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Continuum damage model for Prony-series type viscoelastic solids

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August 6th 2014

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COMPUTATIONAL MECHANICS
at Columbia University

Outline

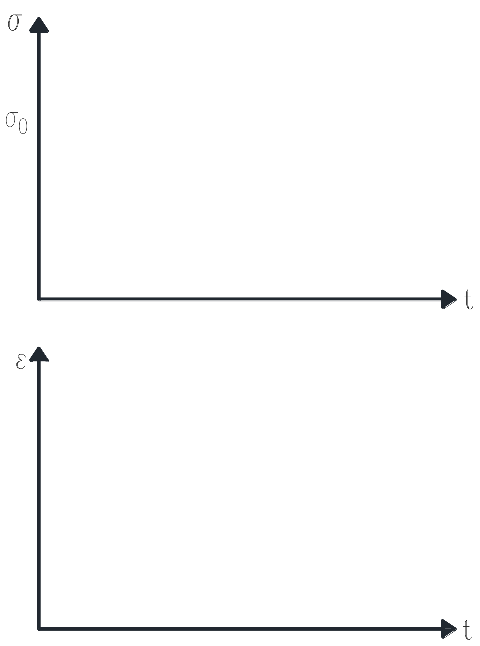
- Introduction and motivation
- Viscoelastic models
- Continuum Damage Mechanics
- Viscoelastic damage and implementation
- Applications and results
- Conclusions
- References

Introduction

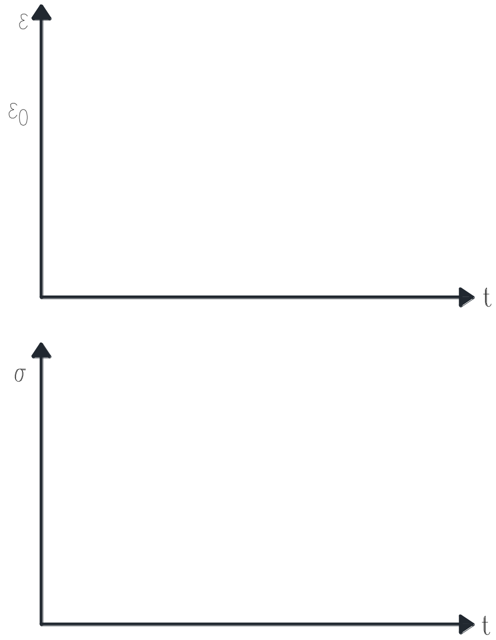
- Adequate computational models is required for engineering applications (Expensive physical testing)
- Materials display time dependent deterioration
- Viscoelastic behavior: Elastic + viscous properties
- Damage growth: Continuum damage mechanics
- Viscoelastic behavior and damage growth effects combined

Viscoelastic model

- Model is phenomenological: no related to chemical composition or molecular structure
- Material experience Creep, Stress relaxation, Recovery



