Nonlocal Dispersive Model For Wave Propagation In Heterogeneous Media. Part 2: Multi-Dimensional Case

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Abstract: Three nondispersive models in multi-dimensions have been developed. The first model consists of a leading-order homogenized equation of motion subjected to the secularity constraints imposing uniform validity of asymptotic expansions. The second, nonlocal model, contains a fourth-order spatial derivative and thus requires C^1 continuous finite element formulation. The third model, which is limited to the constant mass density and a macroscopically orthototropic heterogeneous medium, requires C^0 continuity only and its finite element formulation is almost identical to the classical local approach with the exception of the mass matrix. The modified mass matrix consists of the classical mass matrix (lumped or consistent) perturbed with a stiffness matrix whose constitutive matrix depends on the unit cell solution. Numerical results are presented to validate the present formulations.

1. Introduction

The primary objective of the second part of this manuscript is to extend the one-dimensional nonlocal model developed in the first part [7] to multi-dimensions. We start by developing a mathematical homogenization theory up to the second order with multiple spatial and temporal scales. A variant of the dispersive model consisting of the leading-order homogenized equations of motion subjected to the secularity conditions imposing uniformly valid asymptotic expansion is formulated first. A nonlocal dispersive model is developed by adding together three sets of homogenized equations of motion. The resulting equation is independent of slow time scales, but contains fourth-order spatial derivative and thus requires C^1 continuous finite element formulation. For the case of constant mass density and macroscopically orthotropic heterogeneous medium the fourth-order spatial derivative term can be approximated by a mixed second-order derivative in space and time. The coefficients of the mixed derivative term can be constructed from the solution of the unit cell boundary value problem. Finite element formulation of this model is almost identical to the classical zero-order homogenization theory with an exception of the mass matrix. The modified mass matrix consists of the classical mass matrix (lumped or consistent) perturbed by a stiffness matrix term whose constitutive matrix coefficients depend on the unit cell solution.

2. Nonlocal Model

We consider waves propagating in elastic heterogeneous solid with a periodic microstructure. The problem of elastodynamics on the scale of material heterogeneity can be stated as follows:

$$\rho\ddot{u}_i - \sigma_{ij,j} = 0\,, \qquad \sigma_{ij} = C_{ijkl}e_{kl}\,, \qquad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ on } \Omega$$

$$\text{BCs: } u_i = \mathcal{g}_i \quad \text{on} \quad \Gamma_u\,, \qquad \sigma_{ij}n_j = \mathcal{h}_i \quad \text{on} \quad \Gamma_\sigma$$

$$\text{ICs:} \quad u_i(\boldsymbol{x}, t = 0) = f_i(\boldsymbol{x}) \quad , \qquad \dot{u}_i(\boldsymbol{x}, t = 0) = g_i(\boldsymbol{x}) \quad \text{on } \Omega \tag{1}$$

where Ω denotes the macroscopic domain of interest with boundary Γ; Γ_u and Γ_σ are boundary portions where displacements $\mathcal{g}_i$ and tractions $\mathcal{h}_i$ are prescribed, respectively, such that $\Gamma_u \cap \Gamma_\sigma = \varnothing$ and $\Gamma = \Gamma_u \cup \Gamma_\sigma$; n_i denotes the normal vector on Γ; u_i is the displacement vector, e_{ij} the small strain tensor, σ_{ij} the stress tensor, C_{ijkl} the elasticity tensor and ρ the mass density. The elasticity tensor and the mass density are locally periodic. We assume that micro-constituents possess homogeneous properties. The superposed dot denotes differentiation with respect to time, such that $\dot{u}_i, \ddot{u}_i$ are velocity and acceleration vectors, respectively. The comma followed by a subscript variable denotes the partial derivative. Summation convention over repeated subscripts is adopted, except for subscripts x and y. Bold face letters denote either vector or tensor quantities.

2.1 Asymptotic Analysis with Multiple Spatial-Temporal Scales

As usual in homogenization methods we assume the characteristic size of the macroscopic problem L to be much larger than the dimension of the heterogeneities l, i.e. $\varepsilon = l/L \ll 1$. The existence of two distinct scales introduces two spatial variables $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$. In addition to two spatial scales and the usual time scale denoted as $t_0 = t$, we introduce two slow time scales

$$t_1 = \varepsilon t\,, \quad t_2 = \varepsilon^2 t \tag{2}$$

to capture the long-term behavior of the homogenized solution and to resolve the problem of secularity [5][6]. Using the chain rule, the spatial and temporal derivatives can be expressed as

$$(\ )_{,i} = (\ )_{,x_i} + \varepsilon^{-1}(\ )_{,y_i}\,, \qquad \dot{(\ )} = (\ )_{,t_0} + \varepsilon(\ )_{,t_1} + \varepsilon^2(\ )_{,t_2} \tag{3}$$

The strain and stress tensors then take the following form

$$e_{ij}(\boldsymbol{u}) = e_{xij}(\boldsymbol{u}) + \varepsilon^{-1}e_{yij}(\boldsymbol{u})\,, \qquad \sigma_{ij} = C_{ijkl}[e_{xkl}(\boldsymbol{u}) + \varepsilon^{-1}e_{ykl}(\boldsymbol{u})] \tag{4}$$

where $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ are symmetric gradients with respect to the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively:

$$e_{xij}(\boldsymbol{u}) = u_{(i,x_j)} = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}), \qquad e_{yij}(\boldsymbol{u}) = u_{(i,y_j)} = \frac{1}{2}(u_{i,y_j} + u_{j,y_i}) \tag{5}$$

The equation of motion becomes

$$\rho\left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}\right)\left(\frac{\partial u_i}{\partial t_0} + \varepsilon\frac{\partial u_i}{\partial t_1} + \varepsilon^2\frac{\partial u_i}{\partial t_2}\right) - [\sigma_{ij,x_j} + \varepsilon^{-1}\sigma_{ij,y_j}] = 0 \tag{6}$$

Since the displacement field u_i depends on $\boldsymbol{x}$, $\boldsymbol{y}$, t_0, t_1 and t_2, a multiple-scale asymptotic expansion is employed to approximate the solution

$$u_i(\boldsymbol{x}, \boldsymbol{y}, t) = u_i^0(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) + \varepsilon u_i^1(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) + \varepsilon^2 u_i^2(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) + \ldots \tag{7}$$

Substituting the second equation in (4) into (6) yields

$$\rho[u_{i,t_0t_0} + 2\varepsilon u_{i,t_0t_1} + \varepsilon^2(2u_{i,t_0t_2} + u_{i,t_1t_1}) + 2\varepsilon^3 u_{i,t_1t_2} + \varepsilon^4 u_{i,t_2t_2}] =$$

$$\varepsilon^{-2}L^{-2}(u_i) + \varepsilon^{-1}L^{-1}(u_i) + L^0(u_i) \tag{8}$$

where

$$L^{-2}(u_i) = [C_{ijkl}e_{ykl}(\boldsymbol{u})]_{,y_j}$$

$$L^{-1}(u_i) = [C_{ijkl}e_{ykl}(\boldsymbol{u})]_{,x_j} + [C_{ijkl}e_{xkl}(\boldsymbol{u})]_{,y_j}$$

$$L^0(u_i) = [C_{ijkl}e_{xkl}(\boldsymbol{u})]_{,x_j} \tag{9}$$

The stress expansion can be obtained by substituting (7) into (4)

$$\sigma_{ij} = \varepsilon^{-1}\sigma_{ij}^{-1} + \sigma_{ij}^0 + \varepsilon\sigma_{ij}^1 + \varepsilon^2\sigma_{ij}^2 + \ldots \tag{10}$$

where

$$\sigma_{ij}^{-1} = C_{ijkl}e_{ykl}(\boldsymbol{u}^0), \quad \sigma_{ij}^s = C_{ijkl}[e_{xkl}(\boldsymbol{u}^s) + e_{ykl}(\boldsymbol{u}^{s+1})], \quad s = 0, 1, 2, \ldots \tag{11}$$

