Computational Damage Mechanics for Composite Materials Based on Mathematical Homogenization

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Abstract

This paper is aimed at developing a nonlocal theory for obtaining numerical approximation to a boundary value problem describing damage phenomena in a brittle composite material. The mathematical homogenization method based on double scale asymptotic expansion is generalized to account for damage effects in heterogeneous media. A closed form expression relating local fields to the overall strain and damage is derived. Nonlocal damage theory is developed by introducing the concept of nonlocal phase fields (stress, strain, free energy density, damage release rate, etc.) in a manner analogous to that currently practiced in concrete [7], [8], with the only exception being that the weight functions are taken to be C^0 continuous over a single phase and zero elsewhere. Numerical results of our model were found to be in good agreement with experimental data of 4-point bend test conducted on composite beam made of Blackglas^{TM}/Nextel 5-harness satin weave.

Keywords: damage, composites, homogenization, nonlocal, asymptotic

1.0 Introduction

Damage in composite materials occurs through different mechanisms that are complex and usually involve interaction between microconstituents. During the past two decades, a number of models have been developed to simulate damage and failure process in composite materials, among which the damage mechanics approach is particularly attractive in the sense that it provides a viable framework for the description of distributed damage including material stiffness degradation, initiation, growth and coalescence of microcracks and voids. Various damage models for brittle composites can be classified into micromechanical and macromechanical approaches. In the macromechanical damage approach, composite material is idealized (or homogenized) as an anisotropic homogeneous medium and damage is introduced via internal variable whose tensorial nature depends on assumptions about crack orientation [15], [28], [29], [42], [35], [43], [31]. The micromechanical damage approach, on the other hand, treats each microphase as a statistically homogeneous medium. Local damage variables are defined to represent the state of damage in each phase and phase effective material properties are defined thereafter. The overall response is subsequently obtained by homogenization [1], [30], [44], [45], [46].

From the mathematical formulation stand point, both approaches can be viewed as a two-step procedure. The main difference between the two approaches is in the chronological order in which the homogenization and evolution of damage are carried out. In the macromechanical approach, homogenization is performed first followed by application of damage mechanics principles to homogenized anisotropic medium, while in the micromechanical approach, damage mechanics is applied to each phase followed by homogenization.

The primary objective of the present manuscript is to simultaneously carry out the two steps (homogenization and evolution of damage) by extending the framework of the classical mathematical homogenization theory [3][4][27] to account for damage effects. This is accomplished by introducing a double scale asymptotic expansion of damage parameter (or damage tensor in general). This leads to the derivation of the closed form expression relating local fields to overall strains and damage (Section 2). The second salient feature of our approach is in developing a nonlocal theory by introducing the concept of nonlocal phase fields (stress, strain, free energy density, damage release rate, etc.) in Section 3. Nonlocal phase fields are defined as weighted averages over each phase in the characteristic volume in a manner analogous to that currently practiced in concrete [7], [8] with the only exception being that the weight functions are taken to be C^0 continuous over a single phase and zero elsewhere. On the global (macro) level we limit the finite element size to ensure a valid use of the mathematical homogenization theory and to limit localization. In Sections 4 and 5 we develop a mathematical and numerical model for the case of piecewise constant weight function, which is the simplest variant of the model presented in Section 3. The stress update procedure and the consistent tangent stiffness matrix are then derived. Section 6 compares the results of our numerical model to the experimental data. We consider a 4-point bend test conducted on the composite beam made of Blackglas™/Nextel 5-harness satin weave and compare our numerical simulations to experiments conducted at Rutgers University [14].

2.0 Mathematical Homogenization for Damaged Composites

In this section we extend the classical mathematical homogenization theory [3] for statistically homogeneous composite media to account for damage effects. The strain-based continuum damage theory is adopted for constructing constitutive relations at the level of microconstituents. Closed form expressions of local strain and stress fields in a multi-phase composite medium are derived. Attention is restricted to small deformations.

The microstructure of a composite material is assumed to be locally periodic (Y-periodic) with a period defined by a Statistically Homogeneous Volume Element (SHVE), denoted by Θ, as shown in Figure 1. Let $\boldsymbol{x}$ be a macroscopic coordinate vector in macro domain Ω and $\boldsymbol{y} \equiv \boldsymbol{x}/\varsigma$ be a microscopic position vector in Θ. Here, ς denotes a very small positive number compared with the dimension of Ω, and $\boldsymbol{y} \equiv \boldsymbol{x}/\varsigma$ is regarded as a stretched coordinate vector in the microscopic domain. When a solid is subjected to some load and boundary conditions, the resulting deformation, stresses, and internal variables may vary from point to point within the SHVE due to the high level of heterogeneity. We assume that all quantities have two explicit dependencies: one on the macroscopic level $\boldsymbol{x}$, and the other one on the level of microconstituents $\boldsymbol{y} \equiv \boldsymbol{x}/\varsigma$. For any Y-periodic response function f, we have

$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{k}\hat{\boldsymbol{y}})$ in which vector $\hat{\boldsymbol{y}}$ is the basic period of the microstructure and $\boldsymbol{k}$ is a 3 by 3 diagonal matrix with integer components. Adopting the classical nomenclature, any Y-periodic function f can be represented as

$$f^{\varsigma}(\boldsymbol{x}) \equiv f(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) \tag{1}$$

where superscript ς denotes a Y-periodic function f. The indirect macroscopic spatial derivatives of f^{ς} can be calculated by the chain rule as

$$f^{\varsigma}_{,x_i}(\boldsymbol{x}) = f_{,x_i}(\boldsymbol{x}, \boldsymbol{y}) + \frac{1}{\varsigma} f_{,y_i}(\boldsymbol{x}, \boldsymbol{y}) \tag{2}$$

where the comma followed by a subscript variable x_i denotes a partial derivative with respect to the subscript variable (i.e. $f_{,x_i} \equiv \partial f / \partial x_i$). Summation convention for repeated subscripts is employed, except for subscripts x and y.

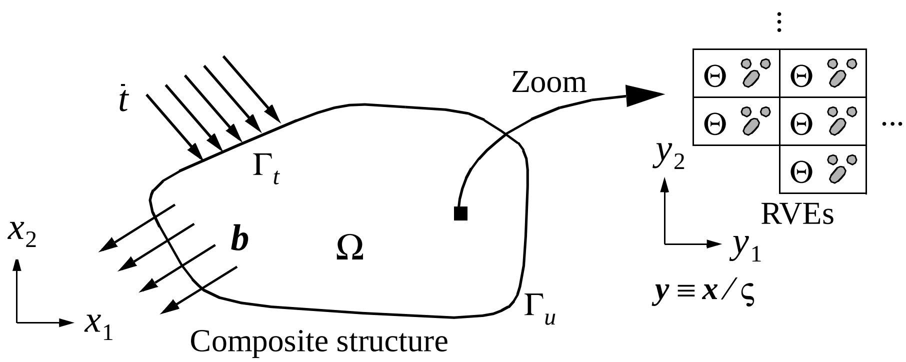

Figure 1: Macroscopic and microscopic structures

The constitutive equation on the microscale is derived from continuum damage theory based on the thermodynamics of irreversible processes and internal state variable theory. To model the isotropic damage process, we define a scalar damage parameter ω^{ς} as a function of microscopic and macroscopic position vectors, i.e., $\omega^{\varsigma} = \omega(\boldsymbol{x}, \boldsymbol{y})$.

Based on the strain-based continuum damage theory, the free energy density has the form of

$$\Psi(\omega^{\varsigma}, \varepsilon^{\varsigma}_{ij}) = (1 - \omega^{\varsigma})\Psi_e(\varepsilon^{\varsigma}_{ij}) \tag{3}$$

where $\omega^{\varsigma} \in [0, 1)$ is the damage parameter. For small deformations, elastic free energy density is given as $\Psi_e(\varepsilon^{\varsigma}_{ij}) = \frac{1}{2} L_{ijkl} \varepsilon^{\varsigma}_{ij} \varepsilon^{\varsigma}_{kl}$. The constitutive equation, thermodynamic force (also known as a damage energy release rate) and dissipative inequality follow from (3)

$$\sigma_{ij}^{\varsigma} = \frac{\partial\Psi(\omega^{\varsigma}, \varepsilon_{ij}^{\varsigma})}{\partial\varepsilon_{ij}^{\varsigma}} = (1-\omega^{\varsigma})L_{ijkl}\varepsilon_{kl}^{\varsigma} \tag{4}$$

$$Y = -\frac{\partial\Psi(\omega^{\varsigma}, \varepsilon_{ij}^{\varsigma})}{\partial\omega^{\varsigma}} = \Psi_e(\varepsilon^{\varsigma}) \tag{5}$$

$$Y\dot{\omega}^{\varsigma} \geq 0 \tag{6}$$

With this brief glimpse into the constitutive theory, we proceed to outlining the strong form of the governing differential equations on the fine scale - the scale of microconstituents. Further details on the evolution of damage are given in Section 4.