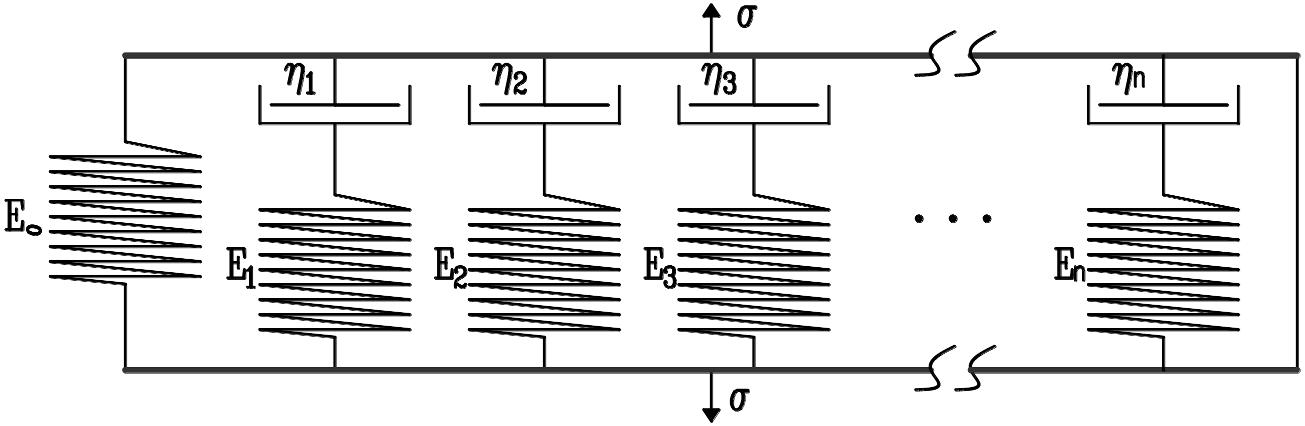
Creep



Relaxation

Viscoelastic model

- **Prony series:**
  - Material modulus expressed in the Prony Series form leads to a generalization of the Maxwell model



$$\sigma = \int_{-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)d\tau \qquad E(t) = E_0 + \sum_{i=1}^{n} E_i e^{-\frac{t}{\lambda_i}} = E\left(\mu_0 + \sum_{i=1}^{n} \mu_i e^{-\frac{t}{\lambda_i}}\right)$$

$$\tau_i = charteristic\ time; \sum_{i=0}^{n} \mu_i = 1$$

Viscoelastic model

- Some materials display viscoelastic behavior on the shear component only,

$$\sigma = \sigma^{vol} + \sigma^{dev} = 3K\, tr(\varepsilon) + 2G(t)\varepsilon^{dev}(t)$$

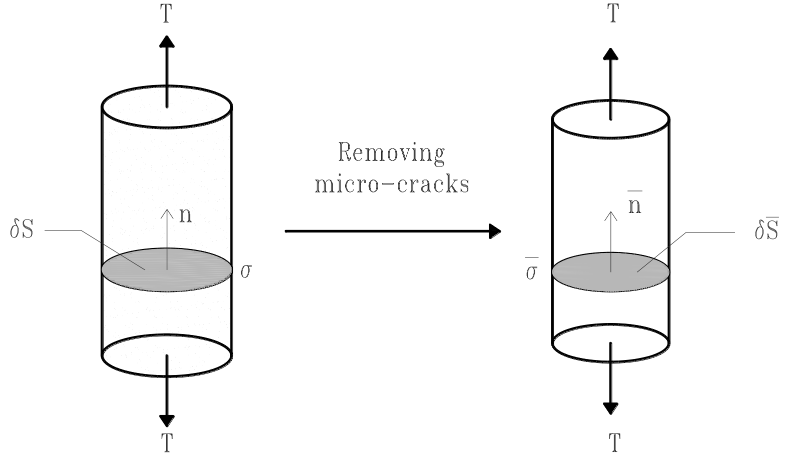
and $$2G(t)\varepsilon^{dev}(t) = 2\int_{-\infty}^{t} G(t-\tau)\dot{\varepsilon}^{dev}(\tau)\, d\tau$$

Prony series of $G(t)$, $$G(t) = G\left(\mu_0 + \sum_{i=1}^{n} \mu_i e^{-\frac{t}{\lambda_i}}\right)$$

$$2G(t)\varepsilon^{dev}\,(t) = 2G\int_{-\infty}^{t}\left[\mu_0 + \sum_{i=1}^{n} \mu_i e^{\left(\frac{-(t-\tau)}{\lambda_i}\right)}\right]\dot{\varepsilon}^{dev}(\tau)\, d\tau$$

Damage model

- Progressive deterioration of material preceding the failure due to accumulation of voids and micro-cracks



Damaged State

Effective Undamaged State

$$D = \frac{\delta S - \delta\tilde{S}}{\delta S}$$

Fig.. Isotropic damage in uniaxial tension (concept of effective stress).