Inserting the asymptotic expansion (7) into (8) yields the following equations of motion for various orders:

$$O(\varepsilon^{-2}) : L^{-2}(u_i^0) = 0 \tag{12}$$

$$O(\varepsilon^{-1}) : L^{-2}(u_i^1) + L^{-1}(u_i^0) = 0 \tag{13}$$

$$O(1):\quad \rho u^0_{i,t_0t_0} = L^{-2}(u^2_i) + L^{-1}(u^1_i) + L^0(u^0_i) \tag{14}$$

$$O(\varepsilon):\quad \rho(u^1_{i,t_0t_0} + 2u^0_{i,t_0t_1}) = L^{-2}(u^3_i) + L^{-1}(u^2_i) + L^0(u^1_i) \tag{15}$$

$$O(\varepsilon^2):\quad \rho(u^2_{i,t_0t_0} + 2u^1_{i,t_0t_1} + 2u^0_{i,t_0t_2} + u^0_{i,t_1t_1}) = L^{-2}(u^4_i) + L^{-1}(u^3_i) + L^0(u^2_i) \tag{16}$$

where based on (9) and (11), the following holds

$$L^{-2}(u^0_i) = \sigma^{-1}_{ij,y_j}, \qquad L^{-2}(u^1_i) + L^{-1}(u^0_i) = \sigma^0_{ij,y_j} + \sigma^{-1}_{ij,x_j}$$

$$L^{-2}(u^{s+2}_i) + L^{-1}(u^{s+1}_i) + L^0(u^s_i) = \sigma^{s+1}_{ij,y_j} + \sigma^s_{ij,x_j}, \quad s = 0, 1, 2 \tag{17}$$

2.2 Resolution of Problems at Different Orders

Consider the $O(\varepsilon^{-2})$ equilibrium equation (12) first. Premultiplying it by u^0_i, integrating over the unit cell domain Y and subsequently integrating by parts yields

$$\int_{\partial_Y} u^0_i \sigma^{-1}_{ij} n_j ds - \int_Y u^0_{(i,y_j)} C_{ijkl} u^0_{(k,y_l)} dY = 0 \tag{18}$$

The boundary integral term in (18) vanishes due to Y-periodicity on the unit cell boundary ∂_Y. Furthermore, since C_{ijkl} is a positive definite fourth-order tensor, we have

$$u^0_{(i,y_j)} = 0 \quad \Rightarrow \quad u^0_i = u^0_i(\boldsymbol{x}, t_0, t_1, t_2) \quad \text{and} \quad \sigma^{-1}_{ij} = 0 \tag{19}$$

We proceed to the $O(\varepsilon^{-1})$ equilibrium equation (13). From (11), (17) and (19) follows

$$\sigma^0_{ij,y_j} + \sigma^{-1}_{ij,x_j} = [C_{ijkl}(e_{ykl}(\boldsymbol{u}^1) + e_{xkl}(\boldsymbol{u}^0))]_{,y_j} = 0 \tag{20}$$

As a consequence of linearity, the general solution to $\boldsymbol{u}^1$ takes the following form

$$u^1_i(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) = U^1_i(\boldsymbol{x}, t_0, t_1, t_2) + H_{ikl}(\boldsymbol{y}) e_{xkl}(\boldsymbol{u}^0) \tag{21}$$

where H_{ikl} is a Y-periodic third-rank tensor, which is symmetric with respect to indices k and l. Substituting (21) into (20) yields

$$[C_{ijkl}(G_{klmn} + \delta_{mk}\delta_{nl})]_{,y_j} e_{xmn}(\boldsymbol{u}^0) = 0 \tag{22}$$

where

$$G_{klmn}(\boldsymbol{y}) = \frac{1}{2}(H_{kmn,y_l} + H_{lmn,y_k}) = H_{(k,y_l)mn} \tag{23}$$

δ_{km} is the Kronecker delta. Since (22) is valid for an arbitrary combination of macroscopic strain field $e_{xmn}(\boldsymbol{u}^0)$, we get the governing equation over the unit cell domain

$$[C_{ijkl}(G_{klmn} + \delta_{km}\delta_{ln})]_{,y_j} = 0 \tag{24}$$

The solution H_{kmn} is sought in the space W defined by:

$$W = \{\boldsymbol{w}|\boldsymbol{w}\ \ Y-periodic, \langle\boldsymbol{w}\rangle = 0\} \tag{25}$$

where

$$\langle\ \cdot\ \rangle = |Y|^{-1}\int_Y \cdot\, dY \tag{26}$$

is the averaging operator. Thus the unit cell boundary value problem can be stated as:

$$C^0_{ijmn,y_j} = 0, \quad C^0_{ijmn}(\boldsymbol{y}) = C_{ijkl}(G_{klmn} + \delta_{km}\delta_{ln}), \quad \langle H_{kmn}(\boldsymbol{y})\rangle = 0 \tag{27}$$

For complex microstructures the finite element method is employed for discretization of $H_{ikl}(\boldsymbol{y})$, which yields a set of linear algebraic equations with six right-hand sides in 3D [8].

2.2.1 $O(1)$ *Homogenization*

Based on (14) and (17) the $O(1)$ perturbation equation can be written as

$$\rho u^0_{i,t_0t_0} = \sigma^1_{ij,y_j} + \sigma^0_{ij,x_j} \tag{28}$$

Applying the averaging operator defined in (26)-(28) and taking into account Y-periodicity of σ^1_{ij} gives

$$\langle\rho\rangle u^0_{i,t_0t_0} - \langle\sigma^0_{ij,x_j}\rangle = 0 \tag{29}$$

From (11) and (21), we have

$$\sigma^0_{ij} = C^0_{ijmn}(\boldsymbol{y})e_{xmn}(\boldsymbol{u}^0) \tag{30}$$

Inserting (30) into (29) yields the $O(1)$ macroscopic equation of motion:

$$\rho_0 u^0_{i,t_0t_0} - D^0_{ijmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_j} = 0 \tag{31}$$

where

$$\rho_0 = \langle\rho\rangle , \qquad D^0_{ijmn} = \langle C^0_{ijmn}(\boldsymbol{y})\rangle \tag{32}$$

The $O(1)$ macroscopic equation of motion is non-dispersive. In order to capture dispersion effects, higher-order terms will be considered in the subsequent sections.

2.2.2 $O(\varepsilon)$ *Homogenization*

Combining (11), (21) and (31), yields

$$\{C_{ijkl}[e_{ykl}(\boldsymbol{u}^2) + e_{xkl}(\boldsymbol{U}^1) + H_{kmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_l}]\}_{,y_j} =$$

$$[(\theta(\boldsymbol{y})D^0_{ijmn} - C^0_{ijmn}(\boldsymbol{y}))e_{xmn}(\boldsymbol{u}^0)]_{,x_j} \tag{33}$$

where

$$\theta(\boldsymbol{y}) = \rho(\boldsymbol{y})/\rho_0 \tag{34}$$

Linearity suggests that u^2_i may be sought in the form

$$u^2_i(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) = U^2_i(\boldsymbol{x}, t_0, t_1, t_2) + H_{ikl}(\boldsymbol{y})e_{xkl}(\boldsymbol{U}^1) + P_{ijmn}(\boldsymbol{y})(e_{xmn}(\boldsymbol{u}^0))_{,x_j} \tag{35}$$

where $P_{ijmn}(\boldsymbol{y})$ is a Y-periodic fourth-rank tensor. Substituting (35) into (33) yields

$$[C_{ijkl}(B_{klpmn} + H_{kmn}\delta_{lp})]_{,y_j}(e_{xmn}(\boldsymbol{u}^0))_{,x_p} = [\theta(\boldsymbol{y})D^0_{ipmn} - C^0_{ipmn}(\boldsymbol{y})](e_{xmn}(\boldsymbol{u}^0))_{,x_p} \tag{36}$$

where

$$B_{klpmn}(\boldsymbol{y}) = \frac{1}{2}(P_{kpmn,y_l} + P_{lpmn,y_k}) = P_{(k,y_l)pmn} \tag{37}$$

Equation (36) is valid for any combination of macroscopic strain gradients $(e_{xmn}(\boldsymbol{u}^0))_{,x_p}$. This yields the following unit cell boundary value problem:

$$C^1_{ijpmn,y_j} = \theta(\boldsymbol{y})D^0_{ipmn} - C^0_{ipmn}(\boldsymbol{y}) , \qquad C^1_{ijpmn}(\boldsymbol{y}) = C_{ijkl}(B_{klpmn} + H_{kmn}\delta_{lp})$$

$$\langle P_{ijmn}(\boldsymbol{y})\rangle = 0 \tag{38}$$

from which $P_{ijmn}(\boldsymbol{y})$ can be determined. The third equation in (38) is the normalization condition similar to (27) for H_{kmn}. Based on (17) the $O(\varepsilon)$ perturbation equation (15) can be written as

$$\rho(u^1_{i,t_0t_0} + 2u^0_{i,t_0t_1}) = \sigma^2_{ij,y_j} + \sigma^1_{ij,x_j} \tag{39}$$