We assume that micro-constituents possess homogeneous properties and satisfy equilibrium, constitutive, kinematics and compatibility equations. The corresponding boundary value problem is governed by the following set of equations:

$$\sigma_{ij;x_j}^{\varsigma} + b_i = 0 \quad \text{in} \quad \Omega \tag{7}$$

$$\sigma_{ij}^{\varsigma} = (1-\omega^{\varsigma})L_{ijkl}\varepsilon_{kl}^{\varsigma} \quad \text{in} \quad \Omega \tag{8}$$

$$\varepsilon_{ij}^{\varsigma} = u_{(i,x_j)}^{\varsigma} \quad \text{in} \quad \Omega \tag{9}$$

$$u_i^{\varsigma} = \bar{u}_i \quad \text{on} \quad \Gamma_u \tag{10}$$

$$\sigma_{ij}^{\varsigma}n_j = \dot{t}_i \quad \text{on} \quad \Gamma_t \tag{11}$$

where ω^{ς} is a scalar damage parameter; σ_{ij}^{ς} and $\varepsilon_{ij}^{\varsigma}$ are components of stress and strain tensors; L_{ijkl} represents components of elastic stiffness satisfying conditions of symmetry

$$L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij} \tag{12}$$

and positivity

$$\exists \;\; C_0 > 0, \qquad L_{ijkl}\xi_{ij}^{\varsigma}\xi_{kl}^{\varsigma} \geq C_0\xi_{ij}^{\varsigma}\xi_{ij}^{\varsigma} \qquad \forall\xi_{ij}^{\varsigma} = \xi_{ji}^{\varsigma} \tag{13}$$

b_i is a body force assumed to be independent of $\boldsymbol{y}$; u_i^{ς} denotes the components of the displacement vector; the subscript pairs with parentheses denote the symmetric gradients defined as

$$u_{(i,x_j)}^{\varsigma} \equiv \frac{1}{2}(u_{i,x_j}^{\varsigma} + u_{j,x_i}^{\varsigma}) \tag{14}$$

Ω denotes the macroscopic domain of interest with boundary Γ; Γ_u and Γ_t are boundary portions where displacements $\bar{u}_i$ and tractions $\bar{t}_i$ are prescribed, respectively, such that $\Gamma_u \cap \Gamma_t = \varnothing$ and $\Gamma = \Gamma_u \cup \Gamma_t$; n_i denotes the normal vector on Γ. We assume that the interface between the phases is perfectly bonded, i.e. $[\sigma^{\varsigma}_{ij}\hat{n}_j] = 0$ and $[u^{\varsigma}_i] = 0$ at the interface, Γ_{int}, where $\hat{n}_i$ is the normal vector to Γ_{int} and $[\bullet]$ is a jump operator.

Clearly, a brute force approach attempting discretization of the entire macro domain with a grid spacing comparable to that of the microscale features is not computationally feasible. Thus, a mathematical homogenization method based on the double-scale asymptotic expansion is employed to account for microstructural effects on the macroscopic response without explicitly representing the details of the microstructure in the global analysis. As a starting point, we approximate the displacement field, $u^{\varsigma}_i(\boldsymbol{x}) = u_i(\boldsymbol{x}, \boldsymbol{y})$, and the damage parameter, $\omega^{\varsigma}(\boldsymbol{x}) = \omega(\boldsymbol{x}, \boldsymbol{y})$, in terms of double-scale asymptotic expansions on $\Omega \times \Theta$:

$$u_i(\boldsymbol{x}, \boldsymbol{y}) \approx u^0_i(\boldsymbol{x}, \boldsymbol{y}) + \varsigma u^1_i(\boldsymbol{x}, \boldsymbol{y}) + \ldots \tag{15}$$

$$\omega(\boldsymbol{x}, \boldsymbol{y}) \approx \omega^0(\boldsymbol{x}, \boldsymbol{y}) + \varsigma\omega^1(\boldsymbol{x}, \boldsymbol{y}) + \ldots \tag{16}$$

Strain expansions on the composite domain $\Omega \times \Theta$ can be obtained by substituting (15) into (9) with consideration of the indirect differentiation rule (2)

$$\varepsilon_{ij}(\boldsymbol{x}, \boldsymbol{y}) \approx \frac{1}{\varsigma}\varepsilon^{-1}_{ij}(\boldsymbol{x}, \boldsymbol{y}) + \varepsilon^0_{ij}(\boldsymbol{x}, \boldsymbol{y}) + \varsigma\varepsilon^1_{ij}(\boldsymbol{x}, \boldsymbol{y}) + \ldots \tag{17}$$

where strain components for various orders of ς are given as

$$\varepsilon^{-1}_{ij} = \varepsilon_{yij}(\boldsymbol{u}^0), \qquad \varepsilon^s_{ij} = \varepsilon_{xij}(\boldsymbol{u}^s) + \varepsilon_{yij}(\boldsymbol{u}^{s+1}), \quad s = 0, 1, \ldots \tag{18}$$

and

$$\varepsilon_{xij}(\boldsymbol{u}^s) = u^s_{(i,x_j)}, \qquad \varepsilon_{yij}(\boldsymbol{u}^s) = u^s_{(i,y_j)} \tag{19}$$

Stresses and strains for different orders of ς are related by the constitutive equation (8)

$$\sigma^{-1}_{ij} = (1 - \omega^0)L_{ijkl}\varepsilon^{-1}_{kl} \tag{20}$$

$$\sigma^s_{ij} = (1 - \omega^0)L_{ijkl}\varepsilon^s_{kl} + \sum_{r=0}^{s} \omega^{s-r+1}L_{ijkl}\varepsilon^{r-1}_{kl}, \qquad s = 0, 1, \ldots \tag{21}$$

The resulting asymptotic expansion of stress is given as

$$\sigma_{ij}(\boldsymbol{x}, \boldsymbol{y}) \approx \frac{1}{\varsigma}\sigma_{ij}^{-1}(\boldsymbol{x}, \boldsymbol{y}) + \sigma_{ij}^{0}(\boldsymbol{x}, \boldsymbol{y}) + \varsigma\sigma_{ij}^{1}(\boldsymbol{x}, \boldsymbol{y}) + \ldots \tag{22}$$

Inserting the stress expansion (22) into equilibrium equation (7) and making use of equation (2) yield the following equilibrium equations for various orders:

$$O(\varsigma^{-2}): \quad \sigma_{ij,y_j}^{-1} = 0 \tag{23}$$

$$O(\varsigma^{-1}): \quad \sigma_{ij,x_j}^{-1} + \sigma_{ij,y_j}^{0} = 0 \tag{24}$$

$$O(\varsigma^{0}): \quad \sigma_{ij,x_j}^{0} + \sigma_{ij,y_j}^{1} + b_i = 0 \tag{25}$$

$$O(\varsigma^{s}): \quad \sigma_{ij,x_j}^{s} + \sigma_{ij,y_j}^{s+1} = 0, \quad s = 1, 2, \ldots \tag{26}$$

We consider the $O(\varsigma^{-2})$ equilibrium equation (23) first. Pre-multiplying it by u_i^0 and integrating over Θ yields

$$\int_{\Theta} u_i^0 \sigma_{ij,y_j}^{-1}\, d\Theta = 0 \tag{27}$$

and subsequently integrating by parts gives

$$\int_{\Gamma_\Theta} u_i^0 \sigma_{ij}^{-1} n_j\, d\Gamma_\Theta - \int_{\Theta} (1-\omega^0) u_{(i,y_j)}^0 L_{ijkl} u_{(k,y_l)}^0\, d\Theta = 0 \tag{28}$$

where Γ_Θ denotes the boundary of Θ. The boundary integral term in (28) vanishes due to Y-periodicity on Γ_Θ, and hence, with the positivity of L_{ijkl} and the assumption of $\omega^0 \in [0, 1)$ (see Section3), we have

$$\varepsilon_{yij}(\boldsymbol{u}^0) = u_{(i,y_j)}^0 = 0 \qquad \Rightarrow \qquad u_i^0 = u_i^0(\boldsymbol{x}) \tag{29}$$

and

$$\sigma_{ij}^{-1}(\boldsymbol{x}, \boldsymbol{y}) = \varepsilon_{ij}^{-1}(\boldsymbol{x}, \boldsymbol{y}) = 0 \tag{30}$$

We proceed to the $O(\varsigma^{-1})$ equilibrium equation (24). From (18) and (20) follow

$$\{(1-\omega^0) L_{ijkl}(\varepsilon_{xkl}(\boldsymbol{u}^0) + \varepsilon_{ykl}(\boldsymbol{u}^1))\}_{,y_j} = 0 \quad \text{on} \quad \Theta \tag{31}$$

To solve for (31) up to a constant we introduce the following separation of variables

$$u_i^1(\boldsymbol{x}, \boldsymbol{y}) = H_{ikl}(\boldsymbol{y})\{\varepsilon_{xkl}(\boldsymbol{u}^0) + d_{kl}^{\omega}(\boldsymbol{x})\} \tag{32}$$

where H_{ikl} is a Y-periodic function. We assume that $d_{kl}^{\omega}(\boldsymbol{x})$ is macroscopic damage-induced strain driven by the macroscopic strain $\bar{\varepsilon}_{kl} \equiv \varepsilon_{xkl}(\boldsymbol{u}^0)$. More specifically we can state that if $\bar{\varepsilon}_{kl} = 0$, then $d_{kl}^{\omega}(\boldsymbol{x}) = 0$ and $\omega^0(\boldsymbol{x}, \boldsymbol{y}) = 0$. Note that vice versa is not true, i.e., if $d_{kl}^{\omega}(\boldsymbol{x}) = 0$ or $\omega^0(\boldsymbol{x}, \boldsymbol{y}) = 0$, the macroscopic strain $\bar{\varepsilon}_{kl}$ may not be necessarily zero. In (32) both H_{ikl} and d_{kl}^{ω} are symmetric with respect to indices k and l.

