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Abstract

We study the cores of non-atomic market games, a class of transferable utility cooperative games introduced by Aumann and Shapley [2], and, more in general, of those games that admit a *na*-continuous and concave extension to the set of ideal coalitions, studied by Einy, Moreno, and Shitovitz [12].

We show that the core of such games is norm compact and we provide some representation results. We also give a Multiple Priors interpretation of some of our results.

1 Introduction

In their studies of values of exchange economies with a continuum of players, Aumann and Shapley [2] introduced (non-atomic) market games, a class of transferable utility (TU) cooperative games that includes those arising from exchange economies. These games have been extensively studied in value theory (see, e.g., [27, Sect. 12]) and our purpose in this paper is to provide a thorough study of their cores, a fundamental solution concept for TU games.

Hart [16] showed that under weak conditions the cores of market games arising from exchange economies are finite dimensional subsets of non-atomic measures. Our main result, Theorem 6, complements his result by showing that, in general, cores of market games are norm compact subsets of non-atomic measures. This implies that cores of these games have a very strong structure, and that the finite dimensional case is actually the most important form that they can take. The result of Hart [16] can thus be viewed as "typical" for the cores of market games. Theorem 6 actually holds for the larger class of games that admit a *na*-continuous and concave extension to the set of ideal coalitions, a class studied by Einy, Moreno, and Shitovitz [12].

We then provide, in Theorem 9, a classification of market games based on the properties of their *na*-extensions and on generalizations of the linear productions games of Owen [28]

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and Billera and Raanan [4]. Proposition 10 shows that the cores of these generalized linear production games admit a convenient representation, and in this way we extend earlier results of Billera and Raanan [4], Einy, Moreno, and Shitovitz [12], and Marinacci and Montrucchio [20]. Finally, in Theorem 13 we provide a full characterization of exact market games.

Some of our results can be interpreted in the Multiple Priors model axiomatized by Gilboa and Schmeidler [14]. In particular, we will show how norm compact sets of priors consisting of non-atomic probability measures have some useful properties that do not hold for general sets of priors. For example, Multiple Priors preferences featuring such sets of priors are never Choquet Expected Utility preferences, unless they are Subjective Expected Utility. In other words, these Multiple Priors preferences have only a "trivial" overlap with Schmeidler [30]'s Choquet Expected Utility ones.

The paper is organized as follows. Section 2 introduces notation and preliminaries, Section 3 reviews some essentially known representation results, while Section 4 establishes the paper's main results, whose proofs are relegated to the Appendix. Section 5 contains some examples. Finally, Section 6 provides the Multiple Priors interpretation of our results, and the reader only interested in this issue can move directly to this section, after having a look at Subsections 2.1 and 2.2 for some notation and terminology.

2 Notation and Preliminaries

2.1 Games and Set Functions

Let Ω be the set of players, Σ the σ-algebra of admissible coalitions, and X a Banach space. A function $\boldsymbol{\nu} : \Sigma \to X$ such that $\boldsymbol{\nu}(\varnothing) = 0$ is called a *set function.* A set function $\boldsymbol{\nu}$ is:[1]

- *bounded* if $\sup_{A \in \Sigma} \|\boldsymbol{\nu}(A)\| < \infty$;
- *additive* (a *vector charge*) if $\boldsymbol{\nu}(A \cup B) = \boldsymbol{\nu}(A) + \boldsymbol{\nu}(B)$ for all pairwise disjoint A and B;
- *continuous* if $\lim_{n\to\infty} \boldsymbol{\nu}(A_n) = \lim_{n\to\infty} (\boldsymbol{\nu}(\Omega) - \boldsymbol{\nu}(A_n^c)) = 0$ whenever $A_n \downarrow \varnothing$;
- *countably additive* (a *vector measure*) if $\boldsymbol{\nu}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \boldsymbol{\nu}(A_i)$ for all countable collections of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$.

The *variation* of a set function $\boldsymbol{\nu} : \Omega \to X$ is the function $|\boldsymbol{\nu}| : \Omega \to [0, \infty]$ defined by

$$|\boldsymbol{\nu}|(A) = \sup \sum_i \|\boldsymbol{\nu}(A_i) - \boldsymbol{\nu}(A_{i-1})\|, \quad \forall A \in \Sigma$$

where the supremum is taken over all finite chains $A_0 \subseteq A_1 \subseteq ... \subseteq A_n = A$. A set function is of *bounded variation* if $|\boldsymbol{\nu}|(\Omega) < \infty$. If the map $\boldsymbol{\nu} \mapsto |\boldsymbol{\nu}|(\Omega)$ defines the *variation norm* on

[1]In the sequel subsets of Ω are understood to be in Σ even where not stated explicitly and they are referred to both as sets and as coalitions.

the vector space $bv(\Sigma, X)$ all X-valued set functions of bounded variation (see [2, Sect. 4]). The variation norm is complete.