Substituting (21) into (39) and applying the averaging operator as well as taking into account Y-periodicity of σ^2_{ij}, yields

$$\rho_0 U^1_{i,t_0t_0} + \langle \rho(\boldsymbol{y}) H_{ikl}(\boldsymbol{y}) \rangle (e_{xkl}(\boldsymbol{u}^0))_{,t_0t_0} + 2\rho_0 u^0_{i,t_0t_1} = \langle \sigma^1_{ij,x_j} \rangle \tag{40}$$

From (11), (21) and (35), we can derive

$$\sigma^1_{ij} = C^0_{ijmn}(\boldsymbol{y}) e_{xmn}(\boldsymbol{U}^1) + C^1_{ijpmn}(\boldsymbol{y})(e_{xmn}(\boldsymbol{u}^0))_{,x_p} \tag{41}$$

Inserting (41) into (40) yields the macroscopic equations of motion at $O(\varepsilon)$:

$$\rho_0 U^1_{i,t_0t_0} - D^0_{ijmn}(e_{xmn}(\boldsymbol{U}^1))_{,x_j} = D^1_{ijkmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_kx_j} - \langle \rho(\boldsymbol{y}) H_{ikl}(\boldsymbol{y}) \rangle (e_{xkl}(\boldsymbol{u}^0))_{,t_0t_0} - 2\rho_0 u^0_{i,t_0t_1} \tag{42}$$

where

$$D^1_{ijkmn} = \langle C^1_{ijkmn}(\boldsymbol{y}) \rangle \tag{43}$$

2.2.3 $O(\varepsilon^2)$ *Homogenization*

u^3_i can be determined from $O(\varepsilon)$ perturbation equation (39). Combining (11), (21), (35), (31) and (42), yields

$$\begin{aligned}\{C_{ijkl}[e_{ykl}(\boldsymbol{u}^3) + e_{xkl}(\boldsymbol{U}^2) + H_{kmn}(e_{xmn}(\boldsymbol{U}^1))_{,x_l} + P_{krmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_rx_l}]\}_{,y_j} = \\ [\theta(\boldsymbol{y}) D^0_{ijmn} - C^0_{ijmn}(\boldsymbol{y})](e_{xmn}(\boldsymbol{U}^1))_{,x_j} + \\ \{[\theta(\boldsymbol{y}) D^1_{ijlmn} - C^1_{ijlmn}(\boldsymbol{y})] + \theta(\boldsymbol{y})[H_{ikl} - \rho_0^{-1}\langle \rho H_{ikl} \rangle] D^0_{kjmn}\}(e_{xmn}(\boldsymbol{u}^0))_{,x_jx_l}\end{aligned} \tag{44}$$

Due to the linearity of (44) u^3_i can be sought in the form

$$u^3_i(\boldsymbol{x}, \boldsymbol{y}, t_0, t_1, t_2) = U^3_i(\boldsymbol{x}, t_0, t_1, t_2) + H_{ikl}(\boldsymbol{y}) e_{xkl}(\boldsymbol{U}^2) + P_{ijkl}(\boldsymbol{y})(e_{xkl}(\boldsymbol{U}^1))_{,x_j} +$$

$$Q_{ijkmn}(\boldsymbol{y})(e_{xmn}(\boldsymbol{u}^0))_{,x_k x_j} \tag{45}$$

Substituting (45) into (44) yields

$$[C_{ijkl}(A_{klprmn} + P_{krmn}\delta_{lp})]_{,y_j}(e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p} = \{[\theta(\boldsymbol{y})D^1_{irpmn} - C^1_{irpmn}(\boldsymbol{y})] +$$

$$\theta(\boldsymbol{y})[H_{ikp} - \rho_0^{-1}\langle\rho H_{ikp}\rangle]D^0_{krmn}\}(e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p} \tag{46}$$

where

$$A_{klprmn}(\boldsymbol{y}) = \frac{1}{2}(Q_{kprmn,y_l} + Q_{lprmn,y_k}) = Q_{(k,\,y_l)prmn} \tag{47}$$

Equation (46) is valid for any combination of macroscopic strain gradients $(e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p}$. Thus $Q_{ijkmn}(\boldsymbol{y})$ can be determined from the solution of the boundary value problem on the unit cell domain

$$C^2_{ijprmn,y_j} = [\theta(\boldsymbol{y})D^1_{irpmn} - C^1_{irpmn}(\boldsymbol{y})] + \theta(\boldsymbol{y})[H_{ikp} - \rho_0^{-1}\langle\rho H_{ikp}\rangle]D^0_{krmn}$$

$$C^2_{ijprmn}(\boldsymbol{y}) = C_{ijkl}(A_{klprmn} + P_{krmn}\delta_{lp})\,, \quad \langle Q_{ijkmn}(\boldsymbol{y})\rangle = 0 \tag{48}$$

Where the last equation in (48) is the normalization condition. The $O(\varepsilon^2)$ perturbation equation (16) can be rewritten as

$$\rho(u^2_{i,t_0t_0} + 2u^1_{i,t_0t_1} + 2u^0_{i,t_0t_2} + u^0_{i,t_1t_1}) = \sigma^3_{ij,y_j} + \sigma^2_{ij,x_j} \tag{49}$$

Substituting (21) and (35) into (49) and applying the averaging operator yields

$$\rho_0 U^2_{i,t_0t_0} + \langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{U}^1))_{,t_0t_0} + \langle\rho P_{ijmn}\rangle(e_{xmn}(\boldsymbol{u}^0))_{,x_j t_0 t_0} + 2\rho_0 U^1_{i,t_0t_1} +$$

$$2\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{u}^0))_{,t_0t_1} + 2\rho_0 u^0_{i,t_0t_2} + \rho_0 u^0_{i,t_1t_1} = \langle\sigma^2_{ij,x_j}\rangle \tag{50}$$

From (11), (35) and (45), we have

$$\sigma^2_{ij} = C^0_{ijmn}(\boldsymbol{y})e_{xmn}(\boldsymbol{U}^2) + C^1_{ijrmn}(\boldsymbol{y})(e_{xmn}(\boldsymbol{U}^1))_{,x_r} + C^2_{ijprmn}(\boldsymbol{y})(e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p} \tag{51}$$

Substituting (51) into (50) gives the $O(\varepsilon^2)$ macroscopic equations of motion:

$$\rho_0 U^2_{i,t_0t_0} - D^0_{ijmn}(e_{xmn}(\boldsymbol{U}^2))_{,x_j} = D^2_{ijprmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p x_j} + D^1_{ijrmn}(e_{xmn}(\boldsymbol{U}^1))_{,x_r x_j} -$$

$$\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{U}^1))_{,t_0t_0} - \langle\rho P_{ijmn}\rangle(e_{xmn}(\boldsymbol{u}^0))_{,x_jt_0t_0} - 2\rho_0 U^1_{i,t_0t_1} -$$

$$2\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{u}^0))_{,t_0t_1} - 2\rho_0 u^0_{i,t_0t_2} - \rho_0 u^0_{i,t_1t_1} \tag{52}$$

where

$$D^2_{ijprmn} = \langle C^2_{ijprmn}(\boldsymbol{y})\rangle \tag{53}$$

Remark 1: The expressions (38), (43), (48), (53) and (35) for $\boldsymbol{D}^1$, $\boldsymbol{D}^2$ and u_i^2 show that

$$\boldsymbol{D}^1 = O(\boldsymbol{C}\hat{l}) , \quad \boldsymbol{D}^2 = O(\boldsymbol{C}\hat{l}^2), \quad \langle\rho\boldsymbol{H}\rangle = O(\rho\hat{l}) , \quad \langle\rho\boldsymbol{P}\rangle = O(\rho\hat{l}^2) \tag{54}$$

where $\hat{l} = l/\varepsilon$ is the unit cell size in the stretched coordinate system $\boldsymbol{y}$. This implies that $\varepsilon\boldsymbol{D}^1 = O(\boldsymbol{C}l)$, $\varepsilon^2\boldsymbol{D}^2 = O(\boldsymbol{C}l^2)$, $\varepsilon\langle\rho\boldsymbol{H}\rangle = O(\rho l)$ and $\varepsilon^2\langle\rho\boldsymbol{P}\rangle = O(\rho l^2)$ can be directly calculated from known geometric and material properties of micro-constituents, independent of the value of ε. Furthermore, when the mass density is constant within the unit cell, the tensors $\langle\rho\boldsymbol{H}\rangle$ and $\langle\rho\boldsymbol{P}\rangle$ vanish and the unit cell boundary value problems can be greatly simplified.