Based on the decomposition given in (32), the $O(\varsigma^{-1})$ equilibrium equation takes the following form:

$$\left\{(1-\omega^0)L_{ijkl}[(I_{klmn}+G_{klmn})\varepsilon_{xmn}(\boldsymbol{u}^0)+G_{klmn}d_{mn}^{\omega}(\boldsymbol{x})]\right\}_{,y_j} = 0 \quad \text{in} \quad \Theta \tag{33}$$

where

$$I_{klmn} = \frac{1}{2}(\delta_{mk}\delta_{nl}+\delta_{nk}\delta_{ml}), \qquad G_{klmn}(\boldsymbol{y}) = H_{(k,y_l)mn}(\boldsymbol{y}) \tag{34}$$

and δ_{mk} is the Kronecker delta, while G_{klmn} is known as a polarization function. It can be shown that the integrals of the polarization functions in Θ vanish due to periodicity conditions. Since equation (33) should be valid for arbitrary macroscopic fields, we may first consider the case of $d_{kl}^{\omega}(\boldsymbol{x}) = 0$ (and $\omega^0 = 0$) but $\bar{\varepsilon}_{kl} \neq 0$, which yields the following equation in Θ:

$$\{L_{ijkl}(I_{klmn}+H_{(k,y_l)mn})\}_{,y_j} = 0 \tag{35}$$

Equation (35) together with the Y-periodic boundary conditions is a linear boundary value problem in Θ. By exploiting the symmetry with respect to the indexes (m, n), the weak form of (35) is solved for 3 right hand side vectors in 2-D and 6 right hand side vectors in 3-D (see for example [20][27]).

In the absence of damage, the asymptotic expansion of strain (17) can be expressed in terms of the macroscopic strain $\bar{\varepsilon}_{ij}$ as follows

$$\varepsilon_{ij} = A_{ijkl}\bar{\varepsilon}_{kl} + O(\varsigma) \tag{36}$$

where A_{ijkl} is termed as the elastic strain concentration function defined as

$$A_{ijkl} = I_{ijkl} + G_{ijkl} \tag{37}$$

The elastic homogenized stiffness $\bar{L}_{ijkl}$ follows from the $O(\varsigma^0)$ equilibrium equation [18]:

$$\bar{L}_{ijkl} \equiv \frac{1}{|\Theta|}\int_{\Theta} L_{ijmn}A_{mnkl}\, d\Theta \;=\; \frac{1}{|\Theta|}\int_{\Theta} A_{mnij}L_{mnst}A_{stkl}\, d\Theta \tag{38}$$

where $|\Theta|$ is the volume of a SHVE.

After solving (35) for H_{imn}, we proceed to find d^{ω}_{mn} from (33). Premultiplying it by H_{ist} and integrating it by parts with consideration of Y-periodic boundary conditions yields

$$\int_{\Theta}(1-\omega^0)G_{ijst}L_{ijkl}(A_{klmn}\varepsilon_{xmn}(\boldsymbol{u}^0) + G_{klmn}d^{\omega}_{mn}(\boldsymbol{x}))\, d\Theta \;=\; 0 \tag{39}$$

from where the expression of the macroscopic damaged induced strain can be shown to be

$$d^{\omega}_{mn}(\boldsymbol{x}) = -\left\{\int_{\Theta}(1-\omega^0)G_{ijst}L_{ijkl}G_{klmn}\, d\Theta\right\}^{-1}\left\{\int_{\Theta}(1-\omega^0)G_{ijst}L_{ijkl}A_{klmn}\, d\Theta\right\}\bar{\varepsilon}_{mn} \tag{40}$$

Let $\vec{\Psi} \equiv \{\psi^{(\eta)}(\boldsymbol{y})\}_1^n$ be a set of C^{-1} continuous functions, then the damage parameter $\omega^0(\boldsymbol{x}, \boldsymbol{y})$ is assumed to have the following decomposition

$$\omega^0(\boldsymbol{x}, \boldsymbol{y}) \;=\; \sum_{\eta = 1}^{n} \psi^{(\eta)}(\boldsymbol{y})\omega^{(\eta)}(\boldsymbol{x}) \tag{41}$$

where $\psi^{(\eta)}(\boldsymbol{y})$ is a damage distribution function on the microscale. Rewriting (40) in terms of strain concentration function A_{ijkl} and manipulating it with (38) and (41) yield

$$d^{\omega}_{mn}(\boldsymbol{x}) \;=\; D_{klmn}(\boldsymbol{x})\bar{\varepsilon}_{mn} \tag{42}$$

where

$$D_{klmn}(\boldsymbol{x}) \;=\; \left(I_{klst} - \sum_{\eta = 1}^{n} B^{(\eta)}_{klst}\omega^{(\eta)}(\boldsymbol{x})\right)^{-1}\left(\sum_{\eta = 1}^{n} C^{(\eta)}_{stmn}\omega^{(\eta)}(\boldsymbol{x})\right) \tag{43}$$

$$B^{(\eta)}_{ijkl} \;=\; \frac{1}{|\Theta|}(\tilde{L}_{ijmn} - \bar{L}_{ijmn})^{-1}\int_{\Theta}\psi^{(\eta)}G_{stmn}L_{stpq}G_{pqkl}\, d\Theta \tag{44}$$

$$C^{(\eta)}_{ijkl} \;=\; \frac{1}{|\Theta|}(\tilde{L}_{ijmn} - \bar{L}_{ijmn})^{-1}\int_{\Theta}\psi^{(\eta)}G_{stmn}L_{stpq}A_{pqkl}\, d\Theta \tag{45}$$

$$\tilde{L}_{ijmn} \;=\; \frac{1}{|\Theta|}\int_{\Theta} L_{ijmn}\, d\Theta \tag{46}$$

In conjunction with (32) and (42), the asymptotic expansion of strain field (17) can be finally cast as

$$\varepsilon_{ij}(\boldsymbol{x},\boldsymbol{y}) = A_{ijmn}(\boldsymbol{y})\bar{\varepsilon}_{mn}(\boldsymbol{x}) + G_{ijkl}(\boldsymbol{y})D_{klmn}(\boldsymbol{x})\bar{\varepsilon}_{mn}(\boldsymbol{x}) + O(\varsigma) \tag{47}$$

where $G_{ijkl}(\boldsymbol{y})$ can be interpreted as a damage strain influence function. Note that the asymptotic expansion of the strain field is given as a sum of mechanical fields induced by the macroscopic strain via elastic strain concentration function and thermodynamical fields governed by damage-induced strain, $d^{\omega}_{kl}(\boldsymbol{x}) = D_{klmn}(\boldsymbol{x})\bar{\varepsilon}_{mn}(\boldsymbol{x})$, through the damage strain influence function.

Finally, we integrate the $O(\varsigma^0)$ equilibrium equation (25) over Θ. The $\int_{\Theta}\sigma^1_{ij,y_j}d\Theta$ term vanishes due to periodicity and we obtain:

$$\left(\frac{1}{|\Theta|}\int_{\Theta}\sigma^0_{ij}\,d\Theta\right)_{,x_j} + b_i = 0 \qquad \text{in} \qquad \Omega \tag{48}$$

Substituting the constitutive relation (20) and the asymptotic expansion of the strain field (47) into (48) yields the macroscopic equilibrium equation

$$\left(\frac{1}{|\Theta|}\int_{\Theta}(1-\omega^0)L_{ijkl}(A_{klmn}\bar{\varepsilon}_{mn} + G_{klmn}d^{\omega}_{mn})\,d\Theta\right)_{,x_j} + b_i = 0 \tag{49}$$

If we define the macroscopic stress $\bar{\sigma}_{ij}$ as

$$\bar{\sigma}_{ij} \equiv \frac{1}{|\Theta|}\int_{\Theta}\sigma^0_{ij}\,d\Theta \tag{50}$$

then the equilibrium equations (48) and (49) can be recast into more familiar form:

$$\bar{\sigma}_{ij,x_j} + b_i = 0 \qquad \text{and} \qquad (\mathcal{L}_{ijmn}\bar{\varepsilon}_{mn})_{,x_j} + b_i = 0 \tag{51}$$

where $\mathcal{L}_{ijmn}$ is an instantaneous secant stiffness given as

$$\mathcal{L}_{ijmn} = \left(\bar{L}_{ijkl} + \sum_{\eta=1}^{n}\frac{\omega^{(\eta)}}{|\Theta|}\int_{\Theta}\psi^{(\eta)}L_{ijst}A_{stkl}\,d\Theta\right)\cdot(I_{klmn} + D_{klmn})$$
$$-\left(\tilde{L}_{ijkl} + \sum_{\eta=1}^{n}\frac{\omega^{(\eta)}}{|\Theta|}\int_{\Theta}\psi^{(\eta)}L_{ijkl}\,d\Theta\right)\cdot D_{klmn} \tag{52}$$

3.0 Nonlocal Damage Model for Multi-phase Materials

Accumulation of damage leads to strain softening and loss of ellipticity. The local approach, stating that in the absence of thermal effects, stresses in a material at a point are completely determined by the deformation and the deformation history at that point, may result in a physically unacceptable localization of the deformation [6]. The principal fault of the local approach, as indicated in [5][6][8], is that the energy dissipation at failure is incorrectly predicted to be zero and the corresponding finite element solution converges to this spurious solution as the mesh is refined. To remedy the situation, a number of approaches have been devised to limit strain localization and to circumvent mesh sensitivity associated with strain softening [16]. One of these approaches is based on the nonlocal damage theory [5], [8], the essence of which is to smear solution variables causing strain softening over the characteristic volume of the material. For other forms of localization limiters including introduction of higher order gradients, artificial (or real) viscosity and elements with embedded localization zones we refer to [5], [10]-[13], [21]-[26], [40].

