A Dispersive Model For Wave Propagation In Periodic Heterogeneous Media Based On Homogenization With Multiple Spatial And Temporal Scales

Wen Chen and Jacob Fish
Department of Civil Engineering and Scientific Computation Research Center,
Rensselaer Polytechnic Institute, Troy, NY 12180, USA

Abstract: A dispersive model is developed for wave propagation in periodic heterogeneous media. The model is based on the higher order mathematical homogenization theory with multiple spatial and temporal scales. A fast spatial scale and a slow temporal scale are introduced to account for the rapid spatial fluctuations as well as to capture the long-term behavior of the homogenized solution. By this approach the problem of secularity, which arises in the conventional multiple-scale higher order homogenization of wave equations with oscillatory coefficients, is successfully resolved. A model initial/boundary value problem is analytically solved and the results have been found to be in good agreement with a numerical solution of the source problem in a heterogeneous medium.

1. Introduction

When a wavelength of a traveling signal in a heterogeneous medium is comparable to the characteristic length of the microstructure, successive reflection and refraction of the waves between the interfaces of the material lead to significant dispersion effect (see for example [1][2][3]). This phenomenon cannot be predicted by the classical homogenization theory and thus prompting a significant interest in the scientific community in attempt to develop a dispersive effective medium theory.

The use of multiple-scale expansions as a systematic tool of averaging for problems other than elastodynamics can be traced to Sanchez-Palencia [5], Benssousan, Lions and Papanicoulau [6], as well as Bakhvalov and Panasenko [7]. The role of higher order terms in the asymptotic expansion has been investigated in statics by Gambin and Kroner [8], and Boutin [9]. In elastodynamics, Boutin and Auriault [10] demonstrated that the terms of a higher order successively introduce effects of polarization, dispersion and attenuation.

There is a substantial number of articles utilizing multiple-scale homogenization techniques for wave propagation problems in periodic media. Most often, a single-frequency time dependence is assumed prior to the homogenization [11]. A notable exception is a recent article of Fish and Chen [12], which investigated the initial/boundary value problem with rapidly varying coefficients by employing the multiple-scale homogenization technique. They showed that while higher order terms are capable of capturing dispersion effects, they introduce secular terms which grow unbounded with time. When the observation time is small, higher order terms introduce the necessary correction to the leading order term capable of resolving the dispersion effect. However, as the time window increases, the higher order terms become close to or larger than the leading order term owing to the existence of secularity. In this case, the asymptotic expansion breaks down as it ceases to be valid. To our knowledge, the present manuscript represents a first attempt to resolve the problem of secularity within the framework of the multiple-scale analysis for wave propagation in composites.

For dynamic problems, described by hyperbolic differential equations, there are at least four scales involved: (1) the scale of the microstructure, (2) the scale of the macrostructure, (3) the shortest wavelength of the signal traveling in the media, and (4) the time scale of observation. The dispersion phenomena become prominent when the time window is large. Therefore, in order to properly model the dispersion effect, it is desirable to construct uniformly valid asymptotic expansions.

The primary objective of the current manuscript is to study the problem of secularity introduced by the higher order multiple-scale approximation of the initial/boundary value problem in periodic heterogeneous media. We first consider fast spatial-temporal scales in addition to the usual space-time coordinates. The resulting unit cell problem is shown to be hyperbolic giving rise to fast time dependence of the solution in the unit cell domain, while the resulting macroscopic equation is the same as in the classical multiple spatial scale analysis and thus failing to resolve dispersion effects. The main contribution of the present paper is given in Section 3.2, where we introduce both fast spatial scale aimed to account for rapid spatial fluctuations of material properties and a slow temporal scale designated to capture the long-term behavior of the homogenized solution. The resulting macroscopic equations of motion are solved analytically in Section 4 for an illustrative initial/boundary value problem.

2. Problem Description

We consider wave propagation normal to the layers of a periodic elastic bilaminate with Ω as the characteristic length (see Figure 1). The governing elastodynamics problem is stated as

$$\rho(x/\varepsilon)u_{;tt} - [E(x/\varepsilon)u_{;x}]_{;x} = 0 \tag{1}$$

with appropriate boundary conditions on the domain boundary and initial conditions

$$u(x, 0) = f(x) , \quad u_{;t}(x, 0) = g(x) \tag{2}$$

where $u(x, t)$ represents the displacement field; $\rho(x/\varepsilon)$ and $E(x/\varepsilon)$ are the mass density and elastic modulus, respectively; $(\ )_{;x}$ and $(\ )_{;t}$ denote differentiation with respect to x and time, respectively; and $0 < \varepsilon \ll 1$ in (1) is used to express a rapid spatial variation of material properties.

The goal is to establish an effective homogeneous model in which the local fluctuations due to the heterogeneities do not appear explicitly and the response of the original heterogeneous material can be approximated by the response of the effective homogeneous medium. This is facilitated by the method of multiple-scale asymptotic expansion.

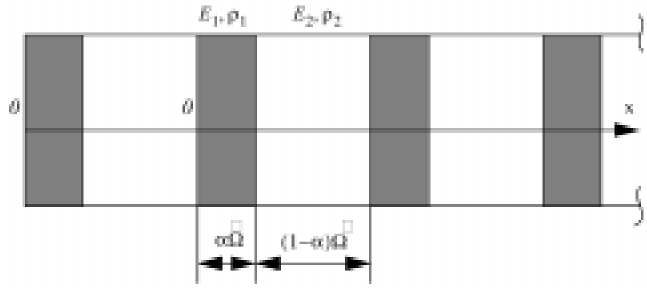

Figure 1: A bilaminate with periodic microstructure

3. Asymptotic Analysis with Multiple Spatial and Temporal Scales

Under the premise that the composite macro reference length $L = \lambda/(2\pi)$ (λ the macroscopic wavelength) [10][17] is much larger than the unit cell dimension Ω, i.e. $\Omega/L = (\omega\Omega)/c = k\Omega \ll 1$, where ω, k and c are the circular frequency, wave number and phase velocity of the macroscopic wave, respectively, it is convenient to introduce a microscopic spatial length variable y such that

$$y = x/\varepsilon \tag{3}$$

In addition to the fast spatial variable, we will consider various time scales

$$\xi = \varepsilon^m t \tag{4}$$

where m is an integer. Since the response quantities u and σ depend on x, $y = x/\varepsilon$, t, and $\xi = \varepsilon^m t$, a two-scale asymptotic expansion is employed

$$u(x, y, t, \xi) = \sum_{i=0}^{\infty} \varepsilon^i u_i(x, y, t, \xi), \quad \sigma(x, y, t, \xi) = \sum_{i=-1}^{\infty} \varepsilon^i \sigma_i(x, y, t, \xi) \tag{5}$$

The homogenization process consists of inserting the asymptotic expansions (5) into the governing equation (1), identifying the terms with the equal power of ε, and then solving the resulting problems.

Following the aforementioned procedure and replacing the spatial derivative $(\ )_{;x}$ by $(\ )_{,x} + \varepsilon^{-1}(\ )_{,y}$ and the time derivative $(\ )_{;t}$ by $(\ )_{,t} + \varepsilon^{m}(\ )_{,\xi}$, we obtain a series of equations in ascending power of ε starting with ε^{-2}.