3. Dispersive Models

The macroscopic equations of motion are given in (31), (42) and (52). The initial and boundary conditions for the above equations are given as

$$\text{ICs:} \quad u_i^0(\boldsymbol{x}, 0, 0, 0) = f_i(\boldsymbol{x}) , \quad \dot{u}_i^0(\boldsymbol{x}, 0, 0, 0) = g_i(\boldsymbol{x})$$

$$U_i^s(\boldsymbol{x}, 0, 0, 0) = 0, \quad \dot{U}_i^s(\boldsymbol{x}, 0, 0, 0) = 0 \quad (s = 1, 2) \tag{55}$$

$$\text{BCs:} \; u_i^0 = \mathscr{g}_i \quad \text{on } \Gamma_u, \qquad [D^0_{ijkl}e_{xkl}(\boldsymbol{u}^0)]n_j = \mathscr{h}_i \quad \text{on } \Gamma_\sigma$$

$$U_i^s = 0 \quad \text{on } \Gamma_u, \quad [D^0_{ijkl}e_{xkl}(\boldsymbol{U}^s)]n_j = 0 \quad \text{on } \Gamma_\sigma \quad (s = 1, 2) \tag{56}$$

It has been shown in [5][6] that the right-hand-side terms in (42) and (52) give rise to secular asymptotic expansions, i.e. higher order terms grow unbounded in time. In order to resolve the problem of secularity, we set the right-hand-side terms to zero. The secularity free solution can be obtained by solving the leading-order macroscopic equations (31):

$$\rho_0 u^0_{i,t_0t_0} - D^0_{ijmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_j} = 0 \tag{57}$$

subjected to the secularity constraints:

$$D^1_{ijkmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_kx_j} - \langle\rho(\boldsymbol{y})H_{ikl}(\boldsymbol{y})\rangle(e_{xkl}(\boldsymbol{u}^0))_{,t_0t_0} - 2\rho_0 u^0_{i,t_0t_1} = 0 \qquad (58)$$

$$D^2_{ijprmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_rx_px_j} + D^1_{ijrmn}(e_{xmn}(\boldsymbol{U}^1))_{,x_rx_j} - \langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{U}^1))_{,t_0t_0} - 2\rho_0 U^1_{i,t_0t_1} -$$

$$\langle\rho P_{ijmn}\rangle(e_{xmn}(\boldsymbol{u}^0))_{,x_jt_0t_0} - 2\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{u}^0))_{,t_0t_1} - 2\rho_0 u^0_{i,t_0t_2} - \rho_0 u^0_{i,t_1t_1} = 0 \qquad (59)$$

Analytical solution of (57)-(59) for one-dimensional problems has been given in [5][6].

3.1 Nonlocal Equations of Motion

In this section we develop an alternative approach by which the three sets of macroscopic equations are combined into a single equation and the dependence on slow time is eliminated. We start by defining the mean displacement as

$$U_i(\boldsymbol{x}, t) = \langle u_i(\boldsymbol{x}, \boldsymbol{y}, t)\rangle = u^0_i + \varepsilon U^1_i + \varepsilon^2 U^2_i + \ldots \qquad (60)$$

Multiplying (42) and (52) by ε and ε^2, respectively, and then adding the resulting equations to the leading-order macroscopic equations (31) gives

$$\rho_0(u^0_i + \varepsilon U^1_i + \varepsilon^2 U^2_i)_{,t_0t_0} - D^0_{ijmn}(e_{xmn}(\boldsymbol{u}^0 + \varepsilon\boldsymbol{U}^1 + \varepsilon^2\boldsymbol{U}^2))_{,x_j} =$$

$$\varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_rx_px_j} + \varepsilon D^1_{ijkmn}(e_{xmn}(\boldsymbol{u}^0 + \varepsilon\boldsymbol{U}^1))_{,x_kx_j} -$$

$$\varepsilon\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{u}^0 + \varepsilon\boldsymbol{U}^1))_{,t_0t_0} - 2\varepsilon^2\langle\rho H_{ikl}\rangle(e_{xkl}(\boldsymbol{u}^0))_{,t_0t_1} -$$

$$\varepsilon^2\langle\rho P_{ijmn}\rangle(e_{xmn}(\boldsymbol{u}^0))_{,x_jt_0t_0} - 2\varepsilon\rho_0(u^0_i + \varepsilon U^1_i)_{,t_0t_1} - \varepsilon^2\rho_0(u^0_{i,t_1t_1} + 2u^0_{i,t_0t_2}) \qquad (61)$$

Exploiting the relations

$$u^0_i + \varepsilon U^1_i + \varepsilon^2 U^2_i = U_i + O(\varepsilon^3)\,,\ \varepsilon^2 u^0_i = \varepsilon^2 U_i + O(\varepsilon^3)\,,\ \varepsilon(u^0_i + \varepsilon U^1_i) = \varepsilon U_i + O(\varepsilon^3)$$

$$U_{i,t_0t_0} + 2\varepsilon U_{i,t_0t_1} + \varepsilon^2(U_{i,t_1t_1} + 2U_{i,t_0t_2}) = \ddot{U}_i + O(\varepsilon^3)$$

$$\varepsilon(U_{i,t_0t_0} + 2\varepsilon U_{i,t_0t_1}) = \varepsilon\ddot{U}_i + O(\varepsilon^3)\,, \quad \varepsilon^2 U_{i,t_0t_0} = \varepsilon^2\ddot{U}_i + O(\varepsilon^3) \qquad (62)$$

and neglecting terms of order $O(\varepsilon^3)$ and higher in (61), we get one set of macroscopic equations of motion with respect to the mean displacement:

$$\rho_0\ddot{U}_i - D^0_{ijmn}(e_{xmn}(\boldsymbol{U}))_{,x_j} - \varepsilon D^1_{ijkmn}(e_{xmn}(\boldsymbol{U}))_{,x_kx_j} - \varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{U}))_{,x_rx_px_j} +$$

$$\varepsilon\langle\rho H_{ikl}\rangle e_{xkl}(\ddot{\boldsymbol{U}}) + \varepsilon^2\langle\rho P_{ijmn}\rangle(e_{xmn}(\ddot{\boldsymbol{U}}))_{,x_j} = 0 \tag{63}$$

For the case of constant mass density within the unit cell, $\langle\rho H_{ikl}\rangle = \langle\rho P_{ijmn}\rangle = 0$ and consequently equation (63) can be simplified as:

$$\rho\ddot{U}_i - D^0_{ijmn}(e_{xmn}(\boldsymbol{U}))_{,x_j} - \varepsilon D^1_{ijkmn}(e_{xmn}(\boldsymbol{U}))_{,x_k x_j} - \varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{U}))_{,x_r x_p x_j} = 0 \tag{64}$$

Moreover, for the case of macroscopically orthotropic materials $D^1_{ijkmn} = 0$ and equation (64) can be further simplified as:

$$\rho\ddot{U}_i - D^0_{ijmn}(e_{xmn}(\boldsymbol{U}))_{,x_j} - \varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{U}))_{,x_r x_p x_j} = 0 \tag{65}$$

In the remainder of this paper, attention is restricted to the approximation and numerical implementation of (65). This equation contains the sixth-rank tensor, which can be evaluated by solving the unit cell boundary value problems up to the second order. The highest spatial derivatives appearing in (65) is fourth order and therefore C^1 continuity is required for the finite element implementation. Moreover, the two sets of physically meaningful boundary conditions are insufficient to define a well-posed initial-boundary value problem. The one-dimensional counterpart of (65) is known as a “bad” Boussinesq equation, which yields meaningless solution for the case of oscillatory loading [7]. To resolve these difficulties we will attempt to approximate the fourth-order spatial derivative in terms of the mixed second-order spatial-temporal derivative. The resulting approximation will be termed as a “Good” Boussinesq problem.

