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Space-time Multiscale Laminated Theory

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ABSTRACT

Multiscale computational techniques in space and time are developed to study the impact response of thin, elastic, laminated composites. The displacement field is approximated using asymptotic expansion in space and time. Using the homogenization procedure in space and time, nonlocal membrane and bending equations of motion are derived. The nonlocal equations are stabilized to filter out the higher frequency content. The multiscale model is verified for membrane and bending problems.

KEY WORDS

non-local; elastodynamics; homogenization; multiple scales; dispersion; impact; asymptotic

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1. INTRODUCTION

In heterogeneous materials elastic waves tend to travel faster in stiffer materials, and therefore, any *a priori* assumption about the through-the-thickness distribution of displacements in a laminate may have a detrimental effect on the solution accuracy. Models based on the piecewise approximation of the solution in thickness direction [1] have the flexibility of capturing the dominant deformation modes, but might unnecessarily overtax computational resources for laminates with a significant number of layers.

These limitations of existing theoretical models motivated the development of the space-time multiscale laminated theory, which makes no assumption about the displacement approximation in thickness direction. The proposed multiscale approach constructs a nearly optimal through-the-thickness solution approximation by solving a sequence of higher-order local equilibrium equations.

Research efforts aimed at developing a nearly optimal laminated plate theory have reached a level of maturity. While there is a consensus that an optimal laminated model depends on loading and boundary conditions in addition to several other factors, the authors of this manuscript are unaware of any prior work attempting to tailor the laminated model to system dynamics.

To this end it is instructive to briefly survey related efforts aimed at constructing solution approximations in the thickness direction. Babuska [2] developed a hierarchic model for laminated composites in which the displacement was expanded using Fourier approximation. Spatial homogenization theory has been applied to thin laminates by Goldenvizer [3], Sanchez Palencia [4], Panasenko and Reztov [5], and Lewinski and Telega [6]. Fish *et al.* [7, 8] and Bakhvalov [9] studied wave dispersion in a periodic, heterogeneous material using higher-order asymptotic terms with spatial and temporal scaling. These higher-order models showed improved results over the classical homogenization theory and serve as a motivation for the present work. Wave dispersion has been also successfully studied by McDevitt *et al.* [10] using multiple spatial scales with an assumed strain field enriched by a linear combination of strain concentration functions obtained from the unit cell solution.

We consider a plate positioned in the x_1x_2 plane with piecewise constant material properties through-the-plate thickness. The domain of the problem is defined as follows (see also Fig. 1):

$$
\begin{aligned}
&\mathrm{X} = \{x_3(-\varepsilon, \varepsilon)\} \\
&\Omega_A = \{x_\alpha\} \\
&\Omega = \Omega_A \times \mathrm{X}
\end{aligned}
$$

The boundary Γ_A consists of Γ_h and Γ_g being displacement boundary and Γ_h the boundary where the tractions are prescribed, such that $\Gamma_\mathrm{h} \cup \Gamma_\mathrm{g} = \emptyset$ and $\Gamma_\mathrm{h} \cap \Gamma_\mathrm{g} = \Gamma_A$. We assume that the thickness of the plate is very small and define the small parameter ε ($\varepsilon \ll 1$)to be half of the thickness of the plate.

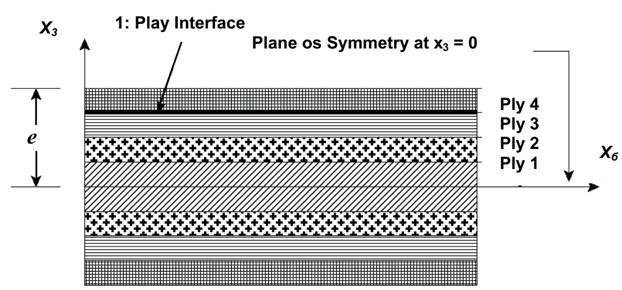

FIGURE 1. The laminate elevation, coordinate system, and layup

The mathematical model on Ω is as follows:

$$\sigma_{ij,j} - \rho u_{i,tt} + b_i = 0 \quad \text{(Equation of motion)}, \tag{1.1}$$

$$\sigma_{ij} = C_{ijkl}\left(x_3\right) e_{kl}\left(\mathbf{u}\right) \quad \text{(Constitutive Relation)}, \tag{1.2}$$

$$e_{kl}\left(\mathbf{u}\right) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \quad \text{(Kinematics Relation)}. \tag{1.3}$$

The plate is assumed to be traction free at its top and bottom surfaces

$$t_i(x_3 = \pm\varepsilon) = \sigma_{i3}(x_3 = \pm\varepsilon)n_3 = 0\,. \tag{1.4}$$

On the edges of the plate, we will consider displacement and traction boundary conditions

$$\begin{aligned} \sigma_{i\alpha}n_\alpha &= t_i \text{ on } \Gamma_{\mathrm{h}} \\ u_i &= \Phi_i \text{ on } \Gamma_{\mathrm{g}}. \end{aligned} \tag{1.5}$$

We are restricting the slippage or relative displacement between plies, so there must be displacement and traction continuity at the ply interface I,

$$[u_k]_{\mathrm{I}} = 0 \quad [\sigma_{i3}n_3]_{\mathrm{I}} = [\sigma_{i3}]_{\mathrm{I}} = 0. \tag{1.6}$$

The initial displacement and velocity conditions are

$$u_i(\mathbf{x}, t = 0) = f_i(\mathbf{x}), \tag{1.7}$$

$$u_{i,t}(\mathbf{x}, t = 0) = g_i(\mathbf{x}). \tag{1.8}$$

The material density $\rho\left(x_\alpha\right)$ and body force $b_i(x_\alpha)$ are assumed to be constant through the thickness of the plate, but vary in the plane of the plate.

We assume the constitutive tensor C_{ijkl} to be piecewise constant through the thickness and orthotropic within each ply. The constitutive tensor is assumed to possess symmetry and positive definiteness

$$C_{ijkl} = C_{klij} = C_{lkij}, \tag{1.9}$$

$$\begin{aligned} &C_{ijkl}\Psi_{ij}\Psi_{kl} \geq 0 \\ &\text{If } \left(C_{ijkl}\Psi_{ij}\Psi_{kl} = 0\right) \text{then} \left(\Psi_{kl} = 0\right). \end{aligned} \tag{1.10}$$

For simplicity, the constitutive tensor is assumed to be symmetric with respect to the mid-plane $(x_3 = 0)$.

2. SOLUTION APPROXIMATION

Since the thickness of the plate is much smaller than the in-plane dimensions, state variables are expected to have a larger variation through the thickness than in-plane directions. The following scaling equation is used to resolve the behavior in the thickness direction:

$$y = \frac{x_3}{\varepsilon}. \tag{2.1}$$

The governing equations will be expressed in terms of three spatial coordinates (x_1, x_2, y_3). The partial derivative terms in x is expressed using the following chain rule:

$$f_{,j} = \partial x_j f + \varepsilon^{-1}\delta_{j3}\partial y\left(f\right), \tag{2.2}$$

where

$$\partial x_j f = \delta_{j\alpha}\frac{\partial f}{\partial x_\alpha} \text{ where } \alpha = 1, 2, \tag{2.3}$$

$$\partial y\left(f\right) = \frac{\partial f}{\partial y}. \tag{2.4}$$

Temporal scaling is used to suppress the unbounded growth of the displacement field as observed in [7, 8].

$$t_s = \varepsilon^s t \tag{2.5}$$

Following [7, 8], two temporal coordinates $t_0 = t$ and $t_1 = \varepsilon^1 t$ are considered.

Thus the temporal derivative is replaced with the following differential operator:

$$f_{,t} = f_{,t_0} + \varepsilon^1 f_{,t_1} + \varepsilon^2 f_{,t_2}. \tag{2.6}$$

Displacements are approximated using the asymptotic expansion in the form

$$u_i = \mathbf{U}_i^0(x_\alpha) + \sum_{s=1}^{\infty}\sum_{t=0}^{\infty} \varepsilon^{s+t} H^s_{ikl\prod(s-1)}(y) D^{kl\prod(s-1)}\left(\mathbf{U}^t(x_\alpha)\right), \tag{2.7}$$

where the spatial gradient in the in-plane direction is defined as:

$$D^{kl\prod(s)}\left(\mathbf{U}^t\right) = \partial x_{\prod_1} ... \partial x_{\prod_s} e_{xkl}\left(\mathbf{U}^t\right). \tag{2.8}$$

In Eq. (2.7) $\prod(s) = \prod_1 ... \prod_s$ is a multi-index and s is its length. The multi-index performs like a series of tensor indices. The indices in the multi-index $\prod(s)$, for example, $\prod_i$ will only span 1 and 2 ($\prod_i = 1$ *or* 2). If s, in the multi-index $\prod(s)$ is less than or equal to zero, then the multi-index becomes a null set of indices, i.e., ($\prod(s) = \emptyset$ for s ≤ 0). The multi-index $\prod(s..s+t)$ represents the indices $\prod_s \prod_{s+1} ... \prod_{s+t}$. For example, the plane strain constitutive relation can be expressed using a multi-index as follows: $\sigma_{\iota\varphi} = C_{\iota\varphi\prod(2)}(x_3)\, e_{\prod(2)}(\mathbf{u})$. This can be expanded as: $\sigma_{\iota\varphi} = C_{\iota\varphi\prod_1\prod_2}(x_3)\, e_{\prod_1\prod_2}(\mathbf{u})$.