3.1 Fast Spatial-Temporal Scales

This case corresponds to $m = -1$. The two time scales are related by

$$\xi = t/\varepsilon = \eta \tag{6}$$

At $O(\varepsilon^{-2})$, we get

$$\rho(y)u_{0,\eta\eta} - [E(y)u_{0,y}]_{,y} = 0 \tag{7}$$

from where it can be easily shown that u_0 is independent of y and η and thus

$$u_0 = U_0(x, t) \tag{8}$$

For $O(\varepsilon^{-1})$ equation we get

$$\rho(y)u_{1,\eta\eta} - [E(y)(u_{0,x} + u_{1,y})]_{,y} = 0 \tag{9}$$

Owing to linearity of the above equation, the solution of u_1 can be sought in the form

$$u_1(x, y, t, \eta) = U_1(x, t) + M(y, \eta)u_{0,x} \tag{10}$$

Substituting (10) into (9) yields

$$\rho(y)M_{,\eta\eta} - [E(y)(1 + M_{,y})]_{,y} = 0 \tag{11}$$

Consider the unit cell in Figure 1. The cell domain consists of subdomains $A^{(1)}$ and $A^{(2)}$, occupied by materials tagged by superscripts *1* and *2*, respectively, such that

$$A^{(1)} = [y|\ 0 < y < \alpha\hat{\Omega}], \quad A^{(2)} = [y|\ \alpha\hat{\Omega} < y < \hat{\Omega}] \tag{12}$$

where $0 \le \alpha \le 1$ is the volume fraction of the unit cell; $\hat{\Omega}$ is the unit cell domain in the stretched coordinate system *y*, such that $\Omega/\hat{\Omega} = \varepsilon$. Since material properties are piece-wise constant over the unit cell, equation (11) can be written as

$$M_{j,\eta\eta} - c_j^2 M_{j,yy} = 0, \quad (j = 1, 2) \tag{13}$$

where

$$c_1 = \sqrt{E_1/\rho_1}, \quad c_2 = \sqrt{E_2/\rho_2} \tag{14}$$

The boundary conditions for the unit cell problem described by (13) are:

$$\begin{aligned} &\text{(a) Periodicity:} \quad u_1(y = 0) = u_1(y = \hat{\Omega}), \ \ \sigma_0(y = 0) = \sigma_0(y = \hat{\Omega}) \\ &\text{(b) Continuity:} \quad [u_1(y = \alpha\hat{\Omega})] = 0, \quad [\sigma_0(y = \alpha\hat{\Omega})] = 0 \end{aligned} \tag{15}$$

where $[\ \ ]$ is the jump operator and

$$\sigma_i = E(y)(u_{i,x} + u_{i+1,y}), \quad i = 0, 1, \ldots, n \tag{16}$$

Substituting (10) into (15) gives

$$M_1(0, \eta) = M_2(\Omega, \eta), \quad E_1\{1 + M_{1,y}(0, \eta)\} = E_2\left\{1 + M_{2,y}(\Omega, \eta)\right\} \tag{17}$$

$$M_1(\alpha\Omega, \eta) = M_2(\alpha\Omega, \eta), \quad E_1\left\{1 + M_{1,y}(\alpha\Omega, \eta)\right\} = E_2\left\{1 + M_{2,y}(\alpha\Omega, \eta)\right\} \tag{18}$$

For simplicity, initial conditions are taken as

$$M_j(y, 0) = M_{j,\eta}(y, 0) = 0, \quad (j = 1, 2) \tag{19}$$

We solve the unit cell problem defined by equations (13) and (17)-(19) using the method of Laplace transform. Taking the Laplace transform of (13) with respect to η and using the initial conditions (19) yields

$$s^2\overline{M}_j(y, s) - c_j^2\frac{\partial^2}{\partial y^2}\overline{M}_j(y, s) = 0, \quad (j = 1, 2) \tag{20}$$

where $M_j(y, s)$ is the Laplace transform of $M_j(y, \eta)$. The general solution of (20) is

$$\overline{M}_1(y, s) = A_1\cosh\frac{sy}{c_1} + A_2\sinh\frac{sy}{c_1}, \quad \overline{M}_2(y, s) = B_1\cosh\frac{sy}{c_2} + B_2\sinh\frac{sy}{c_2} \tag{21}$$

where A_1, A_2, B_1 and B_2 are constants to be determined by the boundary conditions. Taking the Laplace transform of the boundary conditions (17) and (18), and substituting (21) into the transformed boundary conditions yields

$$\begin{gathered}
A_1 = B_1\cosh\frac{s\Omega}{c_2} + B_2\sinh\frac{s\Omega}{c_2} \\
A_1\cosh\frac{s\alpha\Omega}{c_1} + A_2\sinh\frac{s\alpha\Omega}{c_1} = B_1\cosh\frac{s\alpha\Omega}{c_2} + B_2\sinh\frac{s\alpha\Omega}{c_2} \\
E_1\left(\frac{1}{s} + \frac{s}{c_1}A_2\right) = E_2\left(\frac{1}{s} + \frac{s}{c_2}B_1\sinh\frac{s\Omega}{c_2} + \frac{s}{c_2}B_2\cosh\frac{s\Omega}{c_2}\right) \\
E_1\left(\frac{1}{s} + \frac{s}{c_1}A_1\sinh\frac{s\alpha\Omega}{c_1} + \frac{s}{c_1}A_2\cosh\frac{s\alpha\Omega}{c_1}\right) = E_2\left(\frac{1}{s} + \frac{s}{c_2}B_1\sinh\frac{s\alpha\Omega}{c_2} + \frac{s}{c_2}B_2\cosh\frac{s\alpha\Omega}{c_2}\right)
\end{gathered} \tag{22}$$

Solving (22) for constants A_1, A_2, B_1 and B_2, and substituting the result into (21) yields

$$\overline{M}_1(y,s) = c_1\left(1-\frac{E_2}{E_1}\right)\frac{D_1(y,s)}{s^2F(s)}, \quad \overline{M}_2(y,s) = c_1\left(1-\frac{E_2}{E_1}\right)\frac{D_2(y,s)}{s^2F(s)} \tag{23}$$

where

$$D_1(y,s) = \sinh\Phi_1(\cosh\Phi_3-\cosh\Phi_2)+k(\cosh\Phi_1-1)(\sinh\Phi_3+\sinh\Phi_2) \tag{24}$$

$$D_2(y,s) = (\cosh\Psi_1-1)(\sinh\Psi_3-\sinh\Psi_2)+k\sinh\Psi_1(\cosh\Psi_3-\cosh\Psi_2) \tag{25}$$

$$\Phi_1 = \frac{s(1-\alpha)\Omega}{c_2}, \quad \Phi_2 = \frac{s(y-\alpha\Omega)}{c_1}, \quad \Phi_3 = \frac{sy}{c_1} \tag{26}$$

$$\Psi_1 = \frac{s\alpha\Omega}{c_1}, \quad \Psi_2 = \frac{s(y-\alpha\Omega)}{c_2}, \quad \Psi_3 = \frac{s(\Omega-y)}{c_2} \tag{27}$$

$$F(s) = 2k-\frac{(1+k)^2}{2}\cosh(s\theta)+\frac{(1-k)^2}{2}\cosh(s\beta) \tag{28}$$

$$\theta = \frac{\alpha\Omega}{c_1}+\frac{(1-\alpha)\Omega}{c_2}, \quad \beta = \frac{\alpha\Omega}{c_1}-\frac{(1-\alpha)\Omega}{c_2} \tag{29}$$

$$k = \frac{c_1E_2}{c_2E_1} = \frac{\sqrt{E_2\rho_2}}{\sqrt{E_1\rho_1}} \tag{30}$$