Following [6] and [8], the nonlocal damage parameter $\bar{\omega}(\boldsymbol{x})$ is defined as:

$$\bar{\omega}(\boldsymbol{x}) = \frac{1}{|\Theta_C|}\int_{\Theta_C} \varphi(\boldsymbol{y})\omega^0(\boldsymbol{x}, \boldsymbol{y})\, d\Theta \tag{53}$$

where $\varphi(\boldsymbol{y})$ is a weight function; Θ_C is the characteristic volume, and l_C is the characteristic length, defined (for example) as a radius of the largest inscribed sphere in Θ_C. The characteristic length l_C is related to the size of the material inhomogeneity [8], whereas l_H - the radius of the largest inscribed sphere in Θ - primarily depends on the distribution and interaction of inclusions and discrete deffects [9], [39]. Several guidelines for determining the value of characteristic length have been provided in [7] and [26]. l_C, as indicated in [8], is usually smaller than l_H in particular for random microstructures. In the present manuscript, we define the Representative Volume Element (RVE) as the maximum between the statistically homogeneous volume element, for which the local periodicity assumption is valid, and the characteristic volume. Schematically, this can be expressed as

$$l_{RVE} = max\{l_H , l_C\} \tag{54}$$

where l_{RVE} denotes the radius of the largest inscribed sphere in Θ_{RVE}.

We further assume that the microscopic damage distribution function $\psi^{(\eta)}(\boldsymbol{y})$ introduced in (41) is a piecewise function, i.e., it is continuous within the domain of microphase, $\Theta^{(\eta)} \subset \Theta_C \subset \Theta$, but vanishes elsewhere, i.e.

$$\psi^{(\eta)}(\boldsymbol{y}) = \begin{cases} g^{(\eta)}(\boldsymbol{y}) & \text{if } \boldsymbol{y} \in \Theta^{(\eta)} \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

where $\bigcup_{i=1}^{n} \Theta^{(\eta)} = \Theta$ and $\Theta^{(\lambda)} \bigcap \Theta^{(\eta)} = \varnothing$ for $\lambda \neq \eta$, $\eta = 1, 2, \ldots, n$; n is the product of the number of different microphases and the number of characteristic volumes in RVE; $\psi^{(\eta)}(\boldsymbol{y})$ is a distribution function; $g^{(\eta)}(\boldsymbol{y})$ is a C^o continuous function in $\Theta^{(\eta)}$; and

$\omega^{(\eta)}(\boldsymbol{x})$ is a macroscopically variable amplitude. Figure 2 illustrate two possibilities for construction of RVE in a two-phase medium: one for random microstructure where RVE typically coincides with SHVE, and the other one for periodic microstructure, where l_C and l_H are of the same order of magnitude.

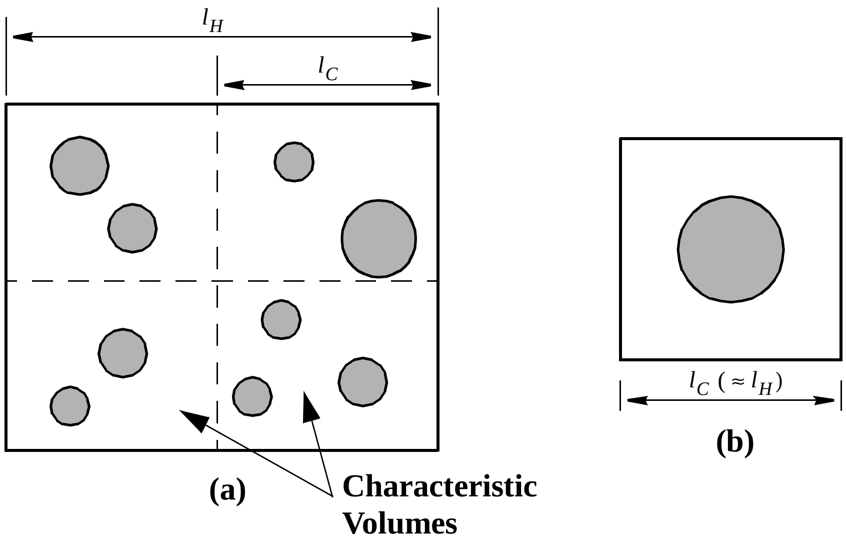

Figure 2: Selection of the Representative Volume Element

We further define the weight function in (53) as

$$\varphi(\boldsymbol{y}) = \mu^{(\eta)}\psi^{(\eta)}(\boldsymbol{y}) \tag{56}$$

where the constant $\mu^{(\eta)}$ is determined by the orthogonality condition

$$\frac{\mu^{(\eta)}}{|\Theta_C|}\int_{\Theta_C} g^{(\lambda)}(\boldsymbol{y})g^{(\eta)}(\boldsymbol{y})\,d\Theta = \delta_{\lambda\eta}, \qquad \lambda,\eta = 1, 2, \ldots, n \tag{57}$$

and $\delta_{\lambda\eta}$ is Kronecker delta. Substituting (41) and (55)-(57) into (53) yields

$$\bar{\omega}(\boldsymbol{x}) = \frac{\mu^{(\eta)}}{|\Theta_C|}\int_{\Theta_C} (g^{(\eta)}(\boldsymbol{y}))^2\omega^{(\eta)}(\boldsymbol{x})\,d\Theta = \omega^{(\eta)}(\boldsymbol{x}) \tag{58}$$

which provides the motivation for the specific choice of the weight function. It can be seen that $\omega^{(\eta)}$ has a meaning of the nonlocal phase damage parameter.

The average strains in each subdomain in RVE are obtained by integrating (47) over $\Theta^{(\eta)}$:

$$\varepsilon_{ij}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} \varepsilon_{ij}\,d\Theta = A_{ijkl}^{(\eta)}\bar{\varepsilon}_{kl} + G_{ijkl}^{(\eta)}D_{klmn}\bar{\varepsilon}_{mn} + O(\varsigma) \tag{59}$$

where

$$A_{ijkl}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} A_{ijkl}\, d\Theta \tag{60}$$

$$G_{ijkl}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} G_{ijkl}\, d\Theta \tag{61}$$

To construct the nonlocal constitutive relation between the phase averages we define the local average stress in $\Theta^{(\eta)}$ as:

$$\sigma_{ij}^{(\eta)} \equiv \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} \sigma_{ij}^{0}\, d\Theta \tag{62}$$

By combining (21), (41), (55), (59)-(61) we get

$$\sigma_{ij}^{(\eta)} = (I_{klmn} - \omega^{(\eta)} N_{klmn}^{(\eta)}) L_{ijkl}^{(\eta)} \varepsilon_{mn}^{(\eta)} \tag{63}$$

where

$$N_{klmn}^{(\eta)} = (\bar{A}_{klst}^{(\eta)} + \bar{G}_{klpq}^{(\eta)} D_{pqst}) \cdot (A_{mnst}^{(\eta)} + G_{mnij}^{(\eta)} D_{ijst})^{-1} \tag{64}$$

$$\bar{A}_{ijkl}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} g^{(\eta)} A_{ijkl}\, d\Theta \tag{65}$$

$$\bar{G}_{ijkl}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|}\int_{\Theta^{(\eta)}} g^{(\eta)} G_{ijkl}\, d\Theta \tag{66}$$

The constitutive equation (63) has a nonlocal character in the sense that it represents the relation between phase averages. The response characteristics between the phases are not smeared as the damage evolution law and thermomechanical properties of phases might be considerably different, in particular when damage occurs in a single phase.

For the isotropic strain-based damage model adopted in this paper, the phase free energy density corresponding to the nonlocal constitutive equation (63) is given as

$$\Psi^{(\eta)}(\omega^{(\eta)}, \varepsilon_{ij}^{(\eta)}) = \frac{1}{2}(I_{klmn} - \omega^{(\eta)} N_{klmn}^{(\eta)}) L_{ijkl}^{(\eta)} \varepsilon_{mn}^{(\eta)} \varepsilon_{ij}^{(\eta)} \tag{67}$$

and the corresponding nonlocal phase damage energy release rate can be expressed as

$$Y^{(\eta)} = -\frac{\partial \Psi^{(\eta)}}{\partial \omega^{(\eta)}} = \frac{1}{2} N_{klmn}^{(\eta)} L_{ijkl}^{(\eta)} \varepsilon_{mn}^{(\eta)} \varepsilon_{ij}^{(\eta)} \tag{68}$$

4.0 Nonlocal Piecewise Constant Damage Model for Two-Phase Materials

As a special case we consider a composite material consisting of two phases, matrix and reinforcement, denoted by $\Theta^{(m)}$ and $\Theta^{(f)}$ such that $\Theta = \Theta^{(m)} \cup \Theta^{(f)}$. Superscripts m and f represent matrix and reinforcement phases, respectively. For simplicity, we assume that damage occurs in the matrix phase only, i.e. $\omega^{(f)} \equiv 0$. The volume fractions for matrix and reinforcement are denoted as $v^{(m)}$ and $v^{(f)}$, respectively, such that $v^{(m)} + v^{(f)} = 1$. The overall elastic properties are given as in [17]

$$\bar{L}_{ijkl} = v^{(m)} L^{(m)}_{ijmn} A^{(m)}_{mnkl} + v^{(f)} L^{(f)}_{ijmn} A^{(f)}_{mnkl} \tag{69}$$

To further simplify the matters, we define the microscopic damage distribution function $\psi^{(\eta)}(\boldsymbol{y})$ (41) as a piecewise constant function

$$\psi^{(\eta)}(\boldsymbol{y}) = \begin{cases} 1 & \text{if } \boldsymbol{y} \in \Theta^{(\eta)} \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

The corresponding weight function becomes piecewise constant function with $\mu^{(\eta)} = |\Theta^{(\eta)}| / |\Theta_C|$. A piecewise constant approximation of damage distribution has been also considered in [32].