Alternatively, it can be expressed in terms of conventional tensor indices that span 1 and 2, such as $\sigma_{\iota\varphi} = C_{\iota\varphi\alpha\beta}(x_3)\, e_{\alpha\beta}(\mathbf{u})$. The term $e_{xkl}\left(\mathbf{U}^t\right)$ denotes the symmetric gradient of $\mathbf{U}^t$ with respect to x_k and x_l. The first three terms of the displacement expansion can be obtained from Eq. (2.7) as

$$u_i = \varepsilon^0\left(U_i^0\right) + \varepsilon^1\left(U_i^1 + H^1_{ikl}(\mathbf{y}) e_{xkl}\left(\mathbf{U}^0\right)\right) + \varepsilon^2\left(U_i^2 + H^1_{ikl}(\mathbf{y}) e_{xkl}\left(\mathbf{U}^1\right) + H^2_{ikl\prod_1}\partial x_{\prod_1} e_{xkl}\left(\mathbf{U}^0\right)\right) + \mathrm{O}(\varepsilon^3)$$

where $\mathbf{U}^t(x_\alpha)$ denotes the through-the-thickness average displacements, i.e., $\partial y_3\left(\mathbf{U}^t(x_\alpha)\right) = 0$, which may vary in the plane of the plate Ω_A.$H^s_{ikl\prod(s-1)}(y_3)$ is a concentration factor, which varies in the thickness direction 3, but is constant in Ω_A , i.e., $\partial x_\alpha\left(H^s_{ikl\prod(s-1)}(y_3)\right) = 0$. Since the strain operator $e_{xkl}\left(\mathbf{U}^t\right)$ is symmetric, the concentration factors possess similar symmetry properties:

$$H^s_{i(kl)\Pi_1\Pi_2...\Pi_{s-1}} = H^s_{i(lk)\Pi_1\Pi_2...\Pi_{s-1}}. \tag{2.9}$$

The constitutive relation and the equation of motion given by Eqs. (1.2) and (1.1), respectively, can be expressed in terms of the asymptotic expansion and spatial and temporal differential operators

$$\frac{\partial u_k}{\partial x_l} = \frac{\partial \mathbf{U}_k^0}{\partial x_l} + \delta_{l3}\partial y H^1_{k(mn)}\frac{\partial \mathbf{U}_m^0}{\partial x_n} + \sum_{s=1}^{\infty}\sum_{t=0}^{\infty}\varepsilon^{s+t}\left(\delta_{l\prod_s} H^s_{k(mn)\prod(s-1)} + \delta_{l3}\partial y H^{s+1}_{k(mn)\prod(s)}\right) \times D^{(mn)\prod(s)}\left(\mathbf{U}^t\right) \tag{2.10}$$

The Cauchy stress can be expanded using various orders of ε

$$\sigma_{ij} = \sum_{R=0}^{\infty}\varepsilon^R \sigma^R_{ij}, \tag{2.11}$$

where

$$\sigma_{ij}^{R}=\sum_{s=0}^{R}C_{ij(mn)\prod(s)}^{s}D^{(mn)\prod(s)}\left(\mathbf{U}^{R-s}\right) \quad (2.12)$$

and the stiffness coefficients are given by

$$C_{ij(mn)}^{0}=C_{ijkl}\left(\delta_{km}\delta_{\ln}+\delta_{l3}\partial yH_{k(mn)}^{1}\right)$$

$$C_{ij(mn)\prod(s)}^{s}=$$

$$C_{ijkl}\left(\delta_{l\prod_{s}}H_{k(mn)\prod(s-1)}^{s}+\delta_{l3}\partial yH_{k(mn)\prod(s)}^{s+1}\right) \quad (2.13)$$

for s ≥ 1.

The symmetry existing in both the constitutive tensor and the macroscopic strain operator gives rise to similar symmetries in the higher-order stiffness tensor, i.e.,

$$C_{(ij)(mn)(\prod(s))}^{s}\ \text{for s}\ \geq 0. \quad (2.14)$$

Substituting the stress expansion (2.11) into the equation of motion (1.1) and using spatial and temporal differential operators yields the following equilibrium equations for various orders of ε:

$$\varepsilon^{-1}\left(\partial y\sigma_{i3}^{0}\right)=0 \quad (2.15)$$

and

$$\sum_{R=0}^{\infty}\varepsilon^{R}\left(\partial x_{\alpha}\sigma_{i\alpha}^{R}+\partial y\sigma_{i3}^{R+1}\right)-\rho\sum_{R=0}^{\infty}\varepsilon^{R}a_{i}^{R}+b_{i}=0,$$

where

$$\begin{aligned}a_{i}^{R}&=\Psi U_{i}^{R}\\&+\sum_{s=1}^{R}\left(H_{imn\prod(s-1)}^{s}D^{mn\prod(s-1)}\Psi\mathbf{U}^{(R-s)}\right),\end{aligned} \quad (2.16)$$

$$\begin{aligned}\Psi\mathbf{U}^{t}&=\mathbf{U}_{,t_0t_0}^{t}+2\mathbf{U}_{,t_0t_1}^{t-1}+2\mathbf{U}_{,t_0t_2}^{t-2}\\&+\mathbf{U}_{,t_1t_1}^{t-2}+2\mathbf{U}_{,t_1t_2}^{t-3}+\mathbf{U}_{,t_2t_2}^{t-4}\end{aligned} \quad (2.17)$$

We define the averaging spatial operator as

$$\langle f\rangle=\frac{1}{2}\int_{-1}^{1}f\,dy \quad (2.18)$$

and apply it to the equation of motion to obtain the macroscopic equations of motion

$$\begin{aligned}&\sum_{R=0}^{\infty}\varepsilon^{R}\left(\partial x_{\alpha}\left\langle\sigma_{i\alpha}^{R}\right\rangle+\left\langle\partial y\sigma_{i3}^{R+1}\right\rangle\right)\\&-\rho\sum_{R=0}^{\infty}\varepsilon^{R}\left\langle a_{i}^{R}\right\rangle+b_{i}=0.\end{aligned} \quad (2.19)$$

The second term in (2.19) can be evaluated using the traction conditions on the top and bottom of the plate defined by (1.14).

$$\left\langle\partial y\sigma_{i3}^{R}\right\rangle=\sigma_{i3}^{R}\big|_{-1}^{1}=0. \quad (2.20)$$

The homogenized (or average) stresses are given by

$$\left\langle\sigma_{ij}^{R}\right\rangle=\sum_{s=0}^{R}D_{ij(mn)\prod(s)}^{s}D^{(mn)\prod(s)}\left(\mathbf{U}^{R-s}\right), \quad (2.21)$$

where the homogenized stiffness tensors of each order are

$$D_{ij(mn)\prod(s)}^{s}=\left\langle C_{ij(mn)\prod(s)}^{s}\right\rangle. \quad (2.22)$$

The concentration factors are normalized so that $\left\langle H_{imn\Pi(s-1)}^{s}\right\rangle=0$ for any $s>0$.

Consequently, the homogenized acceleration reduces to

$$\left\langle a_i^R \right\rangle = \Psi \mathbf{U}_i^R. \tag{2.23}$$

Substituting (2.20)–(2.23) into (2.19) yields a simplified expression for the macroscopic equation of motion

$$\sum_{R=0}^{\infty} \varepsilon^R \left(\partial x_\alpha \left\langle \sigma_{i\alpha}^R \right\rangle\right) - \rho \sum_{R=0}^{\infty} \varepsilon^R \Psi \mathbf{U}_i^R + b_i = 0. \tag{2.24}$$

The macroscopic equation of motion (2.24) is subsequently subtracted from the equation of motion given by Eq. (2.15) to give

$$\sum_{R=0}^{\infty} \varepsilon^R \left(\partial x_\alpha \Delta\left(\sigma_{i\alpha}^R\right) + \partial y \sigma_{i3}^{R+1} - \rho\left(a_i^R - \Psi \mathbf{U}_i^R\right)\right) = 0, \tag{2.25}$$

where the delta function is defined as

$$\Delta(f) = f - \langle f \rangle. \tag{2.26}$$

Exploiting the fact that ε is a small parameter yields one equation for each order of ε,

$$\begin{aligned} &\partial x_\alpha \Delta\left(\sigma_{i\alpha}^R\right) + \partial y \sigma_{i3}^{R+1} - \rho\left(a_i^R - \Psi \mathbf{U}_i^R\right) = 0 \\ &R = 0 \ldots \infty. \end{aligned} \tag{2.27}$$