$M_j(y,\eta)$ can be obtained by taking the inverse Laplace transform of (23), i.e.

$$M_j(y,\eta) = L^{-1}[\overline{M}_j(y,s)] = \frac{c_1}{2\pi i}\left(1-\frac{E_2}{E_1}\right)\int_{(\gamma-i\infty)}^{(\gamma+\infty)}\frac{e^{s\eta}D_j(y,s)}{s^2F(s)}ds, \quad (j = 1, 2) \tag{31}$$

In order to evaluate the above integrals, we first find the singular points of the integrands. Using the hyperbolic identity

$$\cosh(2x) = 2(\sinh x)^2+1 \tag{32}$$

$F(s)$ can be written as

$$F(s) = (1-k)^2\left(\sinh\frac{s\beta}{2}\right)^2-(1+k)^2\left(\sinh\frac{s\theta}{2}\right)^2 = 4F_1(s)F_2(s) \tag{33}$$

where

$$F_1(s) = \sin\lambda\cos(\mu\lambda) + k\cos\lambda\sin(\mu\lambda),\ F_2(s) = k\sin\lambda\cos(\mu\lambda) + \cos\lambda\sin(\mu\lambda) \quad (34)$$

$$\lambda = \frac{is\alpha\Omega}{2c_1}, \quad \mu = \frac{(1-\alpha)c_1}{\alpha c_2} \quad (35)$$

Therefore, the singular points of the integrands in (31) are the high-order pole at $s = 0$ and the simple poles obtained from $F_1(s) = 0$ and $F_2(s) = 0$ (excluding $s = 0$). The roots λ_{1n} and λ_{2n}, $n = 1, 2, 3, \ldots$ are symmetrically located.

In order to evaluate the residues of the integrands at $s = 0$, we expand both numerators and denominators of the integrands into Laurent series [18][19] and apply the binomial theorem to the denominators assuming the value of s is very small. The residues are then the coefficients of $1/s$ in the Laurent expansions. The residues of the integrands at $s = 0$ are given as

$$\left\{\frac{e^{s\eta}D_1(y,s)}{s^2F(s)}\right\}_{s=0} = -\frac{(1-\alpha)E_1(y-\alpha\hat{\Omega}/2)}{c_1[(1-\alpha)E_1+\alpha E_2]} \quad (36)$$

$$\left\{\frac{e^{s\eta}D_2(y,s)}{s^2F(s)}\right\}_{s=0} = \frac{\alpha E_1[y-(1+\alpha)\hat{\Omega}/2]}{c_1[(1-\alpha)E_1+\alpha E_2]} \quad (37)$$

The residues at simple poles are evaluated as

$$\left\{\frac{e^{s\eta}D_j(y,s)}{\frac{d}{ds}[s^2F(s)]}\right\}_{s=-\frac{2ic_1\lambda_{1n}}{\alpha\hat{\Omega}}} = \left\{\frac{e^{s\eta}D_j(y,s)}{4s^2F_2(s)\frac{d}{ds}F_1(s)}\right\}_{s=-\frac{2ic_1\lambda_{1n}}{\alpha\hat{\Omega}}} \quad (j = 1, 2) \quad (38)$$

$$\left\{\frac{e^{s\eta}D_j(y,s)}{\frac{d}{ds}[s^2F(s)]}\right\}_{s=-\frac{2ic_1\lambda_{2n}}{\alpha\hat{\Omega}}} = \left\{\frac{e^{s\eta}D_j(y,s)}{4s^2F_1(s)\frac{d}{ds}F_2(s)}\right\}_{s=-\frac{2ic_1\lambda_{2n}}{\alpha\hat{\Omega}}} \quad (j = 1, 2) \quad (39)$$

In the above two equations, we have exploited the fact that λ_{1n} and λ_{2n} ($n = 1, 2, 3, \ldots$) are roots of $F_1(s) = 0$ and $F_2(s) = 0$, respectively.

Based on the theory of residues, the integrals in (31) can be evaluated as

$$M_j(y,\eta) = c_1\left(1-\frac{E_2}{E_1}\right)\sum Res\left\{\frac{e^{s\eta}D_j(y,s)}{s^2F(s)}\right\} \quad (j = 1, 2) \quad (40)$$

which can be further expressed as

$$M_1(y,\eta) = \frac{(1-\alpha)(E_2-E_1)(y-\alpha\Omega/2)}{(1-\alpha)E_1+\alpha E_2} +$$

$$\frac{\alpha\hat{\Omega}}{4}\left(1-\frac{E_2}{E_1}\right)\sum_{n=1}^{\infty}\left[\frac{W_1(\lambda_{1n})}{G_1(\lambda_{1n})}\cos\frac{2c_1\lambda_{1n}\eta}{\alpha\hat{\Omega}} + \frac{W_1(\lambda_{2n})}{G_2(\lambda_{2n})}\cos\frac{2c_1\lambda_{2n}\eta}{\alpha\hat{\Omega}}\right] \tag{41}$$

$$M_2(y,\eta) = \frac{\alpha(E_1-E_2)[y-(1+\alpha)\Omega/2]}{(1-\alpha)E_1+\alpha E_2} +$$

$$\frac{\alpha\hat{\Omega}}{4}\left(1-\frac{E_2}{E_1}\right)\sum_{n=1}^{\infty}\left[\frac{W_2(\lambda_{1n})}{G_1(\lambda_{1n})}\cos\frac{2c_1\lambda_{1n}\eta}{\alpha\hat{\Omega}} + \frac{W_2(\lambda_{2n})}{G_2(\lambda_{2n})}\cos\frac{2c_1\lambda_{2n}\eta}{\alpha\hat{\Omega}}\right] \tag{42}$$

where

$$W_1(\lambda) = \sin(2\mu\lambda)\left[\cos\frac{2\lambda y}{\alpha\hat{\Omega}} - \cos\frac{2\lambda(y-\alpha\Omega)}{\alpha\hat{\Omega}}\right] +$$

$$k[\cos(2\mu\lambda)-1]\left[\sin\frac{2\lambda y}{\alpha\hat{\Omega}} + \sin\frac{2\lambda(y-\alpha\Omega)}{\alpha\hat{\Omega}}\right] \tag{43}$$

$$W_2(\lambda) = [\cos(2\lambda)-1]\left[\sin\frac{2\mu\lambda(\Omega-y)}{(1-\alpha)\hat{\Omega}} - \sin\frac{2\mu\lambda(y-\alpha\Omega)}{(1-\alpha)\hat{\Omega}}\right] +$$

$$k\sin(2\lambda)\left[\cos\frac{2\mu\lambda(\Omega-y)}{(1-\alpha)\hat{\Omega}} - \cos\frac{2\mu\lambda(y-\alpha\Omega)}{(1-\alpha)\hat{\Omega}}\right] \tag{44}$$

$$G_1(\lambda) = \lambda^2[k\sin\lambda\cos(\mu\lambda)+\cos\lambda\sin(\mu\lambda)]\{\mu[k\cos\lambda\cos(\mu\lambda) -$$

$$\sin\lambda\sin(\mu\lambda)] - [k\sin\lambda\sin(\mu\lambda)-\cos\lambda\cos(\mu\lambda)]\} \tag{45}$$

$$G_2(\lambda) = \lambda^2[\sin\lambda\cos(\mu\lambda)+k\cos\lambda\sin(\mu\lambda)]\{\mu[-k\sin\lambda\sin(\mu\lambda) +$$

$$\cos\lambda\cos(\mu\lambda)] + [k\cos\lambda\cos(\mu\lambda)-\sin\lambda\sin(\mu\lambda)]\} \tag{46}$$

From the solutions of (41) and (42) it can be observed that $M(y,\eta)$ consists of two parts. The first part is fast time independent whereas the second part is fast time dependent.