- No cracks present in the material
- Damage evolution fully phenomenological
- Degree of damage is quantified into the parameter D *(0 ≤ D ≤ 1)*
- Damage might be anisotropic

$$\tilde{\sigma} = \sigma \frac{\delta S}{\delta\tilde{S}} = \frac{\sigma}{(1 - D)} = M\sigma$$

Damage model

- Kachanov-Rabotnov uniaxial creep damage $\dot{d} = B\frac{|\tilde{\sigma}|^r}{(1-d)^{k_\sigma}}$

- Hayhurst's (1972) equivalent stress measure

$$\chi = \alpha\tilde{\sigma}^{(1)} + \beta\sqrt{3\mathrm{II}_{\tilde{\sigma}^{dev}}} + (1-\alpha-\beta)\mathrm{I}_{\tilde{\sigma}}$$

where

$$\mathrm{I}_{\tilde{\sigma}} = \tilde{\sigma}_{ii} \qquad \mathrm{II}_{\tilde{\sigma}^{dev}} = \frac{1}{2}\tilde{\sigma}^{\mathrm{dev}}_{mn}\tilde{\sigma}^{\mathrm{dev}}_{mn} \qquad \tilde{\sigma}^{(1)} = \lambda_1$$

- Murakami & Ohno (1981), Murakami (1983), Murakami (1988)

$$\dot{D} = B\chi^r\{Tr[(1-D)^{-1}(\xi^{(1)}\otimes\xi^{(1)})]\}^k[(1-\gamma)\mathbf{1}+\boldsymbol{\gamma}\xi^{(1)}\otimes\xi^{(1)}]$$

$\xi^{(1)} = \textit{eigenvector related to } \tilde{\sigma}^{(1)}$

$\gamma = \textit{anisotropic parameter}$

- Simplifying for isotropic damage,

$$\dot{D} = B\frac{\langle\chi\rangle^r}{(1-D)^k}$$

$B, k, r =$ **Material parameters**

$\chi =$ **Hayhurst's equiv. stress**

Viscoelastic damage implementation

- Current stress, σ_{n+1}:

$$\sigma_{n+1} = \sigma_{n+1}^{vol} + \sigma_{n+1}^{dev}$$

$$\sigma^{vol} = (1 - D_{n+1})3K\,Tr(\varepsilon)$$

$$\sigma_{n+1}^{dev} = (1 - D_{\substack{n \\ +1}})\left[2G\left\{\mu_0 \varepsilon_{n+1}^{dev} + \sum_{i=1}^{n} \mu_i \left[e^{\left(\frac{-t}{\lambda_i}\right)} \varepsilon_0^{dev} + h_{n+1}^{i}\right]\right\}\right]$$

where,

$$D_{n+1} = D_n + \Delta t\left(B\frac{\langle\chi\rangle^r}{(1 - D_n)^k}\right)$$

$$h^i = e^{\left(\frac{-\Delta t}{\lambda_i}\right)} h_n^i + \Delta h^i$$

$$h_n^i = e^{\left(\frac{-t_n}{\lambda_i}\right)} \int_0^{t_n} e^{\left(\frac{\tau}{\lambda_i}\right)} \dot{\varepsilon}^{dev}(\tau)\, d\tau\,,$$

$$\Delta h^i = \lambda_i\left[1 - e^{\left(\frac{-\Delta t}{\lambda_i}\right)}\right]\frac{\Delta\varepsilon^{dev}}{\Delta t}$$

Viscoelastic damage implementation

- Damage time integration by explicit forward Euler method
- Initial conditions: $t = 0,\ D = 0$
- For the current time step, t_{n+1} : $t_{n+1} = t_n + \Delta t$

1. From previous time step: $t = t_n,\quad \varepsilon(t_n) = \varepsilon_n,\quad D(t_n) = D_n$
2. Strain computation: $u_{n+1} \longrightarrow \varepsilon_{n+1} = \varepsilon^{vol}_{n+1} + \varepsilon^{dev}_{n+1}$
3. Effective stress:

$$\tilde{\sigma}^{dev}_{n+1} = f\left(h^i_n, \Delta h^i, \varepsilon^{dev}_{n+1}, \varepsilon^{dev}_n\right)$$

$$\tilde{\sigma}_{n+1} = \tilde{\sigma}^{vol}_{n+1} + \tilde{\sigma}^{dev}_{n+1}$$

4. Damage update:

$$\dot{D}_{n+1} = f(\tilde{\sigma}_{n+1}) \longrightarrow \Delta D_{n+1} = \Delta t\ \dot{D}_{n+1}$$

$$D_{n+1} = D_n + \Delta D_{n+1}$$

5. Approxim. Stiffness, K_{n+1}:

$$d\sigma = \frac{\partial \sigma}{\partial u} du + \cancel{\frac{\partial \sigma}{\partial D}\frac{\partial D}{\partial u}} du$$