We proceed by inserting the stress (2.11)–(2.13) and the acceleration (2.16) expansions into Eq. (2.27). Manipulating the indices and factoring out the average displacement gradient from the first two terms of (2.27) yields

$$\begin{aligned} &\partial y C_{i3(mn)}^0 D^{(mn)}\left(\mathbf{U}^{R+1}\right) \\ &+ \left(\begin{matrix} \sum_{s=0}^{R} \partial y C_{i3(mn)\prod(s+1)}^{s+1} \\ + \sum_{s=0}^{R} \Delta\left(C_{i\prod_{s+1}(mn)\prod(s)}^s\right) \end{matrix}\right) D^{(mn)\prod(s+1)}\left(\mathbf{U}^{R-s}\right) \\ &-\rho\left(\sum_{s=1}^{R}\left(H_{imn\prod(s-1)}^s\right)\right) D^{mn\prod(s-1)} \Psi \mathbf{U}^{(R-s)} = 0 \\ &R = -1 \ldots \infty. \end{aligned} \tag{2.28}$$

The first term in Eq. (2.28) will vanish by (2.15). The last term will vanish for $R = 0$. Solving the homogenized equilibrium system (2.24) for the acceleration yields

$$\rho \Psi \mathbf{U}_i^R = \partial x_\alpha \left\langle \sigma_{i\alpha}^R \right\rangle \quad R > 0. \tag{2.29}$$

Substituting the stress expansion into Eq. (2.29) yields:

$$\begin{aligned} &\rho \Psi U_i^R = \\ &\sum_{s=0}^{R} D_{i\alpha(mn)\prod(s)}^s D^{(mn)\prod(s)\alpha}\left(\mathbf{U}^{R-s}\right) \ \mathrm{R} > 0 \end{aligned} \tag{2.30}$$

and applying a differential operator to Eq. (2.30) gives

$$\begin{aligned} &\rho D^{(kl)\prod(s-1)} \Psi U_i^{R-s} = \\ &\sum_{p=0}^{R-s} D_{\underline{k}\alpha(mn)\Phi(p)}^p D^{(mn)\underline{l}\alpha\Phi(p)\prod(s-1)}\left(\mathbf{U}^{R-s-p}\right) \end{aligned} \tag{2.31}$$

Symmetry is enforced for indices k and l on the right-hand side of Eq. (2.31). Symmetry with respect to two indices is denoted using the bracket operator around the corresponding indices. Substituting (2.31) into (2.28) yields

$$\sum_{s=0}^{R+1}\left(\partial y C^{s+1}_{i3(mn)\prod(s+1)}+\Delta C^{s}_{i\prod_{s}(mn)\prod(s)}\right)$$
$$\times D^{(mn)\prod(s+1)}\left(\mathbf{U}^{R-s}\right)-\sum_{s=1}^{R} H^{s}_{i(kl)\prod(s-1)}$$
$$\times\sum_{p=0}^{R-s}\left(D^{p}_{k\alpha(mn)\Phi(p)}D^{(mn)l\alpha\Phi(p)\prod(s-1)}\left(\mathbf{U}^{R-s-p}\right)\right)=0. \tag{2.32}$$

Solving for higher-order stiffness gives

$$C^{0}_{i3(mn)\prod(1)}=K^{0}_{i(mn)\prod(1)} \tag{2.33}$$

and

$$C^{s+1}_{i3(mn)\prod(s+1)}=\int_{0}^{y}\begin{pmatrix}\sum_{p=1}^{s}\left(H^{p}_{i(k\prod_1)\prod(2..p)}\right)\\ \left\langle C^{s-p}_{k\prod_{p+1}(mn)\prod(p+2..s+1)}\right\rangle\\ -\Delta C^{s}_{i\prod_{s}(mn)\prod(s)}\end{pmatrix}dy+K^{s+1}_{i(mn)\prod(s+1)}$$

with $s\geq 0$ and $\partial y_3 K^{s}_{i(mn)\prod(s)}=0$.

We proceed by substituting the stress expansion (2.11) into the traction boundary condition at the top and bottom of the plate, which yields

$$\sigma^{R}_{i3}n_3\Big|_{y=\pm1}=\sum_{s=0}^{R}C^{s}_{i3(mn)\prod(s)}D^{(mn)\prod(s)}\left(\mathbf{U}^{R-s}\right)n_3\Big|_{y=\pm1}=0. \tag{2.34}$$

Since U^{R}_{i} is constant through the thickness we have the following boundary condition for the out-of-plane stiffness:

$$C^{s}_{i3(mn)\prod(s)}\Big|_{y=\pm1}=0. \tag{2.35}$$

Combining Eqs. (2.33) and (2.35) yields

$$K^{s}_{i(mn)\prod(s)}=-\int_{0}^{1}\begin{pmatrix}\sum_{p=1}^{s-1}\left(H^{p}_{i(k\prod_1)\prod(2..p)}\right)\\ \left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle\\ -\Delta C^{s-1}_{i\prod_{s}(mn)\prod(s-1)}\end{pmatrix}dy \tag{2.36}$$
$$s\geq 1.$$

Substituting Eq. (2.36) into (2.33) gives

$$C^{0}_{i3(mn)\prod(1)}=0 \tag{2.37}$$

and

$$C^{s}_{i3(mn)\prod(s)}=\Lambda\left(\int_{0}^{y}\begin{pmatrix}\sum_{p=1}^{s-1}\left(H^{p}_{i(k\prod_1)\prod(2..p)}\right)\\ \left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle\\ -\Delta C^{s-1}_{i\prod_{s}(mn)\prod(s-1)}\end{pmatrix}dy\right)$$
$$s\geq 1,$$

where

$$\Lambda(f(y))=f(y)-f(1).$$

The constant terms in Eq. (2.36) are zero for $s=0$ and 1.

3. SOLUTION OF THE LOCAL (THROUGH- THE-THICKNESS) PROBLEM

In this section we derive the closed-form expression for the concentration factors and calculate their average value to be used in macroscopic equations. We start by substituting the stiffness expansion (2.13) into (2.36) and solve for the highest-order concentration factor

$$\left(\partial y_3 H^1_{t(mn)}\right) = -\left[C_{i3t3}\right]^{-1} C_{i3mn} \tag{3.1}$$

and

$$\left(\partial y_3 H^{s+1}_{t(mn)\prod(s)}\right) =$$

$$[C_{i3t3}]^{-1}\Lambda\left(\int_0^y \begin{pmatrix} \sum_{p=1}^{s-1}\left(H^p_{i(k\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{i\prod_s(mn)\prod(s-1)}\end{pmatrix} dy\right)$$

$$-\left[C_{i3t3}\right]^{-1} C_{i3\prod_s k} H^s_{k(mn)\prod(s-1)}$$

for $s \geq 1$.

Integrating the above equation with respect to y yields

$$H^1_{t(mn)} = -\int_0^y \left[C_{i3t3}\right]^{-1} C_{i3mn} + G^1_{t(mn)} \tag{3.2}$$

and

$$H^{s+1}_{t(mn)\prod(s)} =$$

$$\int_0^y\left([C_{i3t3}]^{-1}\Lambda\left\{\int_0^y \begin{pmatrix} \sum_{p=1}^{s-1}\left(H^p_{i(k\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{i\prod_s(mn)\prod(s-1)}\end{pmatrix} dy\right\}dy\right)$$

$$-\int_0^y\left(\left[C_{i3t3}\right]^{-1} C_{i3\prod_s k} H^s_{k(mn)\prod(s-1)}\right)dy$$

$$+G^{s+1}_{t(mn)\prod(s)}$$

for $s \geq 1$.

Applying the normalization condition $\left(\left\langle H^s_{imn\Pi(s-1)}\right\rangle = 0 \,\text{for s} > 0\right)$ to Eq. (3.2) yields the solution for the concentration factors

$$H^1_{t(mn)} = -\int_0^y \left[C_{i3t3}\right]^{-1} C_{i3mn} \tag{3.3}$$

and

$$H^{s+1}_{t(mn)\prod(s)} =$$

$$+\Delta\left(\int_0^y\left([C_{i3t3}]^{-1}\Lambda\int_0^y \begin{pmatrix} l\sum_{p=1}^{s-1}\left(H^p_{i(k\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{i\prod_s(mn)\prod(s-1)}\end{pmatrix} dy\right)dy\right)$$

$$-\Delta\left(\int_0^y\left(\left[C_{i3t3}\right]^{-1} C_{i3\prod_s k} H^s_{k(mn)\prod(s-1)}\right)dy\right)$$

for $s \geq 1$.