Finally, for $O(1)$ equation, we get:

$$\rho(y)(u_{0,tt} + 2u_{1,t\eta} + u_{2,\eta\eta}) - [E(y)(u_{0,x} + u_{1,y})]_{,x} - [E(y)(u_{1,x} + u_{2,y})]_{,y} = 0 \tag{47}$$

For a $\hat{\Omega}$-periodic function $g = g(x, y, t, \xi)$, we define an averaging operator

$$\langle g \rangle = \frac{1}{|\hat{\Omega}|} \int_{\hat{\Omega}} g(x, y, t, \xi) dy \tag{48}$$

Applying the averaging operator to (47) and making use of the solution for u_1, we arrive at

$$\langle \rho(y) \rangle u_{0,tt} + \langle \rho(y) u_{2,\eta\eta} \rangle - \langle E(y)(1 + M_{,y}) \rangle u_{0,xx} = 0 \tag{49}$$

We assume that fast time average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T u_i(x, y, t, \eta) d\eta \tag{50}$$

exists and is finite. Following Francfort [14], we suppose that

$$\lim_{s \to 0} s\bar{u}_i \tag{51}$$

exists and is finite, where $\bar{u}_i$ is the Laplace transform of u_i with respect to the fast time η. Taking the Laplace transform of (49) with respect to η and performing the averaging in the fast time, we get the macroscopic equation of motion at $O(1)$:

$$\rho_0 u_{0,tt} - E_0 u_{0,xx} = 0 \tag{52}$$

where

$$\rho_0 = \langle \rho \rangle = \alpha\rho_1 + (1 - \alpha)\rho_2, \quad E_0 = \frac{E_1 E_2}{(1 - \alpha)E_1 + \alpha E_2} \tag{53}$$

We conclude that the macroscopic equation of motion at $O(1)$ is non-dispersive. Proceeding with the derivation of the higher-order terms reveals that the fast time dependence of the displacement field introduces secular terms at $O(\varepsilon^2)$ and higher.

3.2 Fast Spatial and Slow Temporal Scales

In this section we introduce a fast spatial scale to account for the rapid spatial fluctuations of material properties and a slow temporal scale to capture the long-term behavior of the homogenized solution. We set $m = 2$, i.e.

$$\xi = \varepsilon^2 t = \tau \tag{54}$$

At $O(\varepsilon^{-2})$, we have

$$[E(y)u_{0,y}]_{,y} = 0 \tag{55}$$

The general solution to the above equation is

$$u_0 = a_1(x, t, \tau)\int_{y_0}^{y_0+y} \frac{1}{E(y)}dy + a_2(x, t, \tau) \tag{56}$$

where $a_1(x, t, \tau)$ and $a_2(x, t, \tau)$ are integration constants. Due to periodicity of u_0 $a_1(x, t, \tau)$ vanishes, implying that the leading-order displacement depends only on the macroscale, i.e.

$$u_0 = u_0(x, t, \tau) \tag{57}$$

At the next order $O(\varepsilon^{-1})$, the perturbation equation is

$$[E(y)(u_{0,x} + u_{1,y})]_{,y} = 0 \tag{58}$$

Due to linearity, the general solution of u_1 becomes

$$u_1(x, y, t, \tau) = U_1(x, t, \tau) + N(y)u_{0,x} \tag{59}$$

Substituting (59) into (58) yields

$$[E(y)(1 + N_{,y})]_{,y} = 0 \tag{60}$$

Equation (60) together with the periodicity and continuity conditions of u_1 and σ_0 over the unit cell domain as well as the normalization condition $\langle u_1(x, y, t, \tau)\rangle = 0$ define the the unit cell boundary value problem from which $N(y)$ can be uniquely determined

$$N_1(y) = \frac{(1-\alpha)(E_2 - E_1)}{(1-\alpha)E_1 + \alpha E_2}\left[y - \frac{\alpha\Omega}{2}\right], \quad N_2(y) = \frac{\alpha(E_1 - E_2)}{(1-\alpha)E_1 + \alpha E_2}\left[y - \frac{(1+\alpha)\Omega}{2}\right] \tag{61}$$

It is interesting to note that $N(y)$ is the same as the fast time independent part of $M(y, \eta)$ in the previous section.

At $O(\varepsilon^0)$, the perturbation equation is

$$\rho(y)u_{0,tt} - [E(y)(u_{0,x} + u_{1,y})]_{,x} - [E(y)(u_{1,x} + u_{2,y})]_{,y} = 0 \tag{62}$$

Applying the averaging operator defined in (48) to the above equation and taking into account periodicity of σ_1, we get the non-dispersive macroscopic equation of motion which is identical to equation (52). In order to capture the dispersion effect, we proceed to higher order terms.

3.2.1 $O(\varepsilon)$ homogenization

Higher order correction, u_2, can be determined from $O(\varepsilon^0)$ perturbation equation (62). Substituting (59) and(52) into (62), yields

$$[E(y)(u_{2,y} + U_{1,x} + Nu_{0,xx})]_{,y} = E_0[\rho(y)/\rho_0 - 1]u_{0,xx} \tag{63}$$

Linearity suggests that u_2 may be sought in the form

$$u_2(x, y, t, \tau) = U_2(x, t, \tau) + N(y)U_{1,x} + P(y)u_{0,xx} \tag{64}$$

Substituting the above expression into (63) yields

$$[E(y)(N + P_{,y})]_{,y} = E_0[\rho(y)/\rho_0 - 1] \tag{65}$$

The boundary conditions for the above equation are: periodicity and continuity of u_2 and σ_1 as well as the normalization condition $\langle u_2(x, y, t, \tau)\rangle = 0$. Here we only provide general ideas. For detailed solution of the unit cell boundary value problem, we refer to [12]. Once $P(y)$ is found, we can calculate

$$\langle \rho N\rangle = 0, \quad \langle E(N + P_{,y})\rangle = 0, \quad \langle E(u_{1,x} + u_{2,y})\rangle = E_0 U_{1,x} \tag{66}$$

Consider the equilibrium equation of $O(\varepsilon)$:

$$\rho(y)u_{1,tt} - [E(y)(u_{1,x} + u_{2,y})]_{,x} - [E(y)(u_{2,x} + u_{3,y})]_{,y} = 0 \tag{67}$$

Applying the averaging operator to the above equation, exploiting the periodicity of σ_2 and making use of (66), we arrive at

$$\rho_0 U_{1,tt} - E_0 U_{1,xx} = 0 \tag{68}$$

3.2.2 $O(\varepsilon^2)$ homogenization

Substituting (59), (64) and (68) into $O(\varepsilon)$ equilibrium equation (67) yields

$$[E(y)(u_{3,y} + Pu_{0,xxx} + NU_{1,xx} + U_{2,x})]_{,y} = E_0(\rho(y)/\rho_0 - 1)U_{1,xx} +$$

$$[E_0 N\rho(y)/\rho_0 - E(y)(N + P_{,y})]u_{0,xxx} \tag{69}$$

Due to linearity of the above equation, the general solution of u_3 is as follows

$$u_3(x, y, t, \tau) = U_3(x, t, \tau) + N(y)U_{2,x} + P(y)U_{1,xx} + Q(y)u_{0,xxx} \tag{70}$$