Applications and Results

- Parameters calibration:
  - Constrained Optimization

$$\forall i \ \in [0, n], \mu_i > 0$$

$$\mu_0 + \sum_{i=1}^{n} \mu_i = 1$$

  - Damage Parameters

$$k = k_1 + k_2 \theta$$

$$B = B_1 + B_2 \Theta$$

Where, k_1, k_2 and B_1, B_2 are material parameters from linear fitting and θ and Θ have components of σ, ε or $\dot{\varepsilon}$

- Civil Engineering applications:
  - Polycrystalline Ice
  - Asphalt concrete

Polycrystalline Ice

- Finite Elements implementation: FEAP user element
- Values at central node, Plane Stress

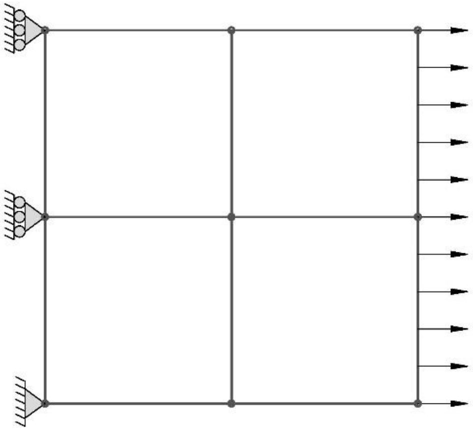
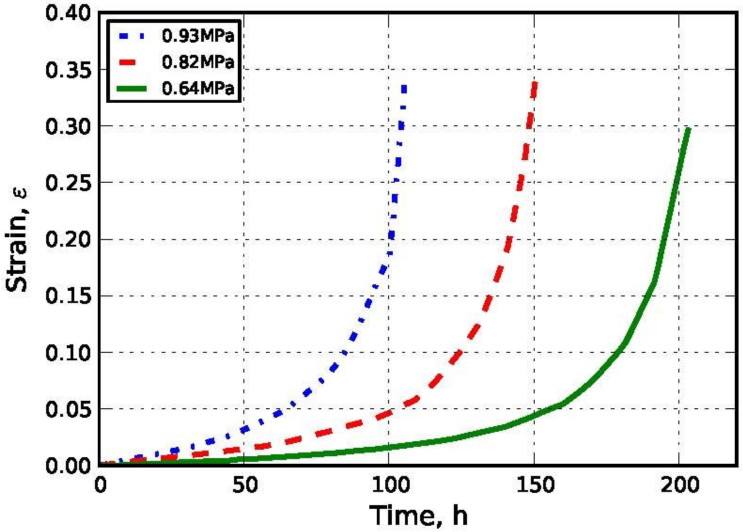


Figure. Plane Stress



- Tensile creep (Mahrenholtz, W. Z. 1992)

$$T = -10°C$$
$$\sigma = 0.93, 0.82, 0.64\ [MPa]$$

Polycrystalline Ice

- Parameters selected

Prony Series

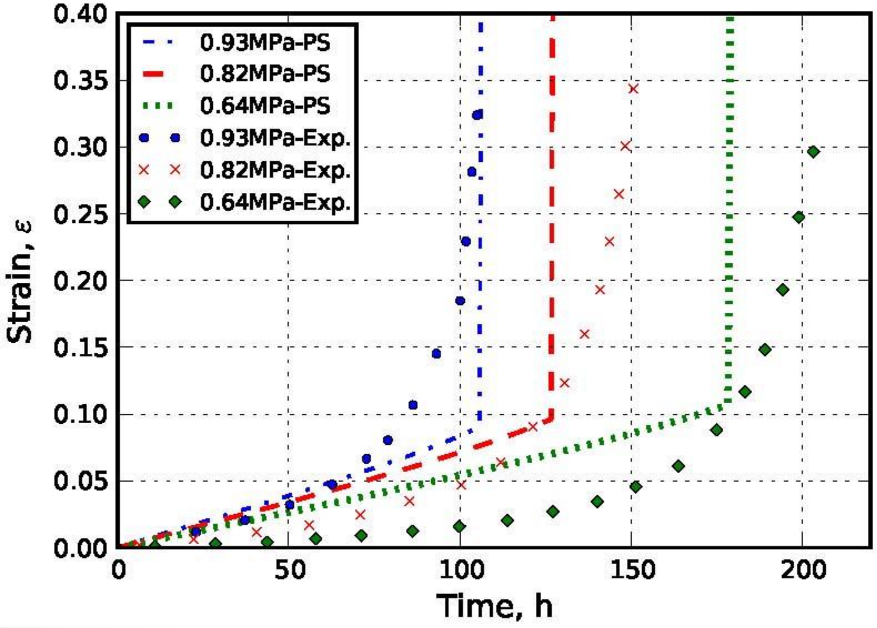
E [MPa]	ν	nvis	μ^1	τ^1
9500	0.35	1.0	0.999	415.0