Given the solutions for the concentration factors, we proceed with the evaluation of the stiffness terms in Eq. (2.13). The solution for the stiffness tensors of various orders is given by

$$C^{s}_{ij(mn)\prod(s)} = C_{ijkl}\left(\delta_{l\prod_s} H^{s}_{k(mn)\prod(s-1)} + \delta_{l3}\partial y_3 H^{s+1}_{k(mn)\prod(s)}\right) \quad (3.4)$$

for $s \geq 1$

and

$$C^{0}_{ij(mn)} = C_{ijkl}\left(\delta_{km}\delta_{\ln} + \delta_{l3}\partial y_3 H^{1}_{k(mn)}\right)$$
$$C^{0}_{i3kl} = 0.$$

The concentration factors are substituted into the stiffness equations to give

$$C^{s}_{ij(mn)\prod(s)} = \left(C_{ijt\prod_s} - C_{ijt3}\left[C_{r3t3}\right]^{-1} C_{r3\prod_s k}\right) H^{s}_{k(mn)\prod(s-1)}$$
$$+ C_{ijt3}\left[C_{t3r3}\right]^{-1}\Lambda\left(\int_0^y \left(\begin{array}{l} \sum_{p=1}^{s-1}\left(H^{p}_{r(k\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{k\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{r\prod_s(mn)\prod(s-1)} \end{array}\right) dy\right)$$

for $s \geq 1$

$$(3.5)$$

and

$$C^{0}_{\iota\varphi(\mu\nu)} = C_{\iota\varphi\mu\nu} - \frac{C_{\iota\varphi 33}C_{33\mu\nu}}{C_{3333}}.$$

The membrane stiffness can be obtained by allowing the indices i and j, in Eq. (3.5), span 1 and 2, which yields

$$C^{s}_{\iota\varphi(mn)\prod(s)} = \left(C^{0}_{\iota\varphi\kappa\prod_s}\right) H^{s}_{\kappa(mn)\prod(s-1)}$$
$$+ \frac{C_{\iota\varphi 33}}{C_{3333}}\Lambda\left(\int_0^y \left(\begin{array}{l} \sum_{p=1}^{s-1}\left(H^{p}_{3(t\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{t\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{3\prod_s(mn)\prod(s-1)} \end{array}\right) dy\right)$$

for $s \geq 1$.

The out-of-plane shear stiffness can be calculated by restricting i to 1 or 2 and forcing j to be 3

$$C^{s}_{\iota 3(mn)\prod(s)} = \Lambda\left(\int_0^y \left(\begin{array}{l} \sum_{p=1}^{s-1}\left(H^{p}_{\iota(t\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{t\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{\iota\prod_s(mn)\prod(s-1)} \end{array}\right) dy\right)$$

for $s \geq 1$.

The transverse normal stiffness can be evaluated by taking i and j in Eq. (3.5) to be 3

$$C^{s}_{33(mn)\prod(s)} = \Lambda\left(\int_0^y \left(\begin{array}{l} \sum_{p=1}^{s-1}\left(H^{p}_{3(t\prod_1)\prod(2..p)}\right) \\ \times\left\langle C^{s-1-p}_{t\prod_{p+1}(mn)\prod(p+2..s)}\right\rangle \\ -\Delta C^{s-1}_{3\prod_s(mn)\prod(s-1)} \end{array}\right) dy\right)$$

for $s \geq 1$.

The zero-order stiffness terms for membrane, shear and transverse normal stresses are:

$$C^0_{\iota\varphi(\mu\nu)} = C_{\iota\varphi\mu\nu} - \frac{C_{\iota\varphi 33} C_{33\mu\nu}}{C_{3333}}, \tag{3.6}$$

$$C^0_{\iota 3(mn)} = 0, \tag{3.7}$$

$$C^0_{33(mn)} = 0. \tag{3.8}$$

The first-order stiffness terms for membrane, shear, and transverse normal stresses are

$$C^1_{\iota\varphi(mn)\prod(1)} = y\left(C^0_{\iota\varphi\kappa\prod_1}\right) A_{\kappa(mn)}, \tag{3.9}$$

$$C^1_{\iota 3\mu\nu\prod(1)} = -\int\limits_0^y \left(\Delta C^0_{\iota\prod_1\mu\nu}\right) dy, \tag{3.10}$$

and

$$C^1_{33(mn)\prod(1)} = 0, \tag{3.11}$$

where

$$\mathrm{A}_{\alpha(mn)} = \mathrm{A}_{\alpha(nm)} \mathrm{and} \mathrm{A}_{\alpha m3} = \delta_{\alpha m}.$$

The second-order stiffness terms for membrane, shear, and transverse normal stresses are

$$\begin{aligned} C^2_{\iota\varphi(\mu\nu)\prod(2)} &= \left(C^0_{\iota\varphi\kappa\prod_2}\right) H^2_{\kappa(\mu\nu)\prod(1)} \\ &+ \tfrac{C_{\iota\varphi 33}}{C_{3333}} \Lambda \left(\int\limits_0^y \begin{pmatrix} H^1_{3\tau\prod_1} D^0_{\tau\prod_2(\mu\nu)} \\ -C^1_{3\prod_2(\mu\nu)\prod(1)} \end{pmatrix} dy \right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} & C^2_{\iota 3(mn)\prod(2)} = \\ & A_{\kappa_m n} \Lambda \left(\int\limits_0^y \left(C^0_{\iota\prod_2\prod(1)\kappa} y_3 \right) dy \right), \end{aligned} \tag{3.13}$$

$$\begin{aligned} & C^2_{33(\mu\nu)\prod(2)} = \\ & \Lambda\left(\int\limits_0^y \left(D^0_{\tau\prod_2(\mu\nu)} H^1_{3(\tau\prod_1)} - C^1_{3\prod_2(\mu\nu)\prod(1)} \right) dy \right). \end{aligned} \tag{3.14}$$

The third-order stiffness terms for membrane, shear, and transverse normal stresses are

$$\begin{aligned} & C^3_{\iota\varphi(mn)\prod(3)} = \\ & \left(C^0_{\iota\varphi\kappa\prod_3}\right) H^3_{\kappa(mn)\prod(2)} \\ & -\tfrac{C_{\iota\varphi 33}}{C_{3333}} \Lambda \left(\int\limits_0^y \left(\Delta C^2_{\prod_3 3(mn)\prod(2)} \right) dy \right), \end{aligned} \tag{3.15}$$

$$\begin{aligned} & C^3_{\iota 3(\mu\nu)\prod(3)} = \\ & \Lambda \left(\int\limits_0^y \begin{pmatrix} H^2_{\iota(t\prod_1)\prod_2} \left\langle C^0_{t\prod_3(\mu\nu)} \right\rangle \\ -\Delta C^2_{\iota\prod_3(\mu\nu)\prod(2)} \end{pmatrix} dy \right), \end{aligned} \tag{3.16}$$

$$C^3_{33(mn)\prod(3)} = -\Lambda \left(\int\limits_0^y \left(\Delta C^2_{3\prod_3(mn)\prod(2)} \right) dy \right). \tag{3.17}$$

The fourth-order stiffness terms for shear and transverse normal stresses are

$$\begin{aligned} & C^4_{\iota 3(mn)\prod(4)} = \\ & -\Lambda \left(\int\limits_0^y \begin{pmatrix} \delta_{\iota\prod_1} y D^2_{\prod_2 3(mn)\prod(3..4)} \\ +C^3_{\iota\prod_4(mn)\prod(3)} \end{pmatrix} dy \right), \end{aligned} \tag{3.18}$$

$$\begin{aligned} & C^4_{33(mn)\prod(4)} = \\ & \Lambda \left(\int\limits_0^y \begin{pmatrix} H^1_{3(t\prod_1)} \left\langle C^2_{t\prod_2(mn)\prod(3..4)} \right\rangle \\ +H^3_{3(t\prod_1)\prod(2..3)} \left\langle C^0_{t\prod_4(mn)} \right\rangle \\ -C^3_{3\prod_4(mn)\prod(3)} \end{pmatrix} dy \right). \end{aligned} \tag{3.19}$$

The transverse normal stiffness for the fifth order is

$$C^5_{33(mn)\prod(5)} = \Lambda\left(\int_0^y \begin{pmatrix} H^2_{33\prod_1\prod_2} D^2_{3\prod_3(mn)\prod(4..5)} \\ -\Delta C^4_{\prod_5 3(mn)\prod(4)} \end{pmatrix} dy\right). \tag{3.20}$$

Remark: The higher-order stiffness terms for membrane, shear, and transverse normal stresses differ from those obtained using static formulation. This is because the dynamic formulation accounts for the fact that the through-the-thickness stress distribution depends on the variation of the wave speed in different layers.