Substituting the above expression into (69) gives

$$[E(y)(P + Q_{,y})]_{,y} = E_0 N\rho(y)/\rho_0 - E(y)(N + P_{,y}) \tag{71}$$

The above equation, together with the periodicity and continuity of u_3 and σ_2 over the unit cell domain as well as the normalization condition $\langle u_3(x, y, t, \tau)\rangle = 0$, fully determines $Q(y)$. After $Q(y)$ is solved for, we can calculate

$$\langle \rho P\rangle = \frac{[\alpha(1-\alpha)]^2(\rho_2 - \rho_1)(E_1\rho_1 - E_2\rho_2)E_0\hat{\Omega}^2}{12\rho_0 E_1 E_2} \tag{72}$$

$$\langle E(P + Q_{,y})\rangle = -\frac{\alpha(1-\alpha)E_0\hat{\Omega}^2}{12\rho_0}\left\{\frac{(E_2 - E_1)[\alpha^2\rho_1 - (1-\alpha)^2\rho_2] + E_0\rho_0}{(1-\alpha)E_1 + \alpha E_2} - \rho_0\right\} \tag{73}$$

Finally, consider the equilibrium equation of $O(\varepsilon^2)$:

$$\rho(y)[u_{2,tt} + 2u_{0,t\tau}] - [E(y)(u_{2,x} + u_{3,y})]_{,x} - [E(y)(u_{3,x} + u_{4,y})]_{,y} = 0 \tag{74}$$

Applying the averaging operator to the above equation, taking into account the periodicity of σ_3 and making use of (72) and (73) lead to

$$\rho_0 U_{2,tt} - E_0 U_{2,xx} = \frac{1}{\varepsilon^2}E_d u_{0,xxxx} - 2\rho_0 u_{0,t\tau} \tag{75}$$

where

$$E_d = \frac{[\alpha(1-\alpha)]^2(E_1\rho_1 - E_2\rho_2)^2 E_0\Omega^2}{12\rho_0^2[(1-\alpha)E_1 + \alpha E_2]^2} \tag{76}$$

E_d characterizes the effect of the microstructure on the macroscopic behavior. It is proportional to the square of the dimension of the unit cell Ω. Note that for a homogeneous material, $\alpha = 0$ or $\alpha = 1$, and in the case of impedance ratio $r = z_1/z_2$, $(z = \sqrt{E\rho})$ equal to one, E_d vanishes.

Remark 1: In absence of slow time scale, the macroscopic equation of motion at $O(\varepsilon^2)$ is

$$\rho_0 U_{2,tt} - E_0 U_{2,xx} = \frac{1}{\varepsilon^2}E_d u_{0,xxxx} \tag{77}$$

In Section 4 we will show that the solution of this equation introduces secular terms.

Remark 2: Alternatively, we could have considered slow time scaling with $m = 1$, i.e., $\xi = \varepsilon t = \zeta$. It can be shown that the homogenized equations of motion in this case are:

$$\rho_0 U_{1,tt} - E_0 U_{1,xx} = -2\rho_0 u_{0,t\zeta} \tag{78}$$

$$\rho_0 U_{2,tt} - E_0 U_{2,xx} = \frac{1}{\varepsilon^2} E_d u_{0,xxxx} - 2\rho_0 U_{1,t\zeta} - \rho_0 u_{0,\zeta\zeta} \tag{79}$$

3.3 Summary of Macroscopic Equations

In this section we summarize various order macroscopic equations of motion which have been derived in the previous section and prescribe initial and boundary conditions. Attention is restricted to slow time scaling with *m=2*.

The macroscopic equations of motion are:

$$O(1): \quad \rho_0 u_{0,tt} - E_0 u_{0,xx} = 0 \tag{52}$$

$$O(\varepsilon): \quad \rho_0 U_{1,tt} - E_0 U_{1,xx} = 0 \tag{68}$$

$$O(\varepsilon^2): \quad \rho_0 U_{2,tt} - E_0 U_{2,xx} = \frac{1}{\varepsilon^2} E_d u_{0,xxxx} - 2\rho_0 u_{0,t\tau} \tag{75}$$

We consider the following problem: a domain composed of an array of bilaminates with fixed boundary at $x = 0$ and free boundary at $x = l$ subjected to the initial disturbance $f(x)$ in the displacement field. The following initial-boundary conditions are considered:

$$\text{ICs:} \quad u_0(x, 0, 0) = f(x), \quad u_{0,t}(x, 0, 0) = g(x) = 0 \tag{80}$$

$$\text{BCs:} \quad u_0(0, t, \tau) = 0, \quad u_{0,x}(l, t, \tau) = 0 \tag{81}$$

The calculation of the field $\varepsilon U_1(x, t, \tau)$ is performed by solving equation of motion (68). The initial and boundary conditions applied to $\varepsilon U_1(x, t, \tau)$ must be such that the global field $u_0(x, t, \tau) + \varepsilon U_1(x, t, \tau)$ meets macroscopic initial conditions and conditions imposed on the boundary, i.e.

$$u_0(x, 0, 0) + \varepsilon U_1(x, 0, 0) = f(x), \quad u_{0,t}(x, 0, 0) + \varepsilon U_{1,t}(x, 0, 0) = g(x) = 0$$

$$u_0(0, t, \tau) + \varepsilon U_1(0, t, \tau) = 0, \quad u_{0,x}(l, t, \tau) + \varepsilon U_{1,x}(l, t, \tau) = 0$$

Considering (80) and (81) the initial and boundary conditions for $\varepsilon U_1(x, t, \tau)$ are

$$\text{ICs:}\quad \varepsilon U_1(x,0,0) = 0\,,\quad \varepsilon U_{1,t}(x,0,0) = 0$$

$$\text{BCs:}\quad \varepsilon U_1(0,t,\tau) = 0\,,\quad \varepsilon U_{1,x}(l,t,\tau) = 0$$

Similarly, the macroscopic field $\varepsilon^2 U_2(x,t,\tau)$ is determined from the equation of motion (75), with the initial and boundary conditions for $\varepsilon^2 U_2(x,t,\tau)$ constructed so that the global field, $u_0(x,t,\tau) + \varepsilon U_1(x,t,\tau) + \varepsilon^2 U_2(x,t,\tau)$, should satisfy macroscopic initial and boundary conditions.