Since damage in the reinforcement phase is neglected, the average strains in the matrix and reinforcement can be written as:

$$\varepsilon^{(\eta)}_{ij} = A^{(\eta)}_{ijmn} \bar{\varepsilon}_{mn} + G^{(\eta)}_{ijkl} D^{(m)}_{klmn} \bar{\varepsilon}_{mn} + O(\varsigma) , \qquad \eta = m, f \tag{71}$$

where

$$D^{(m)}_{klmn} = \left\{ I_{klpq} - B^{(m)}_{klpq} \omega^{(m)} \right\}^{-1} C^{(m)}_{pqmn} \omega^{(m)} \tag{72}$$

$$B^{(\eta)}_{ijkl} = \frac{1}{|\Theta^{(\eta)}|} (\tilde{L}_{ijmn} - \bar{L}_{ijmn})^{-1} \int_{\Theta^{(\eta)}} G_{stmn} L_{stpq} G_{pqkl} \, d\Theta \tag{73}$$

$$C^{(\eta)}_{ijkl} = \frac{1}{|\Theta^{(\eta)}|} (\tilde{L}_{ijmn} - \bar{L}_{ijmn})^{-1} \int_{\Theta^{(\eta)}} G_{stmn} L_{stpq} (I_{pqkl} + G_{pqkl}) \, d\Theta \tag{74}$$

The corresponding nonlocal phase stresses (63) are given as

$$\sigma^{(\eta)}_{ij} = (1 - \omega^{(\eta)}) L^{(\eta)}_{ijkl} \varepsilon^{(\eta)}_{kl} + O(\varsigma) , \qquad \eta = m, f \tag{75}$$

and the overall stresses defined in (50) reduce to

$$\bar{\sigma}_{ij} = v^{(m)}\sigma_{ij}^{(m)} + v^{(f)}\sigma_{ij}^{(f)} \tag{76}$$

The nonlocal energy release rate and the energy dissipation inequality in (67) and (68) become

$$Y^{(m)} = \frac{1}{2}L_{ijkl}^{(m)}\varepsilon_{ij}^{(m)}\varepsilon_{kl}^{(m)} \tag{77}$$

$$Y^{(m)}\dot{\omega}^{(m)} \geq 0 \tag{78}$$

The nonlocal isotropic damage state variable $\omega^{(m)}$ is assumed to be a monotonically increasing function of nonlocal phase deformation history parameter $\kappa^{(m)}$ [15][26][28][29][42], which characterizes the ultimate deformation experienced throughout the loading history. In general, the evolution of matrix damage at time t can be expressed as

$$\omega^{(m)}(\boldsymbol{x}, t) = f(\kappa^{(m)}(\boldsymbol{x}, t)) \tag{79}$$

The nonlocal phase deformation history parameter $\kappa^{(m)}$ is determined by the evolution of nonlocal phase damage equivalent strain, denoted by $\bar{\vartheta}^{(m)}$, as follows

$$\kappa^{(m)}(\boldsymbol{x}, t) = max\{\bar{\vartheta}^{(m)}(\boldsymbol{x}, \tau)|(\tau \leq t), \kappa_i^{(m)}\} \tag{80}$$

where the threshold value for damage initiation in the matrix, $\kappa_i^{(m)}$, represents the extreme value of the equivalent strain prior to the initiation of damage. Equation (80) can be also expressed by the Kuhn-Tucker relations

$$\dot{\kappa}^{(m)} \geq 0, \qquad \bar{\vartheta}^{(m)} - \kappa^{(m)} \leq 0, \qquad \dot{\kappa}^{(m)}(\bar{\vartheta}^{(m)} - \kappa^{(m)}) = 0 \tag{81}$$

In the present manuscript the nonlocal phase damage equivalent strain, $\bar{\vartheta}^{(m)}$, is defined as square root of the nonlocal phase damage energy release rate [42]

$$\bar{\vartheta}^{(m)} \equiv \sqrt{Y^{(m)}} = \sqrt{\frac{1}{2}L_{ijkl}^{(m)}\varepsilon_{ij}^{(m)}\varepsilon_{kl}^{(m)}} \tag{82}$$

Since $L_{ijkl}^{(m)}$ is a positive definite fourth order tensor, it follows that $Y^{(m)} \geq 0$. Consequently, the nonlocal phase energy dissipation inequality (78), together with the definition of damage evolution (79), yield

$$\dot{\omega}^{(m)} = \dot{\kappa}^{(m)}\frac{\partial f(\kappa^{(m)}(\boldsymbol{x}, t))}{\partial \kappa^{(m)}} \geq 0 \tag{83}$$

Combining this inequality with Kuhn-Tucker relations, we arrive at the following two conclusions: 1) the damage evolution law $f(\kappa^{(m)}(\boldsymbol{x}, t))$ is an increasing function of $\kappa^{(m)} \in [\kappa_i^{(m)}, \kappa_u^{(m)}]$ since $\dfrac{\partial f(\kappa^{(m)}(\boldsymbol{x}, t))}{\partial \kappa^{(m)}} > 0$, where $\kappa_u^{(m)}$ is the ultimate equivalent strains at rupture; and 2) the damage evolution condition can be expressed as

$$if\ \ \bar{\vartheta}^{(m)} - \kappa^{(m)} = 0,\ \dot{\kappa}^{(m)} > 0 \ \Rightarrow \ \text{damage process:}\ \dot{\omega}^{(m)} > 0 \tag{84}$$

$$if\ \bar{\vartheta}^{(m)} - \kappa^{(m)} < 0 \ \ \text{or} \ \ if\ \bar{\vartheta}^{(m)} - \kappa^{(m)} = 0,\ \dot{\kappa}^{(m)} = 0 \ \Rightarrow \ \text{elastic process:}\ \dot{\omega}^{(m)} = 0 \tag{85}$$

In accordance with the above thermodynamic considerations, it is possible to construct an appropriate damage evolution law. An extensive review of a variety of damage evolution law has been reported in [26]. In the present manuscript we propose an arctangent form of evolution law to ensure regularity of the tangent stiffness matrices in almost completely damaged state

$$\Phi^{(m)}(\alpha, \beta, \omega^{(m)}, \kappa^{(m)}, \kappa_0^{(m)}) = \omega^{(m)} - \frac{\operatorname{atan}\left(\alpha \dfrac{\kappa^{(m)}}{\kappa_0^{(m)}} - \beta\right) + \operatorname{atan}(\beta)}{\dfrac{\pi}{2} + \operatorname{atan}(\beta)} = 0 \tag{86}$$

where α, β are material parameters; and $\kappa_0^{(m)}$ denotes the threshold of the strain history parameter beyond which the damage will develop very quickly. For simplicity, we set $\kappa_i^{(m)} = 0$. From (86), it can be seen that $\omega^{(m)} \in [0, 1)$ ensures (29) to be the necessary and sufficient conditions for (28). Furthermore, this evolution law accounts for initial microcracks which are often present in ceramic composites.

5.0 Computational issues

In this section, we describe computational aspects of the nonlocal piecewise constant damage model for two-phase materials developed in Section 4.0. Due to the nonlinear character of the problem an incremental analysis is employed. Prior to nonlinear analysis elastic strain concentration factors, $A_{ijkl}(\boldsymbol{y})$, are computed using (35), (37) by either finite element method or if possible by analytically solving an inclusion problem. Subsequently, nonlocal phase elastic strain concentration factors $A_{ijkl}^{(\eta)}$ ($\eta = m, f$) and damage strain concentration factors $G_{ijkl}^{(\eta)}$ are precomputed using (60) and (61), respectively.

The stress update (integration) problem can be stated as follows:

Given: displacement vector ${}_tu_m$; overall strain ${}_t\bar{\varepsilon}_{mn}$; strain history parameter ${}_t\kappa^{(m)}$; damage parameter ${}_t\omega^{(m)}$; and displacement increment Δu_m calculated from the finite element analysis of the macro problem. Here left subscript denotes the increment step, i.e., ${}_{t+\Delta t}\square$ is the variable in the current increment, whereas ${}_{t}\square$ is a converged variable from the previous increment. For simplicity, we will omit the left subscript for the current increment, i.e., $\square \equiv {}_{t+\Delta t}\square$.

<u>Find:</u> displacement vector $u_m \equiv {}_{t+\Delta t}u_m$; overall strain $\bar{\varepsilon}_{mn}$; nonlocal phase strains $\varepsilon_{mn}^{(m)}$ and $\varepsilon_{mn}^{(f)}$; nonlocal strain history parameter $\kappa^{(m)}$; nonlocal phase damage parameter $\omega^{(m)}$; overall stress $\bar{\sigma}_{mn}$ and nonlocal phase stresses $\sigma_{mn}^{(m)}$ and $\sigma_{mn}^{(f)}$.

The stress update procedure consists of the following steps:

i.) Calculate macroscopic strain increment, $\Delta\bar{\varepsilon}_{mn} = \Delta u_{(m,x_n)}$, and then update macroscopic strains through $\bar{\varepsilon}_{mn} = {}_{t}\bar{\varepsilon}_{mn} + \Delta\bar{\varepsilon}_{mn}$.

ii.) Compute the damage equivalent strain $\bar{\vartheta}^{(m)}$ defined by (82) in terms of ${}_{t}\omega^{(m)}$ and $\bar{\varepsilon}_{mn}$.

iii.) Check the damage evolution conditions (84) and (85). Note that $\kappa^{(m)}$ is defined by (80) and $\dot{\kappa}^{(m)}$ is integrated as $\Delta\kappa^{(m)} = \kappa^{(m)} - {}_{t}\kappa^{(m)}$.

If damage process, i.e. $\bar{\vartheta}^{(m)} > {}_{t}\kappa^{(m)}$ *, then* $\kappa^{(m)} = \bar{\vartheta}^{(m)}$ and update for $\omega^{(m)}$.