To obtain the overall or homogenized stiffness tensor we apply the averaging operator to the stiffness tensors of each order. For odd values of s, the stiffness tensors are antisymmetric, and thus, their average is zero. Applying the averaging operator to even-order stiffness tensors yields

$$D^s_{ij(mn)\prod(s)} = \left\langle C^s_{ij(mn)\prod(s)}\right\rangle = \left\langle C_{ijkl}\left(\delta_{l\prod_s} H^s_{k(mn)\prod(s-1)} + \delta_{l3}\partial y_3 H^{s+1}_{k(mn)\prod(s)}\right)\right\rangle \quad \text{for } s \geq 1 \tag{3.21}$$

and

$$D^0_{ij(mn)} = \left\langle C_{ijkl}\left(\delta_{km}\delta_{\ln} + \delta_{l3}\partial y_3 H^1_{k(mn)}\right)\right\rangle$$

$$C^0_{i3kl} = 0.$$

The zero-order homogenized stiffness tensors for membrane, out-of-plane shear and transverse normal stiffness are

$$D^0_{\iota\varphi(\mu\nu)} = \left\langle C_{\iota\varphi\mu\nu} - \frac{C_{\iota\varphi 33}C_{33\mu\nu}}{C_{3333}}\right\rangle, \tag{3.22}$$

$$D^0_{\iota 3(mn)} = 0, \tag{3.23}$$

and

$$D^0_{33(mn)} = 0. \tag{3.24}$$

The second-order, homogenized stiffness tensors are

$$D^2_{\iota\varphi(mn)\prod(2)} = \left\langle \begin{matrix} \left(C^0_{\iota\varphi\kappa\prod_2}\right) H^2_{\kappa(mn)\prod(1)} \\ + \frac{C_{\iota\varphi 33}}{C_{3333}}\Lambda\left(\int_0^y \begin{pmatrix} H^1_{3t\prod_1} D^0_{t\prod_2(mn)} \\ -C^1_{3\prod_2(mn)\prod(1)} \end{pmatrix} dy\right) \end{matrix}\right\rangle, \tag{3.25}$$

$$D^2_{\iota 3(mn)\prod(2)} = A_{\kappa mn}\left\langle \Lambda\left(\int_0^y \left(C^0_{\iota\prod_2\prod(1)\kappa} y_3\right) dy\right)\right\rangle, \tag{3.26}$$

$$D^2_{33(\mu\nu)\prod(2)} = \left\langle \Lambda\left(\int_0^y \begin{pmatrix} H^1_{3(t\prod_1)}\left\langle C^0_{t\prod_2(\mu\nu)}\right\rangle \\ -\Delta C^1_{3\prod_2(\mu\nu)\prod(1)} \end{pmatrix} dy\right)\right\rangle. \tag{3.27}$$

The fourth-order homogenized stiffnesses are

$$D^4_{\iota 3(mn)\prod(4)} = \Lambda\left(\int_0^y \begin{pmatrix} H^1_{\iota(t\prod_1)}\left\langle C^2_{t\prod_2(mn)\prod(3..4)}\right\rangle \\ +H^3_{\iota(t\prod_1)\prod(2..3)}\left\langle C^0_{t\prod_4(mn)}\right\rangle \\ -C^3_{\iota\prod_4(mn)\prod(3)} \end{pmatrix} dy\right), \tag{3.28}$$

$$D^4_{33(mn)\prod(4)} = \Lambda\left(\int_0^y \left(\begin{array}{l} H^1_{3(t\prod_1)}\left\langle C^2_{t\prod_2(mn)\prod(3..4)}\right\rangle \\ +H^3_{3(t\prod_1)\prod(2..3)}\left\langle C^0_{t\prod_4(mn)}\right\rangle \\ -C^3_{3\prod_4(mn)\prod(3)} \end{array}\right) dy\right). \tag{3.29}$$

It can be easily seen that that the membrane, shear, and transverse homogenized stiffness tensors of each order possess the following symmetry properties

$$\begin{array}{l} D^s_{(\iota\varphi)(mn)(\prod(s))} \\ D^s_{(\iota 3)(mn)(\prod(s))} \\ D^s_{33(mn)(\prod(s))}. \end{array} \tag{3.30}$$

4. NONLOCAL EQUATIONS OF MOTION

Substituting the stress expansions (2.12) and (2.13) into the homogenized equations of motion (2.24) yields the homogenized membrane equations of motion

$$\begin{array}{l} \sum\limits_{R=0}^{\infty}\varepsilon^R\left(\sum\limits_{s=0}^{R} D^s_{\iota\alpha(mn)\prod(s)} D^{(mn)\prod(s)\alpha}\left(\mathbf{U}^{R-s}\right)\right) \\ -\rho\sum\limits_{R=0}^{\infty}\varepsilon^R\left\langle a^R_\iota\right\rangle + b_\iota = 0. \end{array} \tag{4.1}$$

The homogenized bending equation of motion is given as

$$\begin{array}{l} \sum\limits_{R=0}^{\infty}\varepsilon^R\left(\sum\limits_{s=0}^{R} D^s_{3\alpha(mn)\prod(s)} D^{(mn)\prod(s)\alpha}\left(\mathbf{U}^{R-s}\right)\right) \\ -\rho\sum\limits_{R=0}^{\infty}\varepsilon^R\left\langle a^R_3\right\rangle + b_3 = 0. \end{array} \tag{4.2}$$

The average through-the-thickness displacement can be obtained by applying the averaging operator to the displacement expansion

$$U_i = \langle u_i(x,y,t)\rangle = \sum_{t=0}^{\infty}\varepsilon^t U^t_i(\mathbf{x}). \tag{4.3}$$

The homogenized equilibrium equations can be expressed in terms of the average displacement. Combining Eqs. (4.2) and (4.3), gives the nonlocal membrane equation of motion

$$\begin{array}{l} D^0_{\iota\alpha(mn)} D^{(mn)\alpha}(\mathbf{U}) \\ +\varepsilon^2 D^2_{(\iota\alpha)(mn)(\prod(2))} D^{(mn)\prod(2)\alpha}(\mathbf{U}) \\ -\rho U_{\iota,tt} + b_\iota = 0. \end{array} \tag{4.4}$$

We proceed by substituting the average displacement (4.3) into bending equation of motion (4.4) and truncate terms of order $O\left(\varepsilon^6\right)$ and higher, which yields the nonlocal bending equation of motion

$$\begin{array}{l} \varepsilon^2 D^2_{3\alpha(mn)\prod(2)} D^{(mn)\prod(2)\alpha}(\mathbf{U}) \\ +\varepsilon^4 D^4_{3\alpha(mn)\prod(4)} D^{(mn)\prod(4)\alpha}(\mathbf{U}) \\ -\rho U_{3,tt} + b_3 = 0. \end{array} \tag{4.5}$$

Note that the second term in Eq. (4.5) has a sixth-order spatial derivative. Regularization (or stabilization) procedure is introduced to reduce the order of differential equation and to filter out the high-frequency content [7, 8].

Following [7, 8] we approximate the second-order membrane stiffness tensor in terms of the zero-order membrane stiffness and unknown fourth-order tensor $V_{(\iota\varphi)\rho\prod_2}$, which represents the through-the-thickness variation

$$D^2_{(\iota\varphi)(\mu\nu)(\prod(2))} \cong V_{(\iota\varphi)\rho\prod_2} D^0_{(\mu\nu)(\prod_1\rho)}. \tag{4.6}$$

A least-squares approximation is used to approximate $V_{(\iota\varphi)\rho\prod_2}$. The resulting zero-order equilibrium equation is given as

$$D^0_{\mu\nu\prod_1\rho} D^{(\mu\nu)\prod(2)\varphi}(\mathbf{U}) = \rho\partial x_\varphi \partial x_{\prod_2} U_{\rho,tt} \\ -\partial x_\varphi \partial x_{\prod_2} b_\rho = O(\varepsilon^1). \tag{4.7}$$

Further assuming that the gradient of the body force is constant in the thickness direction yields:

$$D^0_{\iota\alpha(mn)} \partial x_\alpha e_{xmn}(\mathbf{U}) \\ +\varepsilon^2 V_{\iota\varphi\rho\prod_2} \rho \partial x_\varphi \partial x_{\prod_2} U_{\rho,tt} \\ -\rho U_{\iota,tt} + b_\iota = O(\varepsilon^3). \tag{4.8}$$

Equation (4.8) is referred to as the regularized or stabilized membrane equation of motion.

Consider the second-order terms in Eq. (4.5)

$$\varepsilon^2 D^2_{3\alpha(mn)\prod(2)} D^{(mn)\prod(2)\alpha}(\mathbf{U}) = \\ \rho U_{3,tt} - b_3 = 0. \tag{4.9}$$

Taking two spatial derivatives, while requiring that $b_{3,\alpha\beta} = 0$, yields

$$\varepsilon^2 D^2_{3\alpha(mn)\prod(2)} D^{(mn)\prod(4)\alpha}(\mathbf{U}) = \\ \rho U_{3,tt\prod(3..4)}. \tag{4.10}$$

To express the higher-order term in Eq. (4.5) in terms of the lower-order spatial gradient, the fourth-order bending stiffness $D^4_{3\alpha(mn)\prod(4)}$ is approximated in terms of the second-order bending stiffness $D^2_{3\alpha(mn)\prod(2)}$

$$D^4_{3\alpha(mn)\prod(4)} \sim V^2_{\prod_3\prod_4} D^2_{3\alpha(mn)\prod(2)}. \tag{4.11}$$

Substituting Eq. (4.11) into Eq. (4.5) yields

$$D^2_{3\alpha(mn)\prod(2)} D^{(mn)\prod(2)\alpha}(\mathbf{U}) + \\ \rho V^2_{\prod_3\prod_4} U_{3,\prod_3\prod_4 tt} - \rho U_{3,tt} + b_3 = 0. \tag{4.12}$$

The above is referred to as the regularized (stabilized), bending equation of motion. A least-squares approximation is used to approximate $D^4_{3\alpha(mn)\prod(4)} \sim V^2_{\prod_3\prod_4} D^2_{3\alpha(mn)\prod(2)}$. For more details we refer to [11].