3.2 The “Good” Boussinesq problem

We start by establishing the relation between $\boldsymbol{D}^2$ and $\boldsymbol{D}^0$. From the leading-order unit cell boundary value problem (27), we have the following integral equation over the unit cell domain

$$\int_Y Q_{iprkl} C^0_{ijmn,y_j} dY = \int_{\partial_Y} Q_{iprkl} C^0_{ijmn} n_j ds - \int_Y Q_{iprkl,y_j} C^0_{ijmn} dY = 0 \tag{66}$$

The boundary integral vanishes due to periodicity, and equation (66) becomes

$$\int_Y Q_{iprkl,y_j} C_{ijst}(G_{stmn} + \delta_{ms}\delta_{nt}) dY = 0 \tag{67}$$

from which we have

$$\int_Y A_{ijprkl} C_{ijmn} dY = -\int_Y A_{ijprkl} C_{ijst} G_{stmn} dY \tag{68}$$

where $\boldsymbol{A}$ is the symmetric gradient of $\boldsymbol{Q}$. From (48) and (68) we get

$$\int_Y C^2_{ijprkl} dY = -\int_Y A_{mnprkl} C_{mnst} G_{stij} dY + \int_Y C_{ijpq} P_{qrkl} dY \tag{69}$$

Similarly, from the second-order unit cell boundary value problem (48), we have the following integral equation

$$\int_Y H_{ist} C^2_{ijprmn,y_j} dY = \int_Y H_{ist} \{ [\theta(\boldsymbol{y}) D^1_{irpmn} - C^1_{irpmn}(\boldsymbol{y})] +$$

$$\theta(\boldsymbol{y}) [H_{ikp} - \rho_0^{-1} \langle \rho H_{ikp} \rangle] D^0_{krmn} \} dY \tag{70}$$

Integrating the left-hand side of (70) by parts with consideration of periodicity and inserting the expression for $\boldsymbol{C}^2$ gives

$$-\int_Y A_{mnprkl} C_{mnst} G_{stij} dY = \int_Y [H_{sij} C^2_{stprkl,y_t} + G_{stij} C_{stpq} P_{qrkl}] dY \tag{71}$$

Substituting (71) into (69) and (70), we have

$$\int_Y C^2_{ijprkl} dY = \int_Y P_{qrkl} C^0_{pqij} dY + \int_Y H_{sij} \{ [\theta(\boldsymbol{y}) D^1_{srpkl} - C^1_{srpkl}(\boldsymbol{y})] +$$

$$\theta(\boldsymbol{y}) [H_{spq} - \rho_0^{-1} \langle \rho H_{spq} \rangle] D^0_{qrkl} \} dY \tag{72}$$

From the first-order unit cell boundary value problem (38), we have the following integral equation

$$\int_Y P_{irkl} C^1_{ijpmn,y_j} dY = \int_Y P_{irkl} [\theta(\boldsymbol{y}) D^0_{ipmn} - C^0_{ipmn}(\boldsymbol{y})] dY \tag{73}$$

Similarly

$$\int_Y P_{irkl} C^0_{ipmn}(\boldsymbol{y}) dY = \int_Y \theta(\boldsymbol{y}) P_{irkl} D^0_{ipmn} dY + \int_Y B_{ijrkl} C^1_{ijpmn} dY \tag{74}$$

Substituting (74) into (72) and using the notation for the averaging operator $\langle \ \rangle$ gives

$$D^2_{ijprkl} = \rho_0^{-1} \langle \rho P_{qrkl} \rangle D^0_{pqij} + \rho_0^{-1} \langle \rho H_{sij} \rangle D^1_{srpkl} + \langle B_{mnrkl} C^1_{mnpij} \rangle -$$

$$\langle H_{sij} C^1_{srpkl} \rangle + \rho_0^{-1} \langle \rho H_{sij} H_{spq} \rangle D^0_{qrkl} - \rho_0^{-2} \langle \rho H_{sij} \rangle \langle \rho H_{spq} \rangle D^0_{qrkl} \tag{75}$$

In the case of constant mass density, equation (75) reduces to

$$D^2_{ijprkl} = \langle B_{mnrkl}C^1_{mnpij} - H_{sij}C^1_{srpkl}\rangle + \langle H_{sij}H_{spq}\rangle D^0_{qrkl} \tag{76}$$

Using least square approximation of the first term in (76), $\langle B_{mnrkl}C^1_{mnpij} - H_{sij}C^1_{srpkl}\rangle$ can be approximated by $V_{ijpq}D^0_{qrkl}$ and thus the relation between $\boldsymbol{D}^2$ and $\boldsymbol{D}^0$ can be expressed as:

$$D^2_{ijprkl} = [V_{ijpq} + \langle H_{sij}H_{spq}\rangle]D^0_{qrkl} \tag{77}$$

Utilizing the approximation (77) yields

$$\begin{aligned}\varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{U}))_{,x_r x_p x_j} &= \varepsilon^2[V_{ijpq} + \langle H_{sij}H_{spq}\rangle]D^0_{qrmn}(e_{xmn}(\boldsymbol{U}))_{,x_r x_p x_j} \\ &= \varepsilon^2[V_{ijpq} + \langle H_{sij}H_{spq}\rangle](D^0_{qrmn}e_{xmn}(\boldsymbol{u}^0))_{,x_r x_p x_j} + O(\varepsilon^3)\end{aligned} \tag{78}$$

From the leading-order macroscopic equations of motion (31) and the relations in (62), we have

$$\varepsilon^2 D^0_{qrmn}(e_{xmn}(\boldsymbol{u}^0))_{,x_r} = \varepsilon^2\rho_0 u^0_{q,t_0t_0} = \varepsilon^2\rho_0 U_{q,t_0t_0} + O(\varepsilon^3) = \varepsilon^2\rho_0\ddot{U}_q + O(\varepsilon^3) \tag{79}$$

Inserting (79) into (78) gives

$$\varepsilon^2 D^2_{ijprmn}(e_{xmn}(\boldsymbol{U}))_{,x_r x_p x_j} = \varepsilon^2\rho_0[V_{ijpq} + \langle H_{sij}H_{spq}\rangle]\ddot{U}_{q,x_p x_j} + O(\varepsilon^3) \tag{80}$$

Substituting (80) into (65) and neglecting terms of $O(\varepsilon^3)$ and higher yields the so called “Good” Boussinesq problem in multi-dimensions:

$$\rho_0\ddot{U}_i - D^0_{ijkl}(e_{xkl}(\boldsymbol{U}))_{,x_j} - \rho_0 E^k_{ijkl}(e_{xkl}(\ddot{\boldsymbol{U}}))_{,x_j} = 0 \tag{81}$$

where

$$E^k_{ijpq} = \varepsilon^2[V_{ijpq} + \langle H_{sij}H_{spq}\rangle] \tag{82}$$

Remark 2: For macroscopically isotropic materials the first term in (82) is much smaller than the second and thus $\boldsymbol{E}^k$ can be approximated in terms of the leading-order unit cell boundary value solution and thus eliminating the need for solving higher-order unit cell boundary value problems. It can be seen that $\boldsymbol{E}^k$ introduces the length-scale into the macroscopic equations of motion.

3. Finite Element Formulation

In this section we focus on the finite element semi-discretization of equation (81). Since the highest spatial derivatives appearing in (81) is second-order, the usual C^0 finite element approximation is sufficient. The weak statement of the problem is formulated as follows. For each $t \in (0, T_0]$, find $U_i(\boldsymbol{x}, t) \in H^1(\Omega)$, such that $U_i(\boldsymbol{x}, t) = \mathcal{g}_i$ on Γ_u and

$$\int_{\Omega} \rho_0 w_i \ddot{U}_i d\Omega - \int_{\Omega} w_i D^0_{ijkl}(e_{xkl}(\boldsymbol{U}))_{,x_j} d\Omega - \int_{\Omega} \rho_0 w_i E^k_{ijkl}(e_{xkl}(\ddot{\boldsymbol{U}}))_{,x_j} d\Omega = 0 \tag{83}$$

$$U_i(\boldsymbol{x}, 0) = f_i(\boldsymbol{x}), \qquad \dot{U}_i(\boldsymbol{x}, 0) = g_i(\boldsymbol{x}) \tag{84}$$

for all admissible test functions $w_i(\boldsymbol{x}) \in H^1_0(\Omega)$, where $H^1(\Omega)$ is the Sobolev space defined as

$$H^1(\Omega) = \left\{ \boldsymbol{v}(\boldsymbol{x}) \in L^2(\Omega), \boldsymbol{v}_{,x_i} \in L^2(\Omega) \right\} \tag{85}$$

with $L^2(\Omega)$ denoting the set of square-integrable functions over Ω, and

$$H^1_0(\Omega) = \{ \boldsymbol{w}(\boldsymbol{x}) \in H^1(\Omega) | \boldsymbol{w}(\boldsymbol{x}) = 0 \;\; on \;\; \Gamma_u \} \tag{86}$$