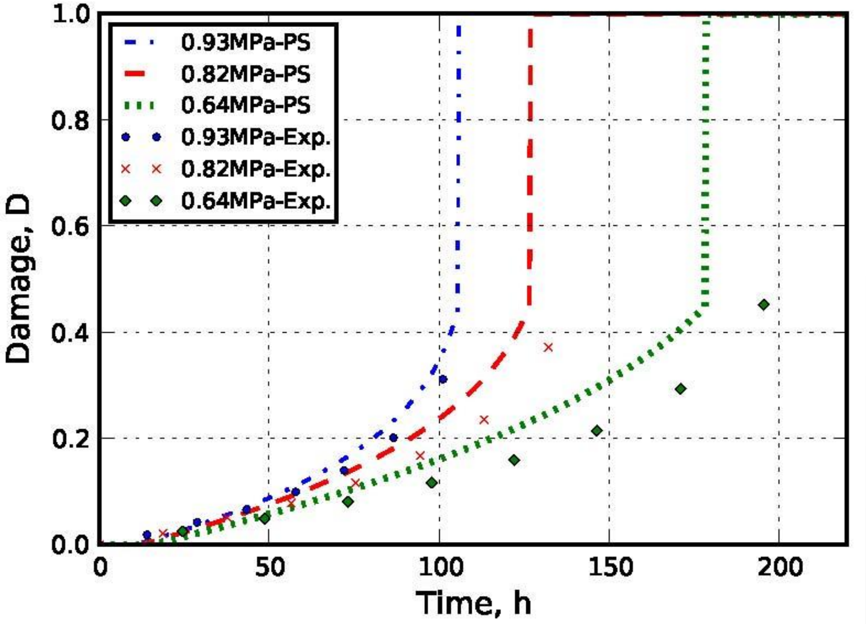
Damage

α	β	B^1	B^2	r	k^1	k^2	γ	$\varepsilon^{threshold}$
0.2	0.63	5.232 × 10−7	0.0	0.43	-2.63	7.24	0.0d0	0.8 × 10−2

$\theta = |\sigma_{ii}|$ $\quad$ $D_c = 0.45$

- Results



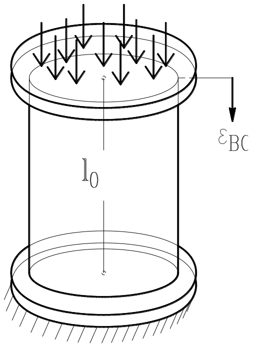


Asphalt Concrete

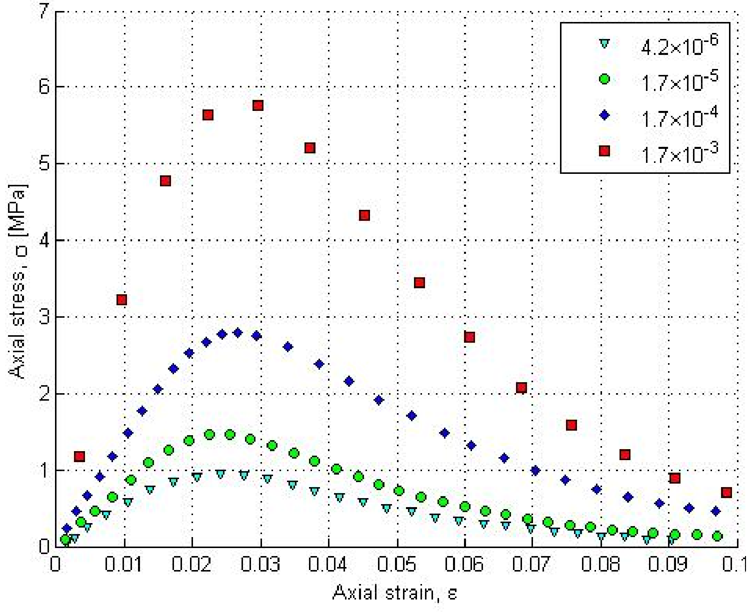
- Unconfined compression:

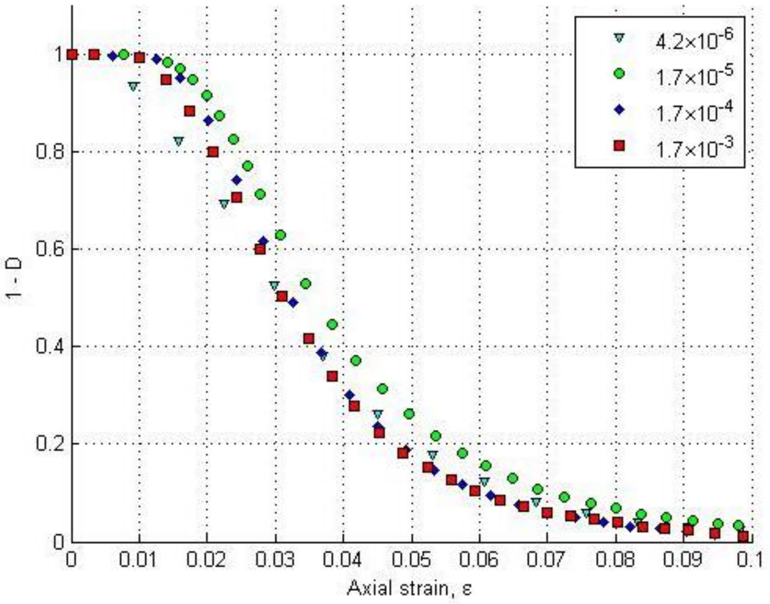
Strain rates applied

$$\dot{\varepsilon} = 4.2 \times 10^{-6},$$
$$\dot{\varepsilon} = 1.7 \times 10^{-5},$$
$$\dot{\varepsilon} = 1.7 \times 10^{-4},$$
$$\dot{\varepsilon} = 1.7 \times 10^{-3}.$$



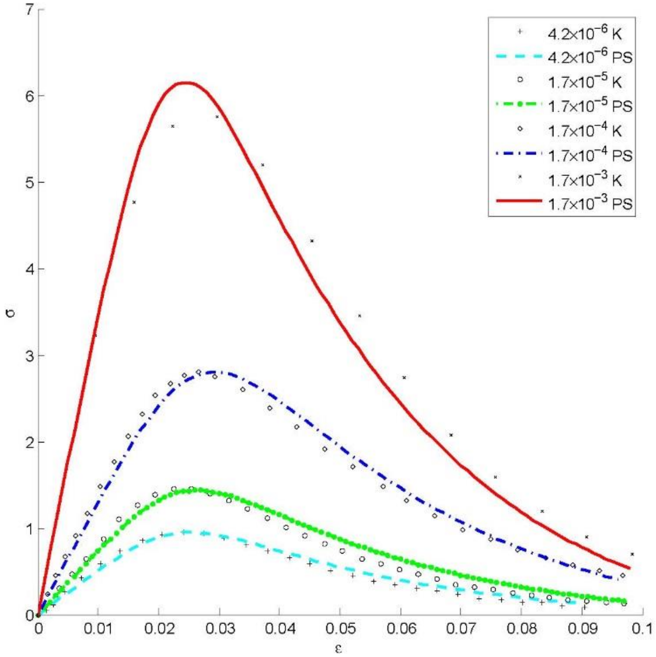
- Experimental data (Katsuki D. & Gutierrez M., 2011)