Given the average displacement field $\mathbf{U}$ and the concentration factors the displacement, strain, and stress through the thickness distribution follow from Eqs. (2.7), (2.12), and (4.4).

5. MODEL VERIFICATION

For model verification we consider a beam problem aligned along the axis 1. Either plane-stress or plane-strain assumptions are made in direction 2. Piecewise homogenous properties are assumed in direction 3. Figure 2 shows the schematics of the beam. For details on the dimensional reduction from the plate equations to the beam problem we refer to [11].

For numerical verification we consider two examples: the membrane (bar) and the bending (beam) problems. We first consider a piecewise homogeneous, isotropic bar such that t is the thickness, E_p is the modulus, and v_p is the

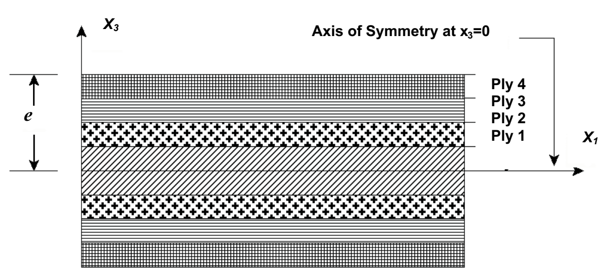

FIGURE 2. Schematics of the beam

Poisson ratio for ply p. The dimensions of the bar are as follows: $\varepsilon = .02m$, $L = 1m$. Material properties are symmetric about the center of the beam.

We consider a composite bar with four plies ($p = 4$). In the first example considered, denoted as bar 1, we study a bar with significantly reduced properties (to mimic existence of damage) in the second and mid-layers. For bar 2 the deviation of material properties is taken to be less pronounced. Material properties for the two bars are as follows:

$$\underline{\textbf{Bar 1}}: \mathbf{E} = \begin{Bmatrix} 200 \\ .01 \\ 50 \\ 1 \end{Bmatrix} Gpa \quad \mathbf{v} = \begin{Bmatrix} .3 \\ .2 \\ .3 \\ .3 \end{Bmatrix}$$

$$\underline{\textbf{Bar 2}}: \mathbf{E} = \begin{Bmatrix} 200 \\ 1 \\ 40 \\ 3 \end{Bmatrix} Gpa \quad \mathbf{v} = \begin{Bmatrix} .3 \\ .3 \\ .3 \\ .3 \end{Bmatrix}$$

We consider the following initial

$$U(x, t = 0) = f(x), \quad U_{,t}(x, t = 0) = 0$$

and boundary

$$U(x = 0, t) = 0, \quad \sigma_{11}(x = L, t > 0) = 0$$

conditions. The initial displacement function is: $f(x) = e_{load}x$ where e_{load} is the initial, constant strain.

We compare three bar models

- M02 – zero-order formulation with piecewise linear approximation in axial direction
- M22 – nonlocal formulation with piecewise linear approximation in axial direction
- M24 – nonlocal formulation with piecewise cubic approximation in axial direction and continuity of derivatives

with a two-dimensional reference solution (denoted as MPS). To obtain the reference solution (MPS) a sequence of two-dimensional meshes was considered to ensure convergence (up to the error of 10–5 in the energy norm) to the elasticity solution in 2D. For all problems considered, implicit time integration (Newmark method with β= 1/4, γ= 1/2) [12] has been employed. For more details on time integration for nonlocal equations, we refer to [7, 8]. Numerical results of the three models are compared to the reference solution in Figs. 3 and 4.

The results show significant improvement of the displacement solution compared to the zero-order theory in case of significant properties reduction in one of the layers. The wave speed for the higher-order solutions, M24 and M22, has been found to be in good agreement with the reference solution.

We proceed with the verification of the beam model. Plane-strain condition is assumed in direction 2. We consider a simply supported beam with length $L = 1$, width $b = 1$, and a uniformly distributed time-dependent load w given as

$$w = F_a \times g(t).$$

The function $g(t)$ is a half-sine function illustrated in Fig. 5 and $F_a = 500\frac{N}{m}$.

The beam is simply supported and composed of eight plies that have mirror symmetry about the mid-plane. All plies have the same thickness and $\varepsilon = .01m$. The following material properties are considered:

$$\mathbf{E} = \begin{Bmatrix} 1 \\ 50 \\ 5 \\ 200 \end{Bmatrix} Gpa \quad \mathbf{v} = \begin{Bmatrix} .3 \\ .3 \\ .3 \\ .3 \end{Bmatrix}.$$