Integrating (83) by parts and accounting for major symmetry of $\boldsymbol{D}^0$ and $\boldsymbol{E}^k$, we have the weak form:

$$\int_{\Omega} \rho_0 w_i \ddot{U}_i d\Omega + \int_{\Omega} e_{xij}(\boldsymbol{w}) D^0_{ijkl} e_{xkl}(\boldsymbol{U}) d\Omega + \int_{\Omega} \rho_0 e_{xij}(\boldsymbol{w}) E^k_{ijkl} e_{xkl}(\ddot{\boldsymbol{U}}) d\Omega =$$

$$\int_{\Gamma_\sigma} w_i h_i ds + \int_{\Gamma_\sigma} \rho_0 w_i n_j E^k_{ijkl} e_{xkl}(\ddot{\boldsymbol{U}}) ds \tag{87}$$

Following *Remark 3* in Part I [7] of this two-part manuscript the second boundary term in (87) can be neglected provided that the wavelengths are significantly larger than the unit cell size. Otherwise it contributes a nonsymmetric term to the mass matrix.

Finite element approximation of the above weak form leads to the semi-discrete equations of motion:

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \tag{88}$$

where $\boldsymbol{d}(t)$ is the vector of nodal displacements; $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{F}$ are the system mass and stiffness matrices as well as the load vector, respectively:

$$\boldsymbol{M} = \sum_{e=1}^{Ne} \boldsymbol{m^e}, \quad \boldsymbol{K} = \sum_{e=1}^{Ne} \boldsymbol{k^e}, \quad \boldsymbol{F} = \sum_{e=1}^{Ne} \boldsymbol{f^e} \tag{89}$$

$$\boldsymbol{m^e} = \int_{\Omega_e} \rho_0 \boldsymbol{N^T N} d\Omega + \int_{\Omega_e} \rho_0 \boldsymbol{B^T E}^k \boldsymbol{B} d\Omega, \quad \boldsymbol{k^e} = \int_{\Omega_e} \boldsymbol{B^T D}^0 \boldsymbol{B} d\Omega, \quad \boldsymbol{f^e} = \int_{\Gamma_{\sigma e}} \boldsymbol{N^T h} ds \tag{90}$$

where $\boldsymbol{N}$ and $\boldsymbol{B}$ are the shape function and the symmetric gradient of $\boldsymbol{N}$; $\boldsymbol{m^e}$, $\boldsymbol{k^e}$, $\boldsymbol{f^e}$ the element mass matrix, stiffness matrix, and force vector, respectively; $\boldsymbol{D}^0$ the homogenized elasticity matrix and $\boldsymbol{E}^k$ the matrix constructed from the elements of E^k_{ijkl}.

Equation (88) is integrated in the time domain using standard time integration schemes.

4. Numerical Results

To validate the proposed nonlocal model, two-dimensional and three-dimensional problems are considered. Numerical examples are restricted to macroscopically isotropic medium with constant mass density. Problems involving randomly or periodically distributed particles, such as concrete, dense polycrystals and short fiber composites, fall into this category.

Problem 1: To compare the solution of the nonlocal model to the solutions of the classical homogenization model and the source heterogeneous problem, a two-dimensional plain strain problem as shown in Figure 1 is considered. The left edge is fixed while the right edge is subjected to impact load $q(t) = a_0(t-T/2)t^4(t-T)^4[1-h(t-T)]$. a_0 is scaled so that $-1 \le q(t) \le 1$; T is the duration of the impact pulse, and $h(t)$ denotes the Heaviside step function. The function $q(t)$ generates a Gaussian-like shape pulse.

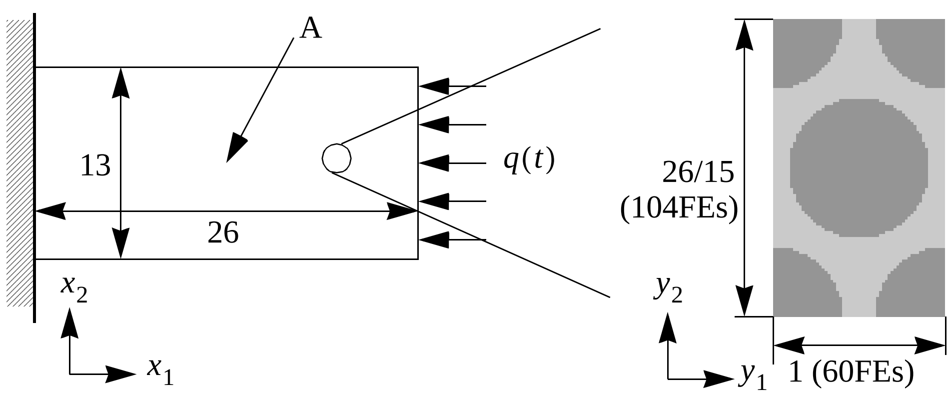


Figure 1: Two-dimensional plain strain multi-scale problem

The microstructure consists of hexagonally arranged circular fibers embedded in matrix material. This configuration is macroscopically isotropic. The volume fraction ratio of fibers is 0.60. The Young's modulus of fibers and matrix are $E_a = 50$, and $E_m = 1$, respectively; the Poisson's ratios are $\nu_a = \nu_m = 0.2$, and mass densities are $\rho_a = \rho_m = 1$. Homogenized properties are evaluated by solving the leading-order microscopic boundary value problem, which yields:

$$\boldsymbol{D}^0 = \begin{bmatrix} 3.64 & 0.89 & 0.00 \\ 0.89 & 3.68 & 0.00 \\ 0.00 & 0.00 & 1.36 \end{bmatrix}, \quad \boldsymbol{E}^k = \begin{bmatrix} 3.11 & -0.10 & 0.00 \\ -0.10 & 3.09 & 0.00 \\ 0.00 & 0.00 & 1.60 \end{bmatrix} \times 10^{-2}.$$