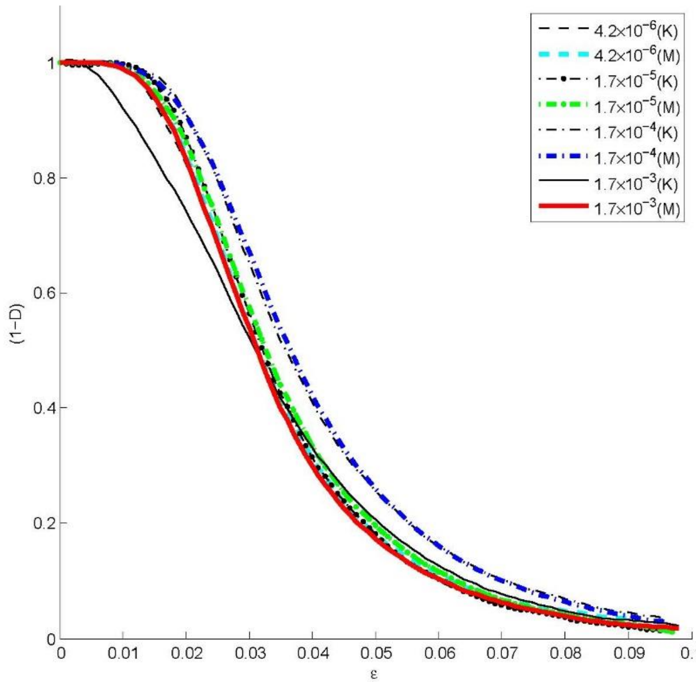




Asphalt Concrete

- Results





- Parameters

α	β	B_1	B_2	r	k_1	k_2	γ	$\varepsilon^{threshold}$
1	0	1.14 × 10−4	−8.655 × 10-3	3.24	-10.986	3.31	0.0d0	0.0

Damage

ν	n	μ^1	τ^1
0.35	1.0	0.999	415.0

Prony Series

$\theta = \varepsilon_{BC}$ $\Theta = \dot{\varepsilon}_{BC}$ $D_c = 0.0$

Conclusions

- Material viscoelastic deterioration can be predicted with simple implementation with proper material selection
- Prediction of material behavior under tension and compression can be calibrated
- Semi-analytical time integration of the constitutive equation and explicit Forward Euler for damage results in fast and accurate prediction of material behavior
- Optimization of the material parameter required careful selection of initial values. Global optimization method could be beneficial
- In the finite elements implementation, the model displayed mesh sensitivity

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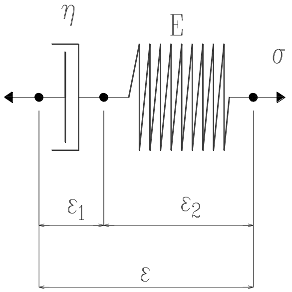
Thank you!

Questions ?

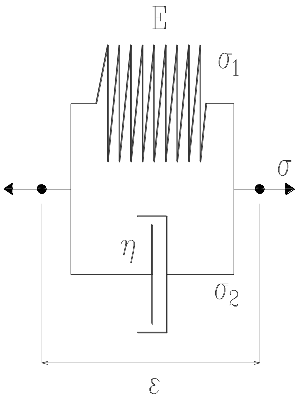
Viscoelastic model

- Springs and dashpots models:
  - Elastic behavior (Hook’s law) $\sigma = E\varepsilon$
  - Viscous behavior (Newton’s Law) $\sigma = \eta\dot{\varepsilon}$
- **Maxwell model:**
  - Represent relaxation very well but not creep or recovery



$$\left.\begin{aligned}\sigma = E\varepsilon_2 = \eta\dot{\varepsilon}_1\\ \varepsilon = \varepsilon_1 + \varepsilon_2\end{aligned}\right\} \longrightarrow \sigma = \frac{\dot{\varepsilon}}{\left(\frac{1}{\eta} + \frac{1}{E}\frac{d}{dt}\right)}$$

- **Kelvin-Voigt model:**
  - Creep and recovery are well represented but not relaxation



$$\left.\begin{aligned}\sigma = \sigma_1 + \sigma_2 = E\varepsilon + \eta\dot{\varepsilon}\\ \varepsilon = \varepsilon_1 = \varepsilon_2\end{aligned}\right\} \longrightarrow \sigma = \left(E + \eta\frac{d}{dt}\right)\varepsilon$$