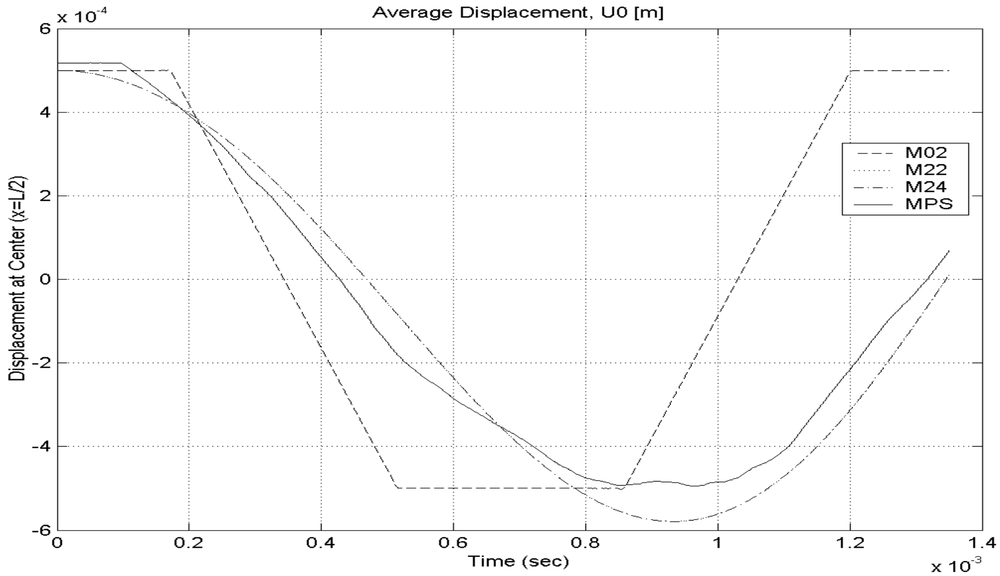

FIGURE 3. Comparison of average displacement 1 at x=L in Bar 1

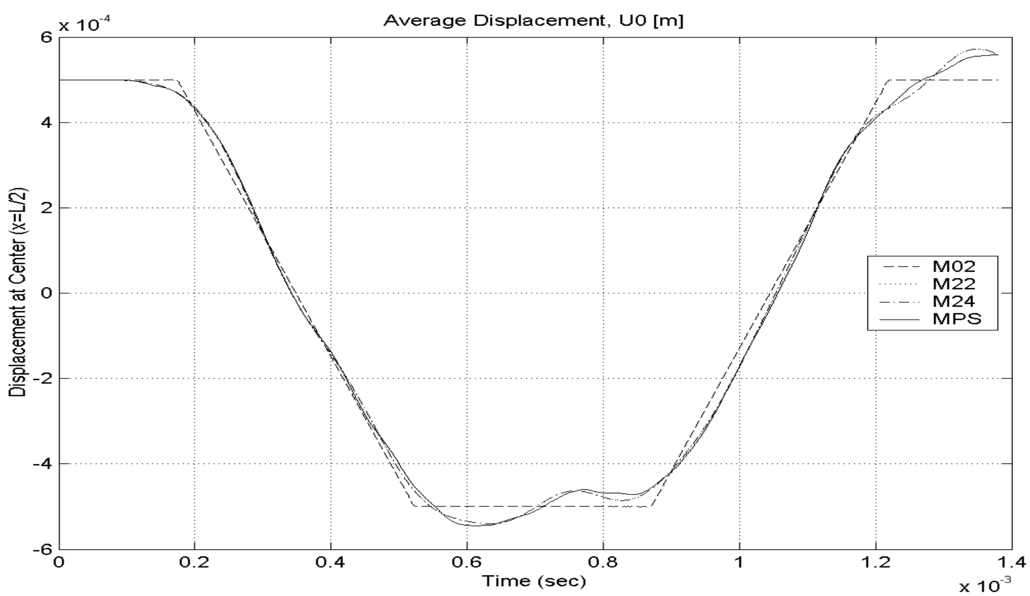

FIGURE 4. Comparison of average displacement 1 at x=L in Bar 2

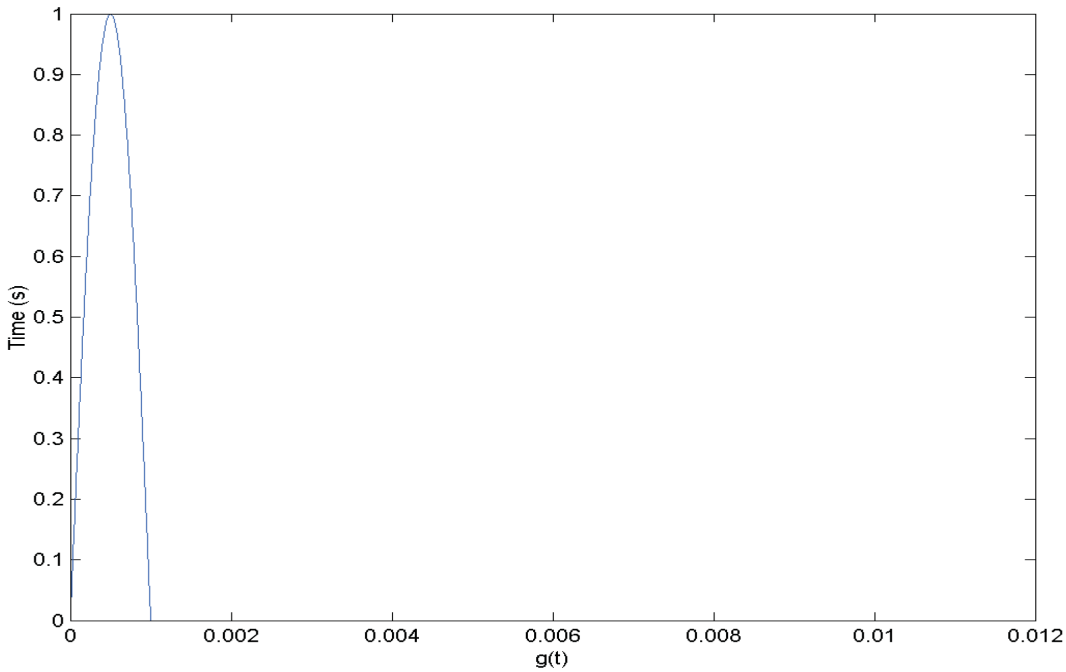

FIGURE 5. Loading function g(t)

Two beam models,

- M3F26 – the classical Kirchoff beam element
- M5VF6 – the proposed nonlocal beam element

are compared to MPS. The results are extracted at $x = L/4$. Figures 6 and 7 illustrate the time history of the axial stress at the top of the beam and the shear stress at the mid-plane, respectively. The nonlocal solution, M5VF26, has been found to be in good agreement with the reference solution. Shear stresses obtained from the classical solution seem to be out of phase with the reference solution.

Figures 8–10 show two snapshots in time for the axial, shear, and transverse normal stress distribution in the thickness direction. Unlike the classical beam model, the time response of the nonlocal formulation matches the reference solution reasonably well.

6. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

The response of the proposed multiscale laminated model has been found to be in good agreement with the reference solution. The multiscale laminated theory developed here provides a significant improvement over classical approaches for highly heterogeneous laminates (caused by ply damage or delamination) subjected to dynamic response.

This paper explores the possibility of tailoring the laminated theory to system dynamics. This concept has been found to be quite promising, but the implementation of the theory and