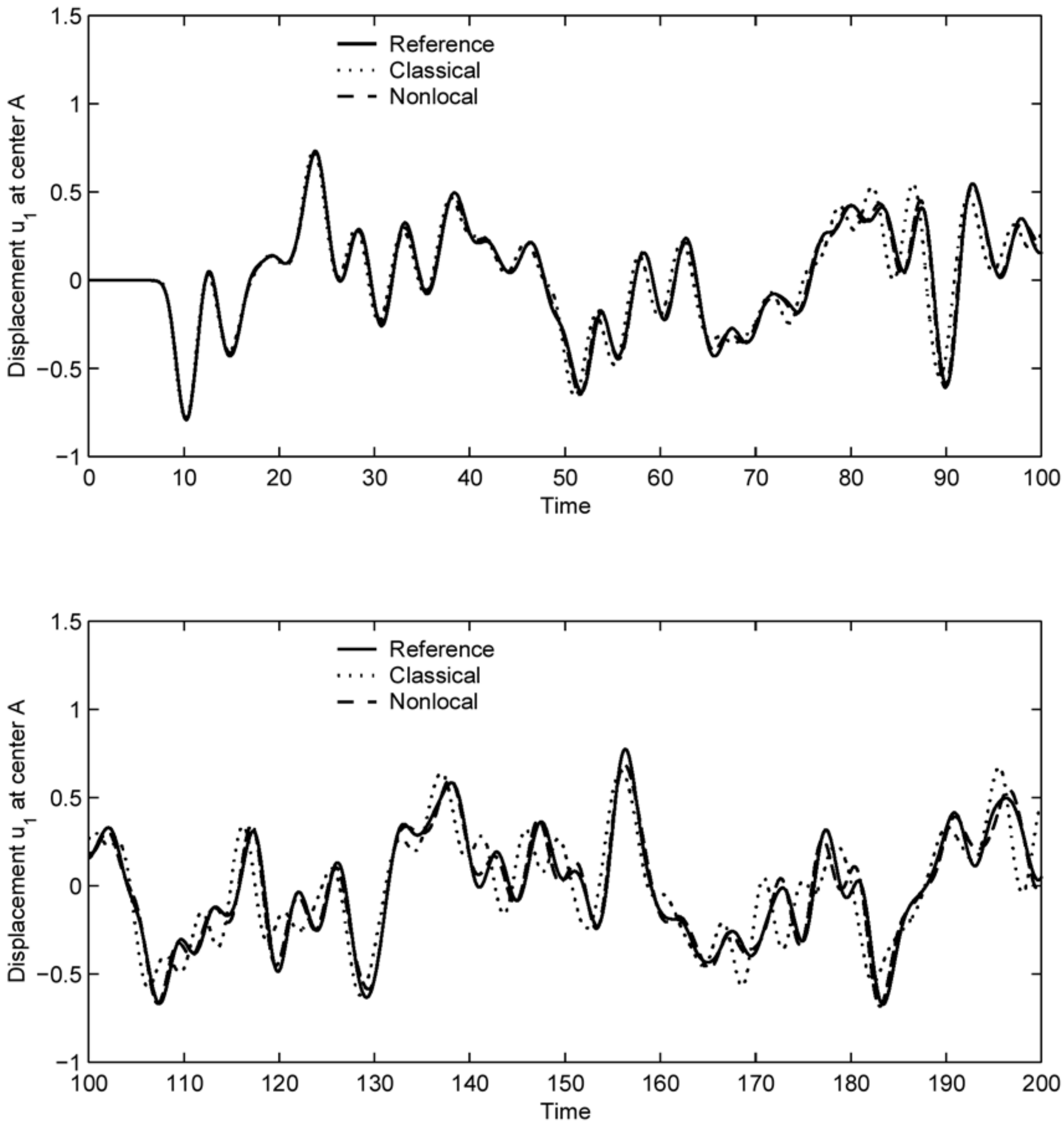


Figure 2: Comparison of responses at the center point A

In this example loading period duration is taken as $T = 7$. The source heterogeneous problem is discretized with 1560×676 bilinear square-shape finite elements. For the classical homogenization and nonlocal methods, the microstructure is discretized with 60×104 bilinear square-shape finite elements, whereas the macro-problem is discretized with 104×52 bilinear square-shape finite elements.

Figure 2 shows the time-varying displacement u_1 at point A (in Figure 1) corresponding to the center of macro-domain. Good agreement between the solution of the nonlocal model and the reference solution of the source problem can be seen. On the other hand, the solution of the classical homogenization model shows significant deviation from the reference solution.

Problem 2: As an example of various three-dimensional problems, an isotropically damaged concrete beam of $450 \times 150 \times 150$ mm^3 shown in Figure 3 is considered. The left face is fixed and the right face is subjected to the impact load $q(t)$ perpendicular to it. A quarter region is discretized with $90 \times 15 \times 15$ trilinear cubic-shape finite elements due to the symmetry. The unit cell model is reconstructed from a three-dimensional digital imaging process [19]. The unit cell consists of 75 *mm*-cubic region, which is three times larger than the maximum size of an aggregate. The unit cell is discretized with 150^3 trilinear cubic-shape finite elements. The volume fraction of aggregates is 0.49; The Young's modulus of aggregates and damaged mortar are $E_a = 55$ *GPa*, $E_m^d = 2.6$ *GPa*, respectively; The Poisson's ratios $\nu_a = 0.15$, $\nu_m = 0.19$; and the mass densities $\rho_a = \rho_m = 2.2 \times 10^{-6}$ Kg/mm^3. Load period duration time is $T = 100\mu s$.

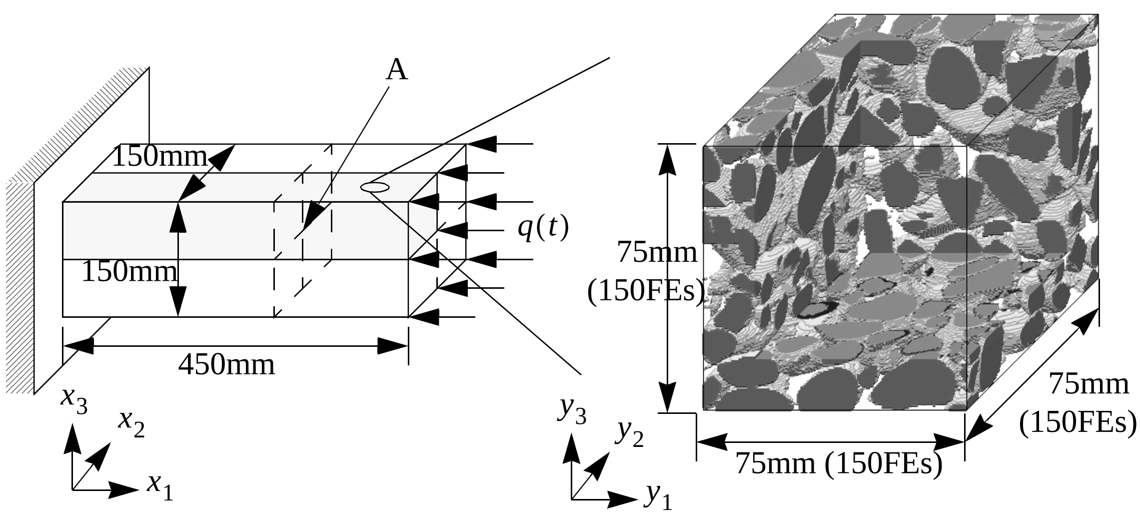


Figure 3: Three-dimensional multi-scale problem of concrete

Figure 4 plots the time-varying displacement u_1 and the normal stress σ_1 as obtained with the nonlocal and the classical homogenization model at the center A in the Figure 3. Figure 5 shows the maximum principal stresses in the unit cell at the center A ($t = 915\mu s$). The stresses in the unit cell are approximated up to $O(1)$. No comparison to the reference solution has been made as that would involve over 100 million degrees-of-freedom.