With this in mind, we obtain the initial and boundary conditions for different order equations of motion

$$\text{ICs:}\qquad u_0(x,0,0) = f(x)\,,\qquad u_{0,t}(x,0,0) = g(x) = 0$$

$$U_i(x,0,0) = 0\,,\qquad U_{i,t}(x,0,0) = 0\qquad (i = 1,2) \tag{82}$$

$$\text{BCs:}\quad U_i(0,t,\tau) = 0\,,\qquad U_{i,x}(l,t,\tau) = 0\qquad (i = 0,1,2) \tag{83}$$

From the above equations of motion and initial/boundary conditions, we can observe that

$$U_1(x,t,\tau) \equiv 0 \tag{84}$$

4. Solution of Macroscopic Equations

4.1 Without Slow Time Scaling

First we show that in absence of slow time scaling, the response of the second order equation contains secular terms. We begin with the zero-order equation of motion (52) and employ separation of variables to solve for this initial/boundary value problem. Let

$$u_0(x,t) = X(x)T(t) \tag{85}$$

Substituting the above equation into (52) and dividing by the product $X \cdot T$ yields

$$\frac{T''}{T} = c^2\frac{X''}{X} = -q^2 \tag{86}$$

where q is the separation constant and

$$c = \sqrt{E_0/\rho_0} \tag{87}$$

The resulting differential equations and corresponding solutions are:

$$T'' + q^2 T = 0, \quad T(t) = A\sin(qt) + B\cos(qt) \tag{88}$$

$$X'' + \frac{q^2}{c^2} X = 0, \quad X(x) = K\sin\frac{qx}{c} + D\cos\frac{qx}{c} \tag{89}$$

where A, B, K and D are constants of integration. Substituting the above solutions into the boundary conditions (83) gives

$$D = 0, \quad K\cos\frac{ql}{c} = 0 \tag{90}$$

The second condition in the above equation gives

$$q_n = (2n-1)\frac{\pi c}{2l}, \quad (n = 1, 2, 3, \ldots) \tag{91}$$

Due to linearity of the differential equation, the total solution to the problem is the sum of individual solutions. Hence, we may write

$$u_0(x, t) = \sum_{n=1}^{\infty} \sin\frac{q_n x}{c}[A_n \sin(q_n t) + B_n \cos(q_n t)] \tag{92}$$

The coefficients A_n and B_n can be determined by the initial conditions. Substituting (92) into the initial conditions (82) gives

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}, \quad \sum_{n=1}^{\infty} A_n \frac{(2n-1)\pi c}{2l} \sin\frac{(2n-1)\pi x}{2l} = 0 \tag{93}$$

Multiplying both sides of the above equations by $\sin(2m-1)\pi x/(2l)dx$ and integrating in space between $x = 0$ and $x = l$, yields

$$A_n = 0, \quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{(2n-1)\pi x}{2l}dx \tag{94}$$

Substituting (94) into (92) gives the final expression for $u_0(x, t)$

$$u_0(x, t) = \sum_{n=1}^{\infty} B_n \cos\frac{(2n-1)\pi ct}{2l} \sin\frac{(2n-1)\pi x}{2l} \tag{95}$$

Next, we proceed to solve for the second order macroscopic equation. Taking the fourth derivative of $u_0(x, t)$ with respect to x and substituting the result into (77) gives

$$U_{2,tt} - c^2 U_{2,xx} = \frac{E_d}{\varepsilon^2 \rho_0} \sum_{n=1}^{\infty} B_n \left(\frac{q_n}{c}\right)^4 \cos(q_n t) \sin\left(\frac{q_n x}{c}\right) \tag{96}$$

Equation (96) can be solved by using either the method of separation of variables or Laplace transform. We employ the latter. Taking the Laplace transform of (96) with respect to time, *t*, and making use of the initial conditions (82) yields

$$s^2 \overline{U}_2(x, s) - c^2 \overline{U}_{2,\,xx} = \frac{E_d}{\varepsilon^2 \rho_0} \sum_{n=1}^{\infty} B_n \left(\frac{q_n}{c}\right)^4 \frac{s}{s^2 + q_n^2} \sin\left(\frac{q_n x}{c}\right) \tag{97}$$

where $\overline{U}_2(x, s)$ is the Laplace transform of $U_2(x, t)$. The general solution of (97) is

$$\overline{U}_2(x, s) = b_1 \cosh\frac{sx}{c} + b_2 \sinh\frac{sx}{c} + \frac{E_d}{\varepsilon^2 \rho_0} \sum_{n=1}^{\infty} B_n \left(\frac{q_n}{c}\right)^4 \frac{s}{(s^2 + q_n^2)^2} \sin\left(\frac{q_n x}{c}\right) \tag{98}$$

where b_1 and b_2 are constants. Substituting the above equation into the boundary conditions (83), gives

$$b_1 = b_2 = 0 \tag{99}$$

Inserting the above equation into (98) and taking the inverse Laplace transform to the resulting equation yields the solution for $U_2(x, t)$

$$U_2(x, t) = \frac{E_d t}{2\varepsilon^2 \rho_0 c^4} \sum_{n=1}^{\infty} B_n q_n^3 \sin(q_n t) \sin\left(\frac{q_n x}{c}\right) \tag{100}$$

It can be readily observed that the solution of $U_2(x, t)$ is linear in time and will grow unbounded as the time approaches infinity. Obviously, this does not reflect the physics of the problem. Hence, the solution is only valid when the time window is very small. In an attempt to construct an uniformly valid solution, we will consider a slow time scale in the next section.

4.2 Dispersive Solution

We consider the solution of the macroscopic equations of motion (52) and (75). Assume the following separation of variables for $u_0(x, t, \tau)$

$$u_0(x, t, \tau) = Y(x)\Theta(t, \tau) \tag{101}$$

Substituting (101) into (52) and dividing by the product $Y \cdot \Theta$ yields

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial t^2} = c^2\frac{Y''}{Y} = -p^2 \tag{102}$$

where p is a separation constant. The resulting differential equations and corresponding solutions are

$$\frac{\partial^2\Theta}{\partial t^2} + p^2\Theta = 0, \qquad \Theta(t, \tau) = S(\tau)\sin(pt) + R(\tau)\cos(pt) \tag{103}$$

$$Y'' + \frac{p^2}{c^2}Y = 0, \qquad Y(x) = h_1\sin\frac{px}{c} + h_2\cos\frac{px}{c} \tag{104}$$

where h_1 and h_2 are integration constants, $S(\tau)$ and $R(\tau)$ are undetermined functions. Substituting the above solutions into the boundary conditions (83) gives

$$h_2 = 0, \qquad h_1\cos\frac{pl}{c} = 0 \tag{105}$$

The second condition in the above equation leads to

$$p_n = (2n-1)\frac{\pi c}{2l}, \qquad (n = 1, 2, 3, \dots) \tag{106}$$

Due to linearity of the differential equation, the total solution can be written as the sum of individual solutions, i.e.

$$u_0(x, t, \tau) = \sum_{n=1}^{\infty} \sin\frac{p_n x}{c}[S_n(\tau)\sin(p_n t) + R_n(\tau)\cos(p_n t)] \tag{107}$$

Inserting the above solution into the second order macroscopic equation of motion (75) gives

$$\begin{aligned} U_{2,tt} - c^2U_{2,xx} = \sum_{n=1}^{\infty} \frac{p_n}{c}\sin\frac{p_n x}{c}\Bigg\{&\left[\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{p_n}{c}\right)^3 S_n(\tau) + 2cR'_n(\tau)\right]\sin(p_n t) + \\ &\left[\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{p_n}{c}\right)^3 R_n(\tau) - 2cS'_n(\tau)\right]\cos(p_n t)\Bigg\} \end{aligned} \tag{108}$$

The forcing terms in (108) will typically generate secular terms. In order to eliminate secular terms the forcing terms are set to zero, i.e.

$$\frac{E_d}{\varepsilon^2\rho_0}\left(\frac{p_n}{c}\right)^3 S_n(\tau) + 2cR'_n(\tau) = 0, \qquad \frac{E_d}{\varepsilon^2\rho_0}\left(\frac{p_n}{c}\right)^3 R_n(\tau) - 2cS'_n(\tau) = 0 \tag{109}$$