Since $\bar{\vartheta}^{(m)}$ is governed by the current average strains in the matrix phase, which in turn depend on the current damage parameter, it follows that the damage evolution law (86) is a nonlinear function of $\omega^{(m)}$. Using Newton's method, we construct an iterative process for the damage parameter:

$$ {}^{k+1}\omega^{(m)} = {}^{k}\omega^{(m)} - \left(\frac{\partial\Phi^{(m)}}{\partial\omega^{(m)}}\right)^{-1}\Phi^{(m)}\bigg|_{{}^{k}\omega^{(m)}} \quad (87) $$

The derivative in (87) can be evaluated by (71), (82), (86) as

$$ \frac{\partial\Phi^{(m)}}{\partial\omega^{(m)}} = 1 - \frac{\alpha\kappa_0^{(m)} \cdot \dfrac{\partial\bar{\vartheta}^{(m)}}{\partial\omega^{(m)}}}{(\pi/2 + \operatorname{atan}(\beta)) \cdot \left\{(\kappa_0^{(m)})^2 + (\alpha\bar{\vartheta}^{(m)} - \beta\kappa_0^{(m)})^2\right\}} \quad (88) $$

where

$$\frac{\partial \bar{\vartheta}^{(m)}}{\partial \omega^{(m)}} = \frac{1}{2\bar{\vartheta}^{(m)}} \cdot \varepsilon_{ij}^{(m)} L_{ijkl}^{(m)} G_{klst}^{(m)} R_{stmn}^{(m)} \bar{\varepsilon}_{mn} \tag{89}$$

with

$$R_{stmn}^{(m)} = (I_{stpq} - B_{stpq}^{(m)} \omega^{(m)})^{-2} C_{pqmn}^{(m)} \tag{90}$$

Otherwise for elastic process: $\omega^{(m)} = {}_t\omega^{(m)}$.

vi.) Update the nonlocal strains $\varepsilon_{kl}^{(m)}$ and $\varepsilon_{kl}^{(f)}$ using (71) and update the nonlocal strain history parameter $\kappa^{(m)}$ in (80).

v.) Update macroscopic stresses $\bar{\sigma}_{ij}$ defined by (76) and calculate nonlocal phase stresses $\sigma_{ij}^{(m)}$ and $\sigma_{ij}^{(f)}$ using (75).

To this end we focus on the computation of a consistent tangent stiffness matrix needed for the Newton method on the macro level. We start by substituting (71) into (75) and then taking the material derivative of the incremental form of (75) in the matrix domain, i.e. $\eta = m$

$$\dot{\sigma}_{ij}^{(m)} = P_{ijmn}^{(m)} \dot{\bar{\varepsilon}}_{mn} + Q_{ijmn}^{(m)} \bar{\varepsilon}_{mn} \dot{\omega}^{(m)} \tag{91}$$

where

$$P_{ijmn}^{(m)} = (1 - \omega^{(m)}) L_{ijkl}^{(m)} (A_{klmn}^{(m)} + G_{klst}^{(m)} D_{stmn}^{(m)}) \tag{92}$$

$$Q_{ijmn}^{(m)} = (1 - \omega^{(m)}) L_{ijkl}^{(m)} G_{klst}^{(m)} R_{stmn}^{(m)} - L_{ijkl}^{(m)} (A_{klmn}^{(m)} + G_{klst}^{(m)} D_{stmn}^{(m)}) \tag{93}$$

In order to obtain $\dot{\omega}^{(m)}$, we take the material derivative of damage evolution law (86), $\dot{\Phi}^{(m)} = 0$, and make use of (75), (82), and (88), which yields

$$\dot{\omega}^{(m)} = -\aleph_{kl}^{(m)} \dot{\sigma}_{kl}^{(m)} \tag{94}$$

where

$$\aleph_{kl}^{(m)} = \gamma \varepsilon_{kl}^{(m)} \tag{95}$$

and γ is a scalar given as

$$\gamma = \left\{-(1-\omega^{(m)})(\pi/2 + \operatorname{atan}(\beta))\left\{(\kappa_0^{(m)})^2 + (\alpha\bar{\vartheta}^{(m)} - \beta\kappa_0^{(m)})^2\right\} + \alpha\bar{\vartheta}^{(m)}\kappa_0^{(m)}\right\}^{-1} \cdot \frac{\alpha\kappa_0^{(m)}}{2\bar{\vartheta}^{(m)}} \tag{96}$$

Substituting (94) into (91) and manipulating the indices, we get the following relation between the rate of overall strain and nonlocal phase stresses in the matrix domain

$$\dot{\sigma}_{ij}^{(m)} = \wp_{ijmn}^{(m)}\dot{\bar{\varepsilon}}_{mn} \tag{97}$$

where

$$\wp_{ijmn}^{(m)} = (\delta_{ik}\delta_{jl} + \aleph_{ij}^{(m)}Q_{klst}^{(m)}\bar{\varepsilon}_{st})^{-1}P_{klmn}^{(m)} \tag{98}$$

By using Sherman-Morrison formula (98) reduces to

$$\wp_{ijmn}^{(m)} = \left(\delta_{ik}\delta_{jl} - \frac{\aleph_{ij}^{(m)}Q_{klst}^{(m)}\bar{\varepsilon}_{st}}{1 + \aleph_{ij}^{(m)}Q_{ijst}^{(m)}\bar{\varepsilon}_{st}}\right)P_{klmn}^{(m)} \tag{99}$$

A similar result relating the rate of the nonlocal reinforcement stress and the overall strain rate, can be obtained by substituting (71) into (75) and then taking the material derivative of (75) in the reinforcement domain:

$$\dot{\sigma}_{ij}^{(f)} = \wp_{ijmn}^{(f)}\dot{\bar{\varepsilon}}_{mn} \tag{100}$$

where

$$\wp_{ijmn}^{(f)} = P_{ijmn}^{(f)} - Q_{ijst}^{(f)}\bar{\varepsilon}_{st}\aleph_{kl}^{(m)}\wp_{klmn}^{(m)} \tag{101}$$

and

$$P_{ijmn}^{(f)} = L_{ijkl}^{(f)}(A_{klmn}^{(f)} + G_{klst}^{(f)}D_{stmn}^{(m)}) \tag{102}$$

$$Q_{ijst}^{(f)} = L_{ijkl}^{(f)}G_{klmn}^{(f)}R_{mnst}^{(m)} \tag{103}$$

Finally, the overall consistent tangent stiffness is constructed by substituting (97) and (100) into the rate form of the overall stress-strain relation (76)

$$\dot{\bar{\sigma}}_{ij} = \wp_{ijmn}\dot{\bar{\varepsilon}}_{mn} \tag{104}$$

$$\wp_{ijmn} = v^{(m)}\wp_{ijmn}^{(m)} + v^{(f)}\wp_{ijmn}^{(f)} \tag{105}$$

6.0 Numerical Examples

6.1 Qualitative Examples for Two-phase Fibrous Composites Under Uniaxial Loading

The first numerical example is aimed at qualitative study of the behavior of the proposed non-local piecewise constant damage model for two-phase materials. We consider a macro domain in the shape of a block discretized with a single brick element and a periodic fibrous micro-structure as shown in Figure 3. The block is subjected to the state of constant macro-strain field in the axial (parallel to the fibers) and transverse (normal to the fibers) directions. The axial direction is aligned along the Z axis whereas the two transverse directions coincide with the X and Y axes. The phase properties of microconstituents are as follows:

Matrix: Volume fraction = 0.733; Young's modulus = 69GPa; Poisson's ratio = 0.33.

Fiber: Volume fraction = 0.267; Young's modulus = 379GPa; Poisson's ratio = 0.21.

The parameters of the damage evolution law are chosen as $\alpha = 8.2$, $\beta = 10.2$ and $\kappa_0^{(m)} = 0.05$. The corresponding damage evolution law is depicted in Figure 4.

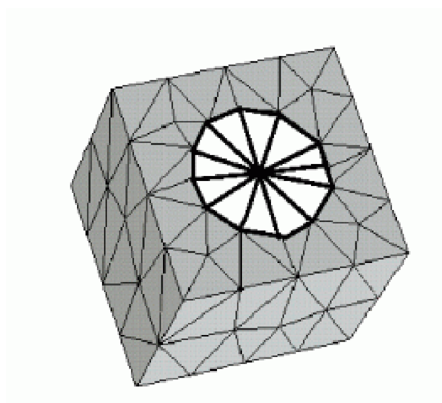

Figure 3: Finite Element Mesh of the RVE for Fibrous Microstructure

The uniaxial stress-strain curves for the axial and transverse tension problems are illustrated in Figures 4 and 6, respectively. Figure 5 shows a rapid loss of stiffness as the damage in the matrix phase accumulates and in the limit as the matrix material is completely damaged the axial loading capacity of composite is provided by the fiber only. Our numerical model is in good agreement with the limit solution which gives $\lim_{\omega^{(m)} \to 1.0} \sigma_{33} = v^{(f)} E^{(f)} \varepsilon_{33}$. Results of the transverse tension problem are shown in Figure 6. It can be seen that when the matrix is totally damaged, it fails to transfer the load into the fiber and consequently, the entire load carrying capacity of the fibrous composite is lost in the transverse direction, i.e. $\lim_{\omega^{(m)} \to 1.0} \sigma_{11} = 0$. In both figures, we also demonstrate the evolution of the damage parameter in the matrix phase. Referring back to the damage evolution curve shown in Figure 4, it can be seen that the sudden drop in load carrying capacity in both axial and transverse directions occurs when the damage parameter reaches $\omega^{(m)} \approx 0.1$ beyond which the damage parameter grows sharply.