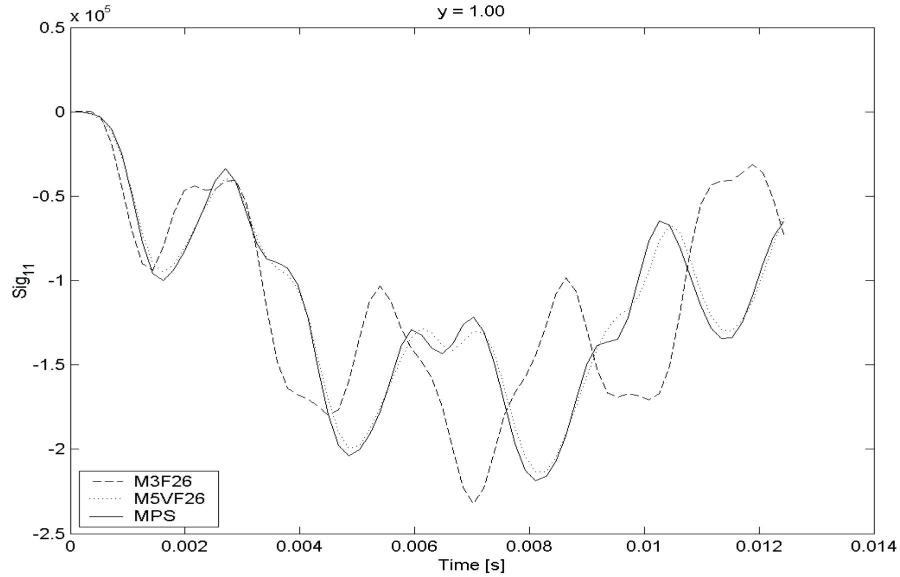

FIGURE 6. Axial stress at the top of the beam

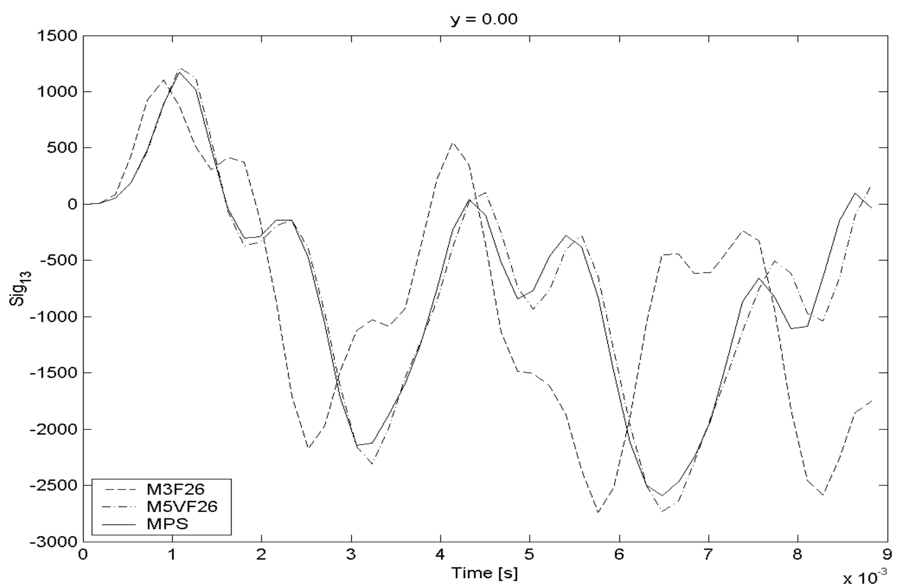

FIGURE 7. Shear stress at the mid-plane

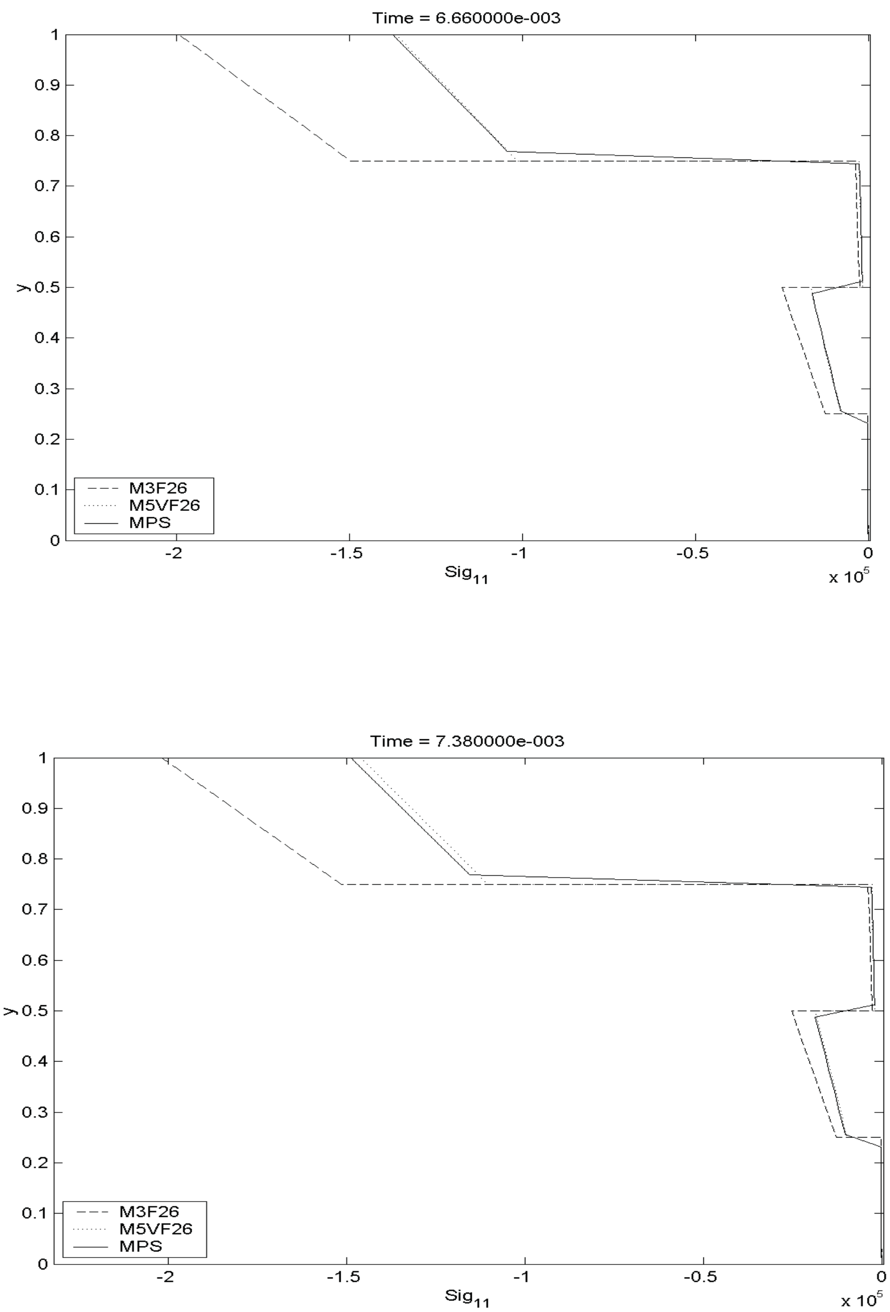

FIGURE 8. Axial stress at different snapshots in time

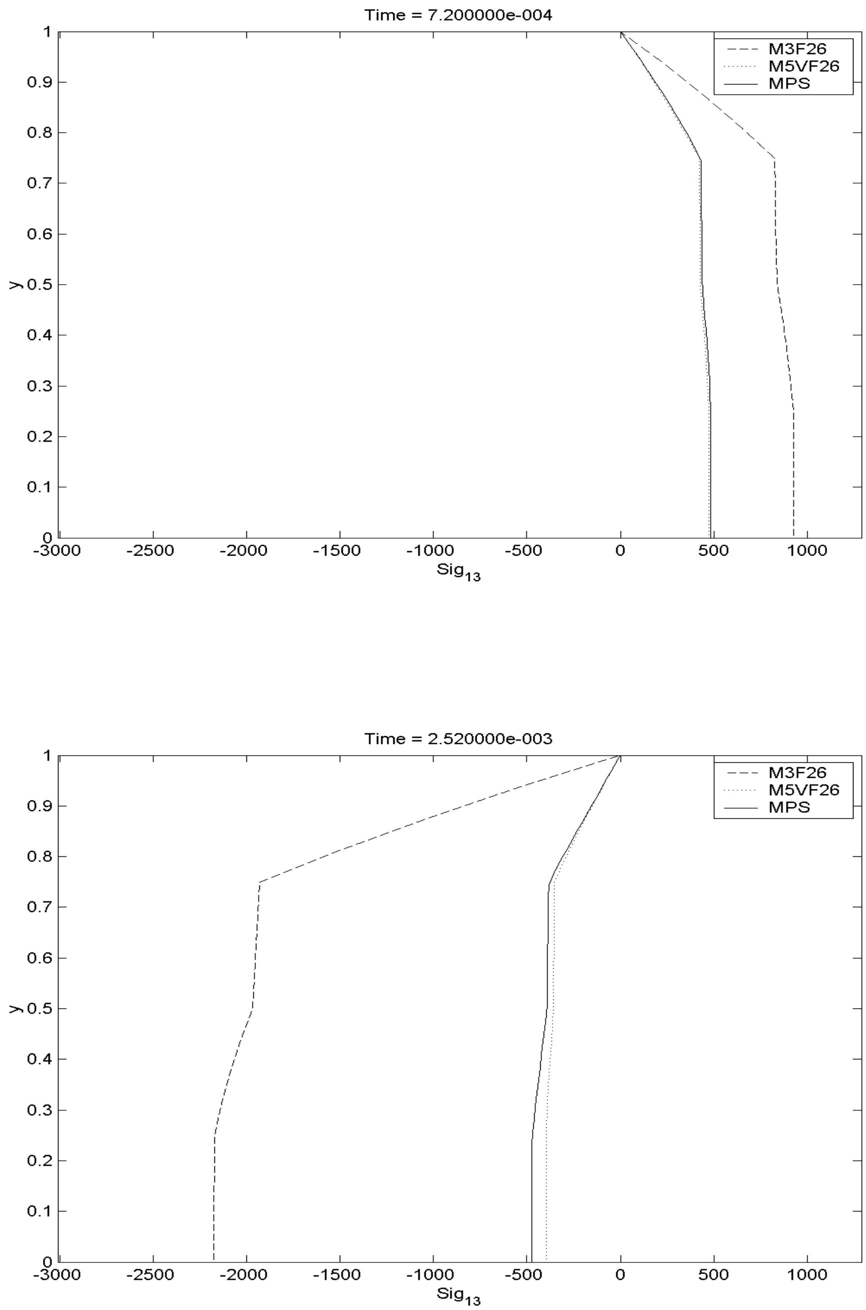

FIGURE 9. Shear stress at different snapshots in time g(t)

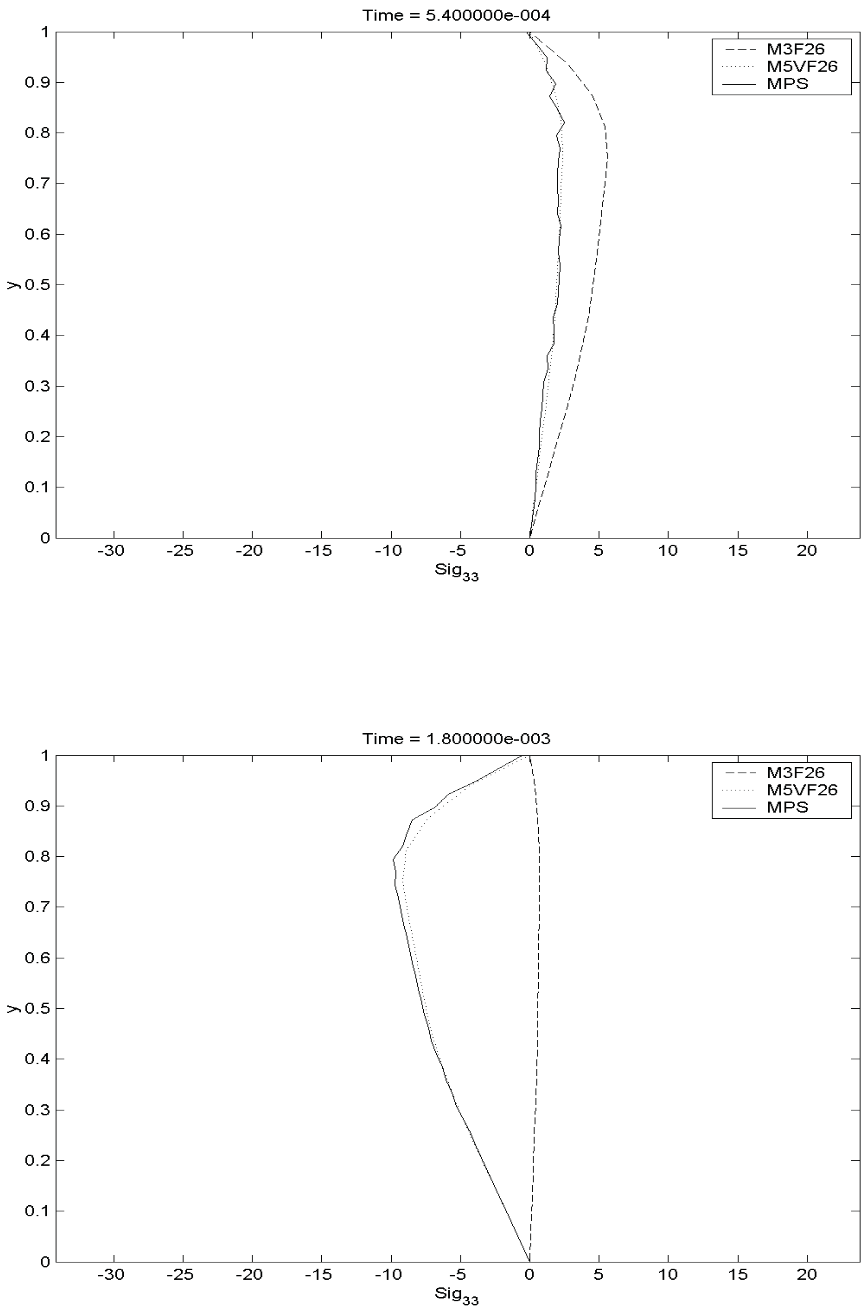

FIGURE 10. Normal stress at different snapshots in time

verification studies so far have been limited to simple model problems of beams and bars. Implementation of the theory to plates and shells as well the verification studies will be pursued in our future investigation.

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