Let

$$\omega_n = \frac{E_d}{2c\rho_0}\left(\frac{p_n}{c}\right)^3 = \frac{[(2n-1)\pi]^3 E_d}{16\rho_0 c l^3} \tag{110}$$

Then (109) can be written as

$$\varepsilon^2 R'_n(\tau) + \omega_n S_n(\tau) = 0, \qquad \varepsilon^2 S'_n(\tau) - \omega_n R_n(\tau) = 0 \tag{111}$$

Differentiating the first equation in (111) and inserting the second equation into the resulting equation lead to

$$\varepsilon^4 R''_n(\tau) + \omega_n^2 R_n(\tau) = 0 \tag{112}$$

Likewise, differentiating the second equation in (111) and inserting the first equation into the resulting equation yields

$$\varepsilon^4 S''_n(\tau) + \omega_n^2 S_n(\tau) = 0 \tag{113}$$

Solutions of (112) and (113) are

$$R_n(\tau/\varepsilon^2) = d_1 \sin(\omega_n\tau/\varepsilon^2) + d_2\cos(\omega_n\tau/\varepsilon^2)$$

$$S_n(\tau/\varepsilon^2) = d_3 \sin(\omega_n\tau/\varepsilon^2) + d_4\cos(\omega_n\tau/\varepsilon^2) \tag{114}$$

where d_1, d_2, d_3 and d_4 are constants of integration. The above solutions must satisfy (111). Inserting (114) into (111) gives

$$d_1 = -d_4, \qquad d_2 = d_3 \tag{115}$$

Substituting (114) and (115) into (107) and utilizing initial conditions gives

$$d_1 = 0, \qquad d_2 = B_n \tag{116}$$

and thus the dispersive solution up to the second order, denoted here as $u_d(x, t, \tau/\varepsilon^2)$, is given as

$$u_d(x, t, \tau/\varepsilon^2) = \sum_{n=1}^{\infty} B_n \sin\frac{p_n x}{c}\cos\left(\omega_n\frac{\tau}{\varepsilon^2} - p_n t\right) \tag{117}$$

For function evaluation we insert $t = \tau/\varepsilon^2$ which yields

$$u_d(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}\cos\left\{\left[\left[\frac{[(2n-1)\pi]^2 E_d}{8\rho_0 c^2 l^2} - 1\right]\frac{(2n-1)\pi ct}{2l}\right\}\right. \qquad (118)$$

5. Numerical Results

To assess the accuracy of the proposed model, we construct a reference solution by utilizing a very fine finite element mesh and employ an explicit time integration scheme. We consider the following initial disturbance in the displacement field:

$$f(x) = f_0 a_0[x-(x_0-\delta)]^4[x-(x_0+\delta)]^4\{1-H[x-(x_0+\delta)]\}[1-H(x_0-\delta-x)]$$

where $a_0 = 1/\delta^8$ and $H(x)$ is the Heaviside step function; f_0, x_0 and δ are the magnitude, the location of the maximum value and the half width of the initial pulse. Several pulses with $f_0 = 1m$ and different half pulse width, δ, are plotted in Figure 2.

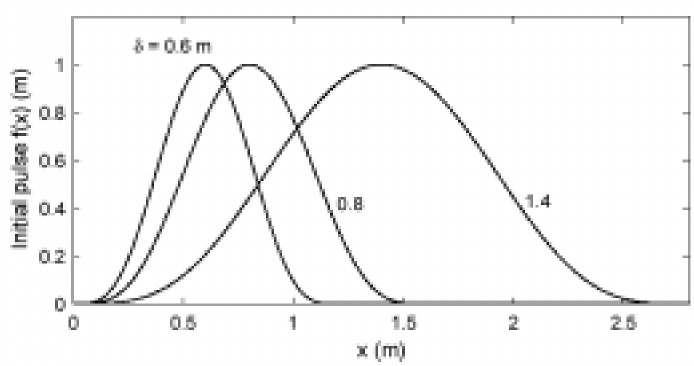

Figure 2: The initial disturbance in displacement with different half pulse widths

It can be seen that this pulse is similar in shape to the Gaussian distribution function. Substituting the initial disturbance $f(x)$ into (94) and integrating it analytically, yields

$$B_n = \frac{2}{l}\int_{x_0-\delta}^{x_0+\delta} f_0 a_0[x-(x_0-\delta)]^4[x-(x_0+\delta)]^4 \sin\frac{(2n-1)\pi x}{2l}dx$$

$$= \frac{49152 l^4 f_0}{\delta^8[(2n-1)\pi]^9}\{[1680 l^4 - 180((2n-1)\pi\delta l)^2 + ((2n-1)\pi\delta)^4]\sin\frac{(2n-1)\pi x_0}{2l}\sin\frac{(2n-1)\pi\delta}{2l} + [20l((2n-1)\pi\delta)^3 - 840(2n-1)\pi\delta l^3]\sin\frac{(2n-1)\pi x_0}{2l}\cos\frac{(2n-1)\pi\delta}{2l}\right\}$$

We choose material properties as $E_1 = 120$ *GPa*, $E_2 = 6$ *GPa*, $\rho_1 = 8000$ *Kg/m*^{3}, $\rho_2 = 3000$ *Kg/m*^{3}, and volume fraction $\alpha = 0.5$. The dimension of the macro-domain and that of the unit cell are set as $l = 40$ *m* and $\Omega = 0.2$ *m*, respectively. The homogenized material properties are calculated as $E_0 = 11.43$ *GPa*, $\rho_0 = 5500$ *Kg/m*^{3} and $E_d = 1.76 \times 10^7$ *N*. In this case, $E_1 / E_2 = 20$ and the ratio of the impedances of the two material constituents is $r = 7.30$. The initial pulse is centered at the midpoint of the domain, i.e. $x_0 = 20$ *m*, with the magnitude $f_0 = 1.0$ *m*.