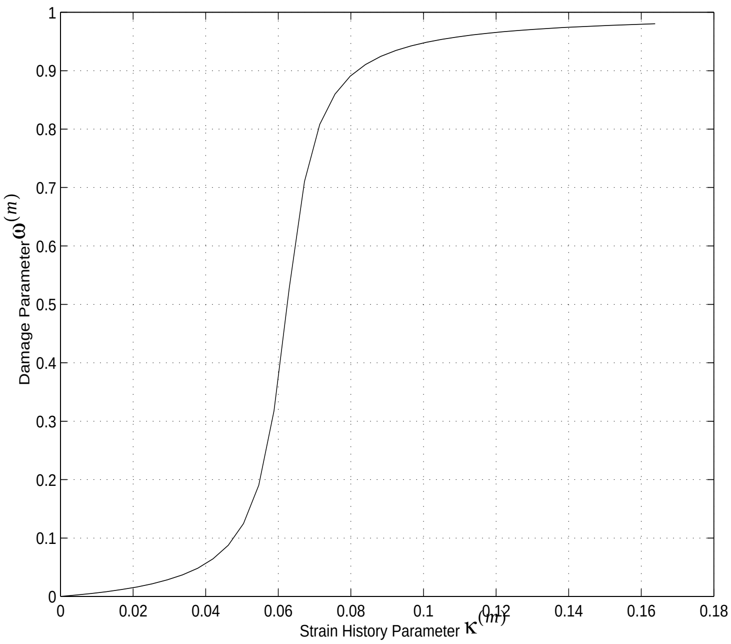

Figure 4: Damage Evolution Law for the Titanium Matrix

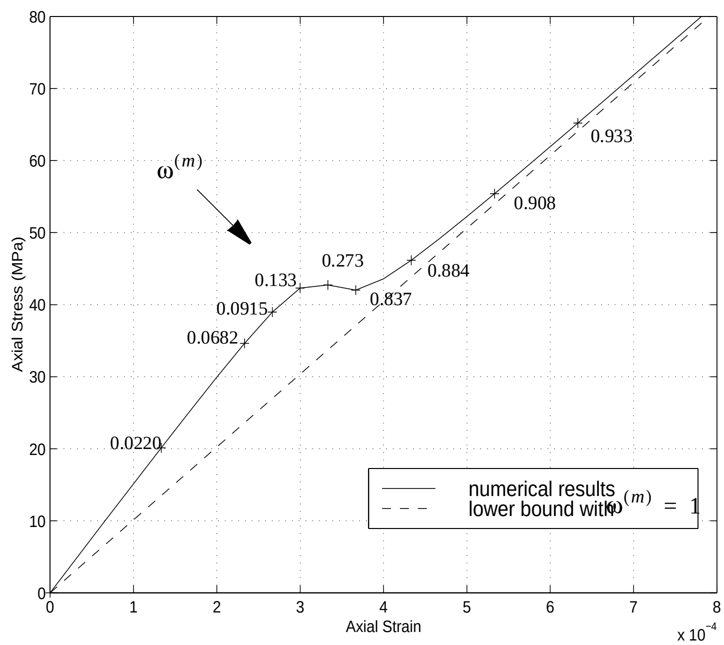

Figure 5: Loading Capacity in the Axis (Z axis) Direction

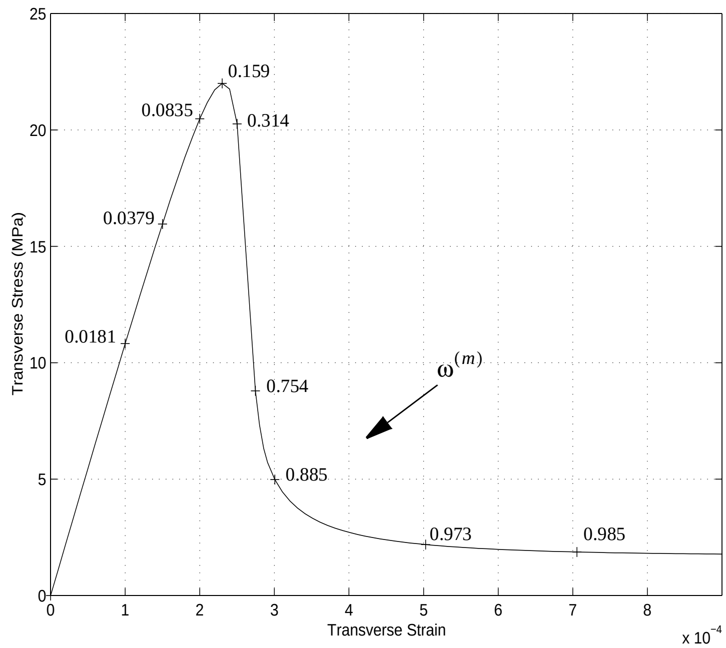

Figure 6: Loading Capacity in the Transverse (X axis) Direction

6.2 4-Point Bending Problem for Woven Composite

We next consider a 4-point bending problem carried out on a composite beam made of Blackglas^{TM}/Nextel 5-harness satin weave as shown in Figure 7. The fabric designs used 600 denier bundles of Nextel^{TM} 312 fibers, spaced at 46 threads per inch, and surrounded by Blackglas^{TM} matrix material. The bundle is assumed to be linear elastic throughout the analysis. The average transversely isotropic elastic properties were computed by the Mori-Tanaka method. We will refer to this material system as AF10. The micrograph in Figure 7 was produced at Northrop-Grumman [14]. In this set of numerical examples, the nonlocal piecewise constant damage model is employed and we assume that $l_{RVE} = l_H = l_C$. The phase properties of RVE are summarized below:

Blackglas^{TM} Matrix: volume fraction = 0.548; Young's modulus = 9.653GPa;
Poisson's ratio = 0.244.

Nextel^{TM} 312 Fiber: volume fraction = 0.452; Young's modulus = 151.7GPa;
Poisson's ratio = 0.26.

The microstructure of RVE is discretized with 6857 elements totaling 10608 degrees of freedom as shown in Figure 8. The issues of the automatic extraction, construction and linking of the geometry and attributes, automatic construction of matched meshes have been described in [47]. The configuration of the composite beam is shown in Figure 9 where the loading direction (normal to the plane of the weave) is aligned along the Y axis. The finite element model of the beam (macrostructure) is composed of 1856 brick elements totaling 7227 degrees of freedom. Figure 10 depicts the damage evolution law for Blackglas™ matrix with $\alpha = 7.1$, $\beta = 10.1$ and $\kappa_0^{(m)} = 0.22$, which are calibrated to the tensile and shear test data.

Comparison between tensile test data and the numerical simulation for the uniaxial tension is shown in Figure 11. It can be seen that the ultimate experimental stress/strain values in the uniaxial tension test are $\sigma_u = 150 \pm 7\text{MPa}$ and $\varepsilon_u = 2.5\times10^{-3} \pm 0.3\times10^{-3}$, while the numerical simulation gives $\sigma_u = 152\text{MPa}$ at $\varepsilon_u = 3.2\times10^{-3}$.

Numerical simulation results as well as the test data for 4-point bending problem are shown in Figures 12 and 13. Experiments have been conducted on five identical beams and the scattered experimental data of force versus the displacement at the point of load application in the beam are shown by the gray area in Figure 12. It can be seen that the numerical simulation results are in good agreement with the experimental data in terms of predicting the overall behavior (Figure 12) and the dominant failure mode. Both numerical simulation and experimental data predict that the dominant failure mode is tension/compression (so-called bending induced failure). Figure 13 illustrates the distribution of the damage parameter in the composite beam at the peak load (Point A in Figure 12).

To this end we note that since bundles have been modeled as linear elastic spurious increase in load carrying capacity of the weave in the in-plane tension/compression eventually takes place. Remedies are discussed I Section 7

