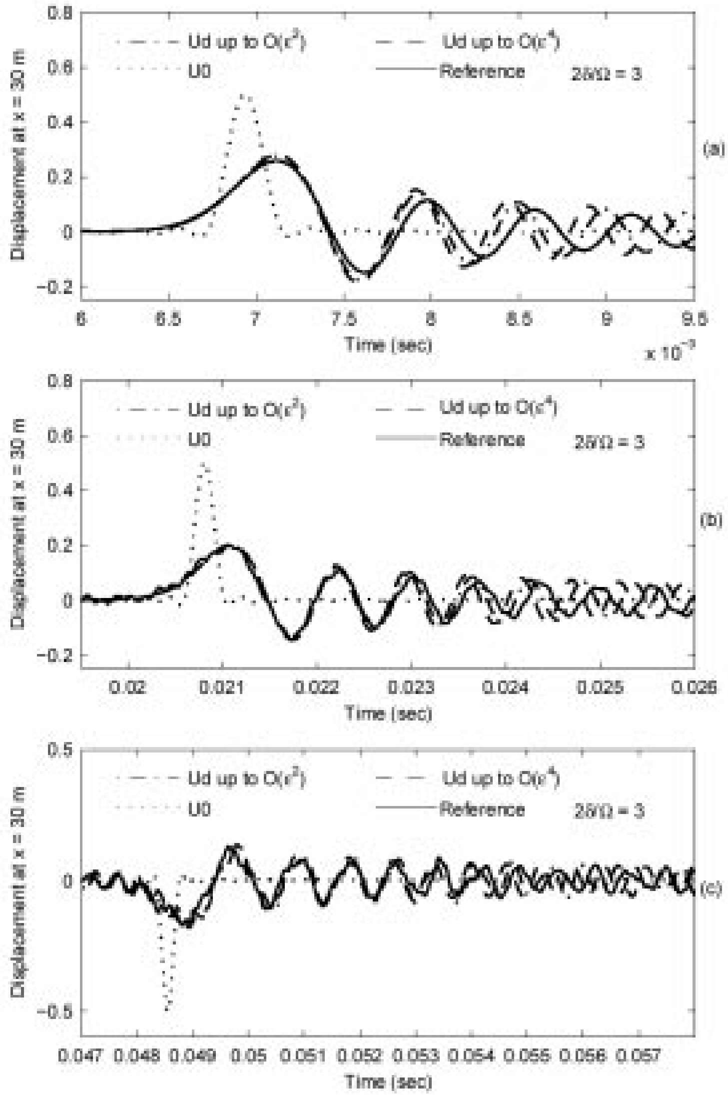


Figure 4: Displacements at $x = 30m$ for the initial half pulse width $\delta = 0.3m$

## 6. Concluding Remarks

Mathematical homogenization theory with multiple spatial and temporal scales have been investigated. This work is motivated by our recent studies [11][12] which suggested that in absence of multiple time scaling, higher order mathematical homogenization method gives rise to secular terms which grow unbounded with time. In attempt to develop a uniformly valid dispersive model up to an arbitrary order we extend the theory developed in [12] to fast spatial scale and a series of slow temporal scales.

In our future work we will focus on the following two issues: (i) generalization to the multidimensional case, and (ii) a finite element implementation.

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