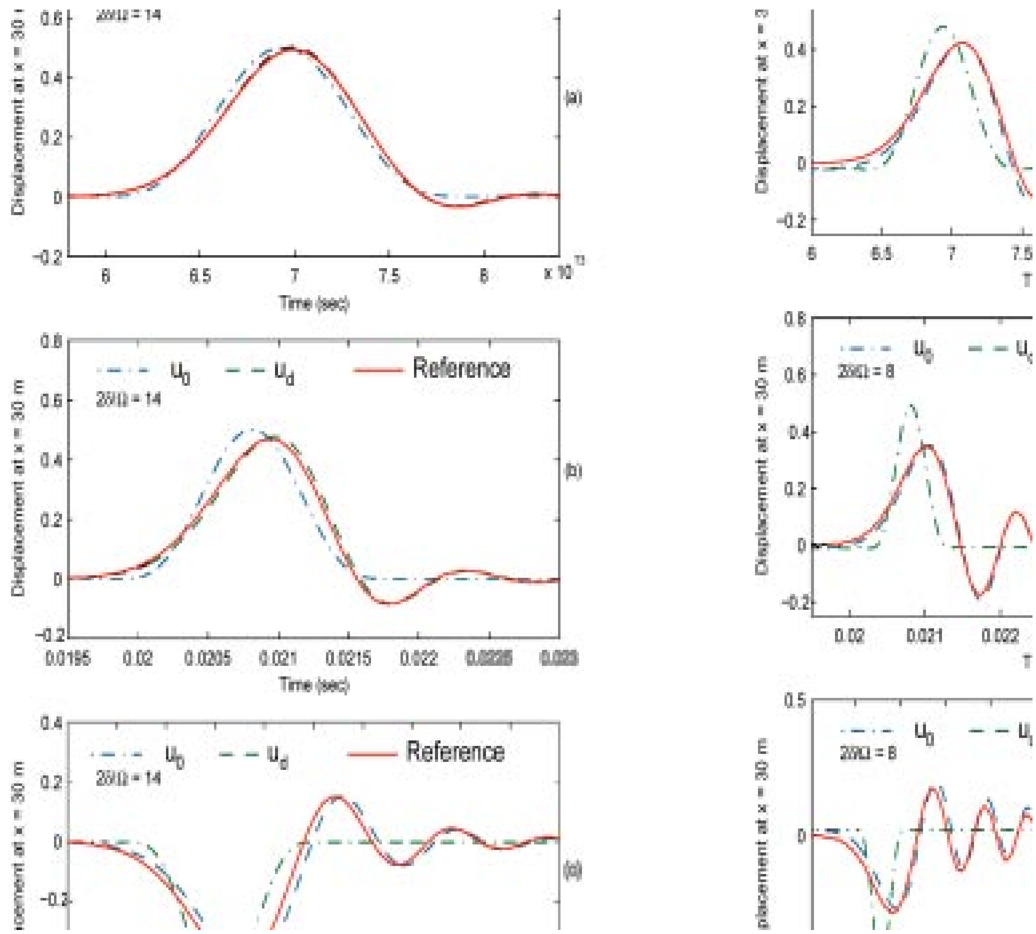

Figure 3: Displacements at $x = 30m$ for the normalized pulse width $2\delta / \Omega = 14$

Figures 3-5 show the evolution of displacements at $x = 30$ *m* for different values of pulse width: $\delta = 1.4$ *m*, $\delta = 0.8$ *m* and $\delta = 0.6$ *m*. The corresponding ratios between the pulse width and the unit cell dimension, $2\delta / \Omega$, are: 14, 8 and 6, respectively. In each of the Figures 3-5, there are three plots denoted as (a)-(c), which correspond to the numerically exact solution of the original heterogeneous problem, the analytical nondispersive solution $u_0(x, t)$ obtained by the classical homogenization theory and the analytical dispersive solution $u_d(x, t)$.

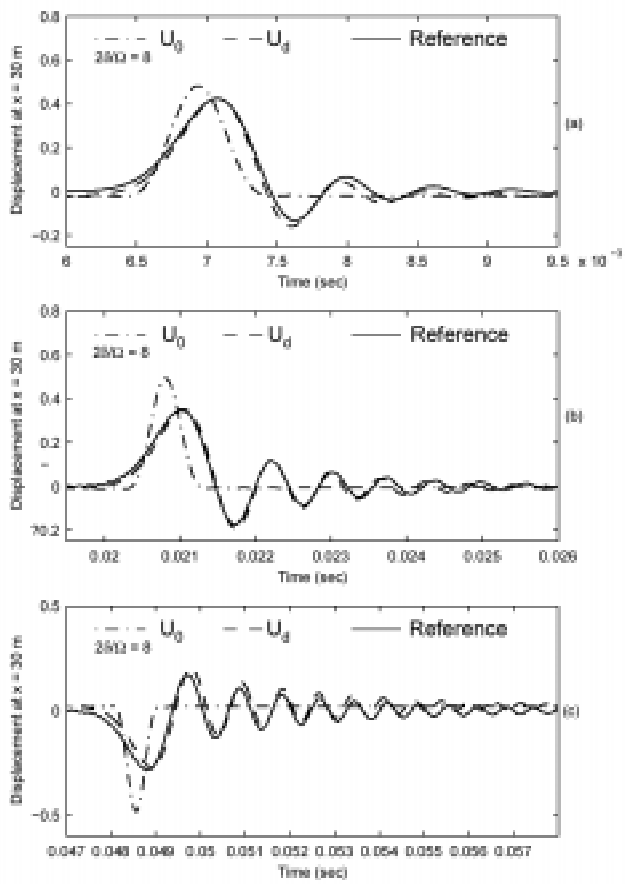

Figure 4: Displacements at $x = 30m$ for the normalized pulse width $2\delta/\Omega = 8$

The dispersion phenomenon can be clearly seen from Figures 3-5. In the low frequency case, depicted in Figure 3, the pulse almost maintains its initial shape except for some small wiggles at the wavefront. In this case, the zero-order homogenization theory provides a reasonable approximation to the response of the heterogeneous media. However, when the pulse width of the initial disturbance is comparable to the dimension of the unit cell and the observation time is large, which are the cases shown in Figures 4 and 5, the wave becomes strongly dispersive and the zero-order homogenization errs badly. It can be also seen that our dispersive model is in good agreement with the reference solution of the heterogeneous media.

6. Concluding Remarks

Mathematical homogenization theory with multiple spatial and temporal scales have been investigated. This work is motivated by our recent study [12] which suggested that in absence of multiple time scaling, higher order mathematical homogenization method gives rise to secular terms which grow unbounded with time. In the present manuscript we have demonstrated that multiple scale analysis based on fast spatial and temporal scales gives rise to nondispersive $O(1)$ model. On the other hand, the combination of fast spatial and slow temporal scales successfully captures dispersion effects.

In our future work we will focus on the following three issues: (i) developing an uniformly valid mathematical model, (ii) generalization to the multidimensional case, and (ii) a finite element implementation.

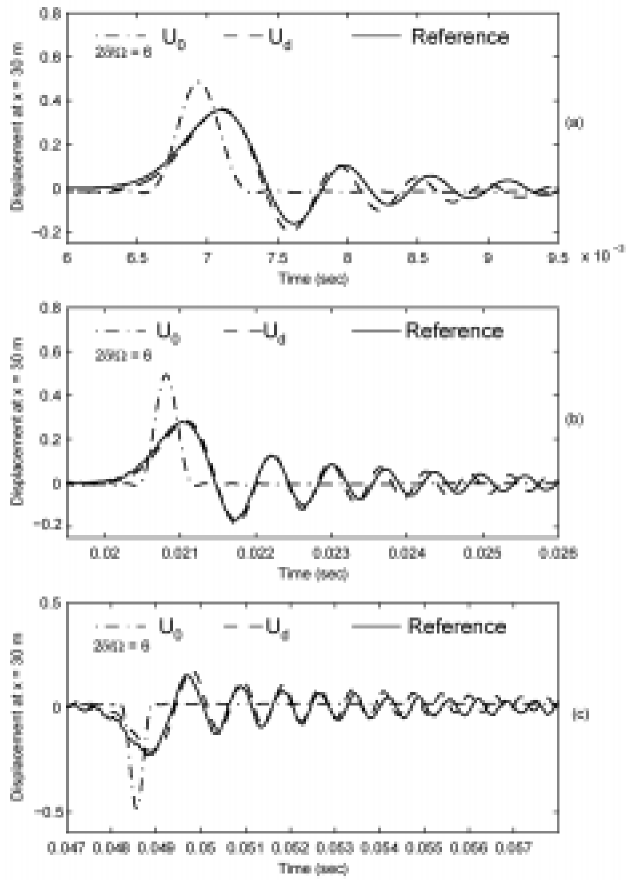

Figure 5: Displacements at $x = 30m$ for the normalized pulse width $2\delta/\Omega = 6$

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