Figure 7: Blackglas™/Nextel 5-harness Satin Weave

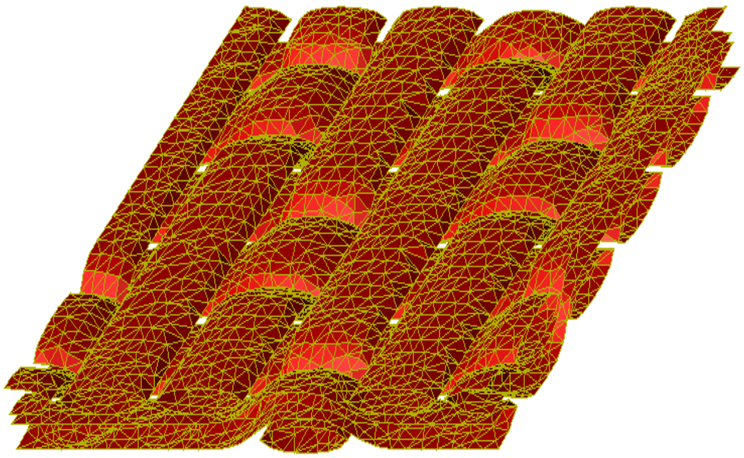

Figure 8: Microstructure of AF10 Woven Composites

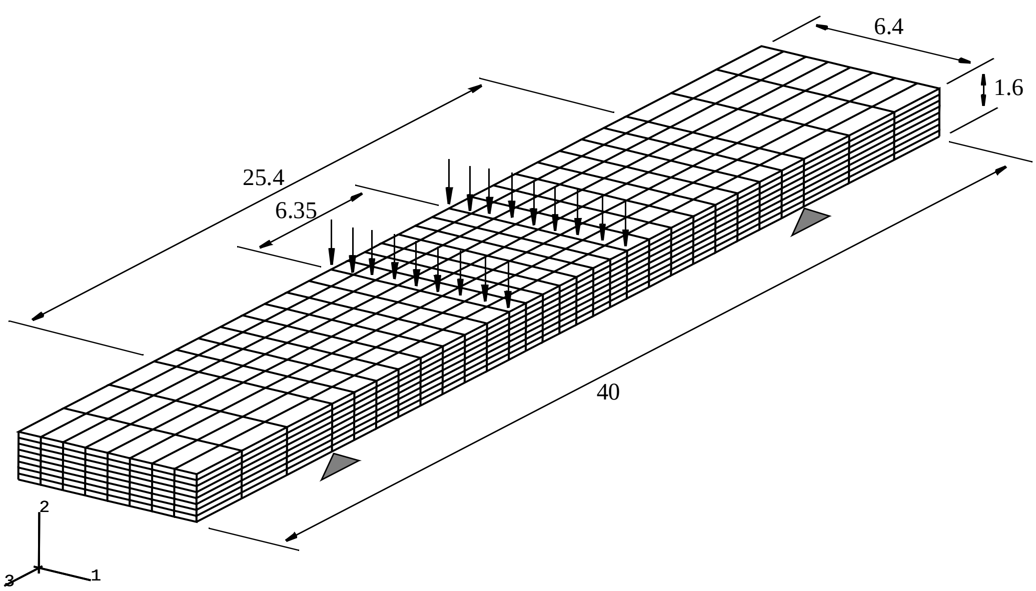

Figure 9: Configuration and FE Mesh of 4-Point Bending Problem

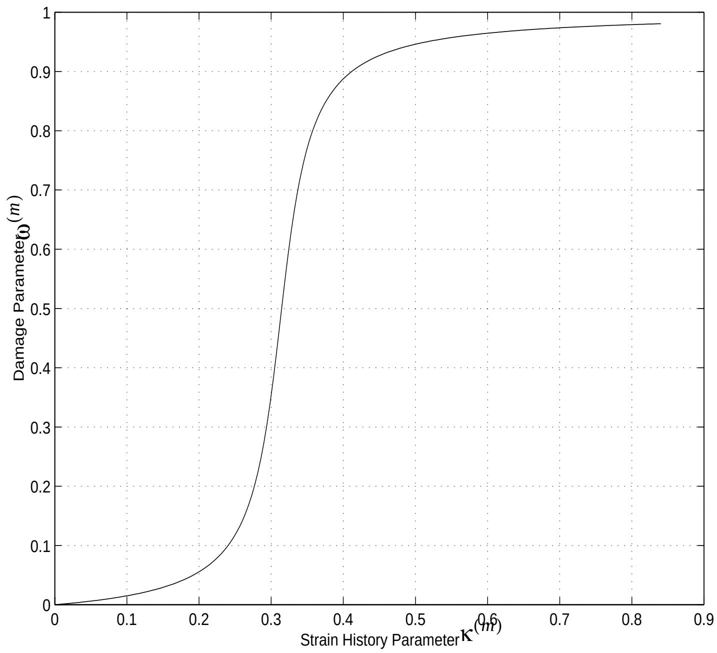

Figure 10: Damage Evolution Law for the Blackglas™ Matrix

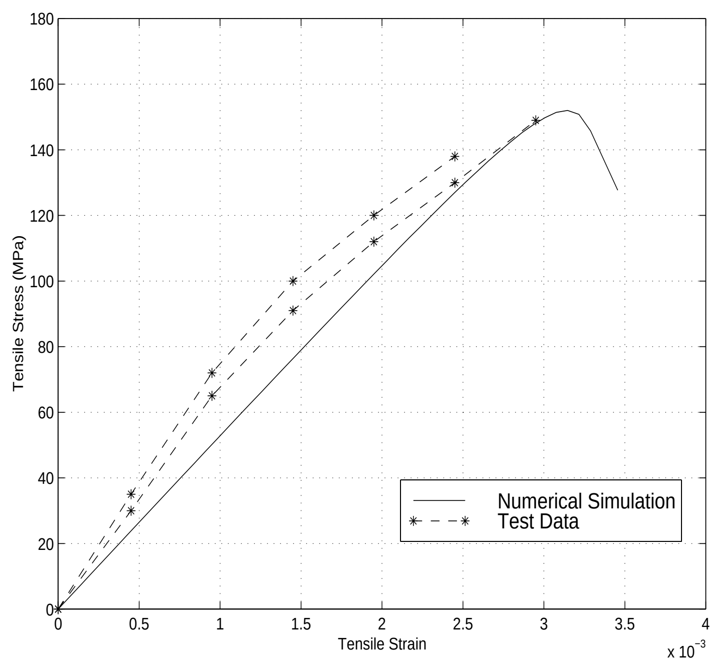

Figure 11: Strain-Stress Curves for the Uniaxial Tension Problem

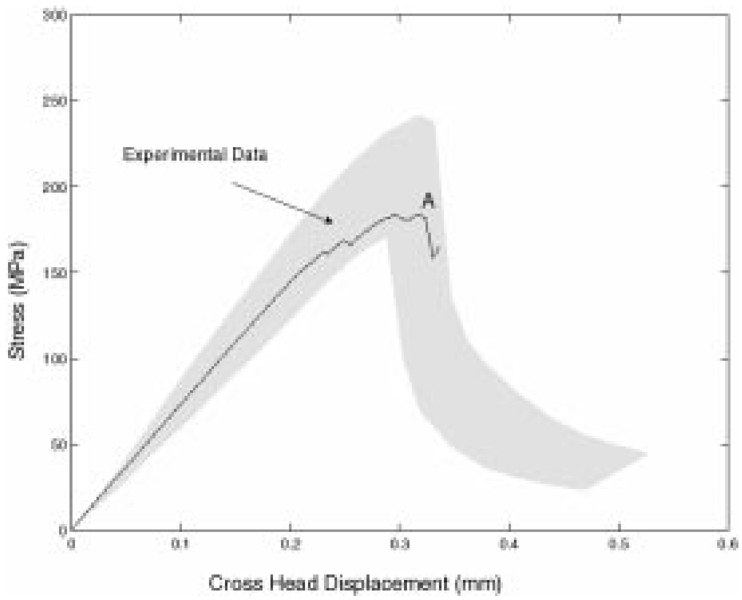

Figure 12: Force versus Displacement for the 4-Point Bending Problem

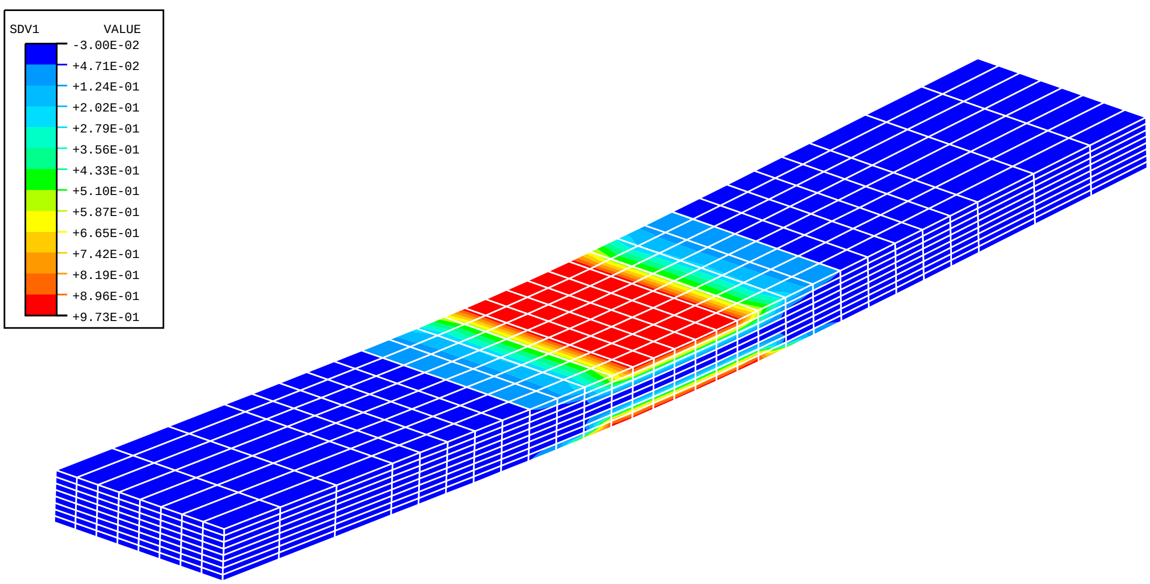

Figure 13: Damage Distribution at the Peak Load in the 4-Point Bending Problem

7.0 Summary and future research directions

A nonlocal damage theory for brittle composite materials based on double scale asymptotic expansion of damage has been developed. A closed form expression relating local fields to the overall strains and damage has been derived. The concept of nonlocal phase fields (stress, strain, free energy density, damage release rate, etc.) has been introduced via weighting functions defined over the microphase. Numerical results revealed an excellent performance of the method.

The present work by no means represents a complete account of all theoretical and numerical issues related to damage in composites and we apologize if some important works have been omitted. We note that the assumptions of periodicity and uniformity of macroscopic fields, which are embedded in our formulation, may yield inaccurate solutions in the vicinity of boundary layers. The remedies to this phenomenon range from changing the RVE size [2] to carrying out an iterative global-local analysis [37], [38], [33], [24], [25]. Moreover, various failure modes other than matrix cracking, such as damage at the interface and in the bundle domain, coupled plasticity-damage effects, different responses in tension and compression have not been accounted for in the present manuscript. These are just few of the issues that will be investigated in our future work.

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