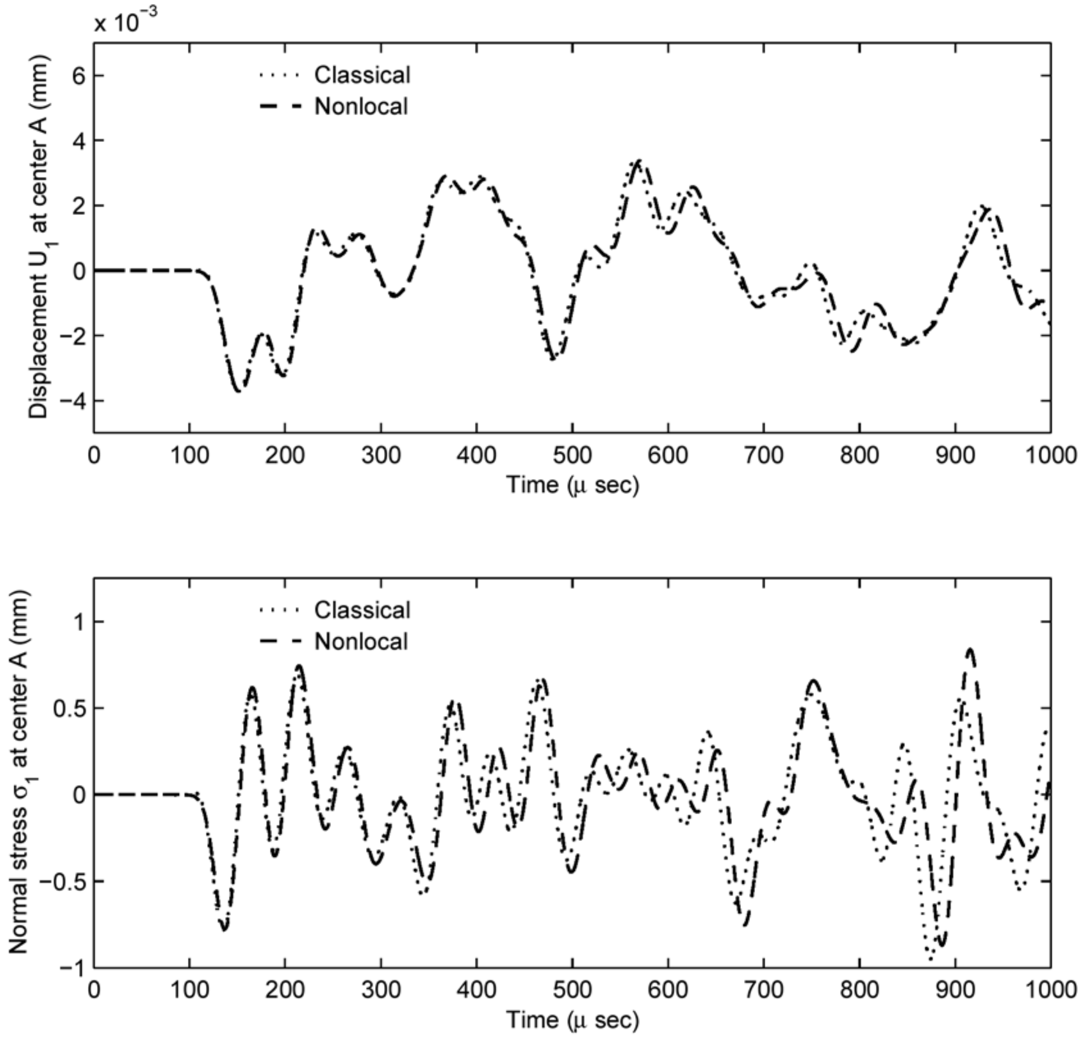
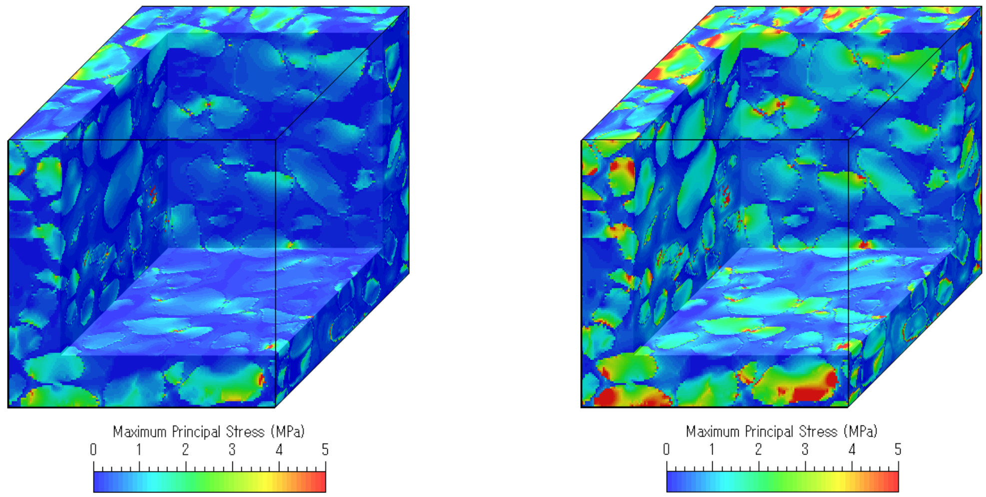


Figure 4: Comparison of responses at the center point A (Concrete sample)



Classical Homogenization   Nonlocal Model

Figure 5: Comparison of maximum stresses in the unit cell at $t = 915\mu s$

5. Summary and Future Research Directions

Three dispersive models for wave propagation in heterogeneous media have been developed for one- and multi-dimensional problems. This work is motivated by our recent studies for one-dimensional problems [4][5] which suggested that in absence of multiple time scaling, higher-order homogenization method gives rise to secular terms which grow unbounded with time and the problem of secularity can be successively resolved with the introduction of slow time scales. The first model consists of a leading-order homogenized equation of motion subjected to secularity constraints imposing uniform validity of asymptotic expansions. The second, nonlocal model, contains fourth-order spatial derivative and thus requires C^1 continuous finite element formulation. The third model, which has been implemented for constant mass density and macroscopically isotropic heterogeneous medium, requires C^0 continuity only and its finite element formulation is almost identical to the classical local approach with the exception of the mass matrix. The modified mass matrix consists of the classical mass matrix (lumped or consistent) perturbed with a stiffness matrix whose constitutive matrix depends on the unit cell solution.

Several issues, however, have not been addressed:

1. Only the special case with constant mass density and macroscopically isotropic heterogeneous medium have been implemented and validated. For general macroscopically anisotropic materials $\boldsymbol{D}^1 \neq \mathbf{0}$ and thus the issue of C^1 continuity has to be resolved.

2. For macroscopically orthotropic materials higher order unit cell problems have to be solved.

3. The present model requires implicit time integration. Lumping of the additional term in the mass matrix, $\int_{\Omega} \rho_0 \boldsymbol{B}^{\boldsymbol{T}} \boldsymbol{E}^k \boldsymbol{B} d\Omega$, is identically zero. Therefore various mass matrix splitting procedures have to be investigated in the context of explicit methods.

4. For the finite element formulation of the dispersive model involving secularity constraints (57)-(59) time integration procedures with multiple time scales have to be developed.

These issues will be investigated in our future work.

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References

1 Sanchez-Palencia, E., “Non-homogeneous Media and Vibration Theory”, Springer, Berlin, 1980

2 Benssousan, A., Lions, J.L. and Papanicoulau, G., “Asymptotic Analysis for Periodic Structures”, North Holland, Amsterdam, 1978.

3 Boutin, C., and Auriault, J.L, “Rayleigh scattering in elastic composite materials”, Int. J. Engng. Sci. 31(12), 1993, pp.1669-1689

4 Fish, J., and Chen, W., “High-order homogenization of initial/boundary-value problems”, accepted by ASCE Journal of Engineering Mechanics, 2000.

5 Chen, W., and Fish, J., “A dispersive model for wave propagation in periodic heterogeneous media based on homogenization with multiple spatial and temporal scales”, accepted by ASME Journal of Applied Mechanics, 2000.

6 J. Fish and W.Chen, “Uniformly Valid Multiple Spatial-Temporal Scale Modeling for Wave Propagation in Heterogeneous Media,” Mechanics of Composite Materials and Structures, Vol. 8, pp. 1-19, (2001).

7 J. Fish, W.Chen and G. Nagai, “Nonlocal Dispersive Modeling For Wave Propagation In Heterogeneous Media. Part 1: One-Dimensional Case”submitted to International Journal for Numerical Methods in Engineering, (2001)

8 Fish, J., Nayak, P., and Holmes, M.H., Macroscale reduction error indicators and estimators for a periodic heterogeneous medium, Comput. Mech. 14, 1994.

9 Boutin, C., “Microstructural effects in elastic composites”, Int. J. Solids Struct. 33(7),

1996, pp.1023-1051.

10 Guedes, J., M., and Kikuchi, N., “Preprocessing and postprocessing for materials based on the homogenization method with adaptive finite element methods”, Comput. Methods Appl. Mech. Engrg. 83, 1990, pp. 143-198.

11 Wagrowska, M., and Wozniak, C., “On the modeling of dynamic problems for viscoelastic composites”, Int. J. Engng Sci. 34(8), 1996, pp.923-932.

12 Santosa, F., and Symes. W.W., “A dispersive effective medium for wave propagation in periodic composites”, SIAM J. Appl. Math., 51(4), 1991, pp. 984-1005.

13 Mei, C.C., Auriault J.L., and Ng, C.O., “Some applications of the homogenization theory”, in: Advances in Applied Mechanics, Vol. 32, Hutchinson, J.W., and Wu, T.Y., eds., Academic Press, Boston, 1996, pp. 277-348.

14 Boutin, C., and Auriault, J.L., “Dynamic behavior of porous media saturated by a viscoelastic fluid. Application to bituminous concretes”, Int. J. Engng. Sci., 28(11), 1990, pp.1157-1181.

15 Carey, G.F., and Oden, J.T., “Finite Elements, A Second Course”, Vol. II, Prentice-Hall, New Jersey, 1983.

16 Carey, G.F., and Oden, J.T., “Finite Elements, Mathematical Aspects”, Vol. IV, Prentice-Hall, New Jersey, 1983.

17 Maugin, G.A., “Physical and mathematical models of nonlinear waves in solids”, in: Nonlinear Waves in Solids, Jeffrey, A., and Engelbrecht, J., eds., Springer, Wien-New York, 1994.

18 Whitham, G.B., “Linear and Nonlinear Waves”, John Wiley & Sons, New York, 1974.

19 Nagai, G., Yamada, T., and Wada, A, “Accurate Modeling and Fast Solver for the Stress Analysis of Concrete Materials Based on Digital Image Processing Technique”, International Journal for Computational Civil and Structural Engineering, Vol. 1, 2000