Given a set function $\boldsymbol{\nu} : \Sigma \to X$, a coalition N is $\boldsymbol{\nu}$*-null*, or simply *null*, if $\boldsymbol{\nu}(A \cup N) = \boldsymbol{\nu}(A)$ for all A in Σ. An *atom* of $\boldsymbol{\nu}$ is a non-null coalition A such that for every $B \subseteq A$, either B or $A \setminus B$ is null. If $\boldsymbol{\nu}$ has no atoms, $\boldsymbol{\nu}$ is called *non-atomic* (see, [2, p. 14] and [21, p. 55]).

We call *(transferable utility) game* a real valued set function, and *measure* (*charge*, resp.) a real valued vector measure (*vector charge*, resp.). A game ν is:

- *positive* if $\nu(A) \geq 0$ for all A;
- *superadditive* if $\nu(A \cup B) \geq \nu(A) + \nu(B)$ for all pairwise disjoint A and B;
- *convex* if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all A and B;

The set $ba(\Sigma)$ of all charges of bounded variation, as well as its subsets $ca(\Sigma)$ of all measures and $na(\Sigma)$ of all non-atomic measures, is a closed subspace of $bv(\Sigma) = bv(\Sigma, \mathbb{R})$. As well known, $ba(\Sigma)$ is (isometrically isomorphic to) the norm dual of the space $B(\Sigma)$ of all bounded and measurable functions (endowed with the supnorm), the duality being $\langle g, \mu \rangle = \int g d\mu$ for all g in $B(\Sigma)$ and μ in $ba(\Sigma)$. We will sometimes write $\mu(g)$ instead of $\int g d\mu$. For all $\alpha \in \mathbb{R}$, the closed and convex subset of $ba(\Sigma)$ consisting of all measures taking value α at the grand coalition Ω is denoted by $ba_\alpha(\Sigma)$; $ca_\alpha(\Sigma)$ and $na_\alpha(\Sigma)$ are defined analogously.

2.2 Exact Games

The *core* of a game $\nu : \Sigma \to \mathbb{R}$ is the set

$$core(\nu) = \{\mu \in ba(\Sigma) : \mu(\Omega) = \nu(\Omega) \text{ and } \mu(A) \geq \nu(A) \text{ for all } A \in \Sigma\}.$$

The core is a weak* compact subset of $ba(\Sigma)$. Games having non-empty cores are called *balanced*.

A balanced game ν is *exact* if

$$\nu(A) = \min_{\mu \in core(\nu)} \mu(A), \quad \forall A \in \Sigma.$$

An exact game ν is naturally extended to $B(\Sigma)$ by the function

$$\nu_e(g) = \min_{\mu \in core(\nu)} \mu(g), \quad \forall g \in B(\Sigma). \tag{1}$$

Clearly, a game is exact if and only if it is the lower envelope

$$\nu_K(A) = \inf_{\phi \in K} \phi(A), \quad \forall A \in \Sigma \tag{2}$$

of a non-empty bounded subset K of $ba_\alpha(\Sigma)$.[2] In this case, the weak* closed and convex hull $\overline{co}^*(K)$ of K is contained in $core(\nu_K)$. Notice that in general the inclusion is strict, namely, $\overline{co}^*(K) \neq core(\nu_K)$ (see Examples 6, 7, and 8 of Section 5).

The next proposition is basically proved in [21, p. 54-58].

[2] When we consider subsets of a normed space, by *bounded* we mean *norm bounded*.

Proposition 1 *Let K be a non-empty bounded subset of $ba_\alpha(\Sigma)$. The game ν_K is continuous if and only if K is a relatively weak compact subset of $ca_\alpha(\Sigma)$. In this case, ν_K is non-atomic if and only if $K \subseteq na_\alpha(\Sigma)$.*

2.3 Market and Pre-Market Games

An *ideal coalition* is an element of $B(\Sigma)$ taking values in $[0,1]$, and the set of all ideal coalitions is denoted by $B_1(\Sigma)$. The set $B_1(\Sigma)$ can be endowed with the *na-topology* due to Aumann and Shapley [2], which is the coarsest topology that makes continuous all the functionals $g \mapsto \mu(g)$ with $\mu \in na(\Sigma)$.[3] By the Lyapunov Theorem, the characteristic functions are na-dense in $B_1(\Sigma)$. Therefore, any game ν, when viewed as a function $1_A \mapsto \nu(A)$ over the characteristic functions, has at most one na-continuous extension to $B_1(\Sigma)$. Following Aumann and Shapley [2], we denote this extension by ν^*.

We say that a game ν is a *(non-atomic) market game* if it is superadditive and admits a positively homogeneous na-continuous extension ν^* (see, e.g., Mertens [24] for a similar definition). This name is justified by the fact that, as it will be seen momentarily in Proposition 5, under suitable conditions exchange economies with a continuum of agents can be modelled as market games. For this reason, market games play an important role in value theory.[4]

Aumann and Shapley [2, Prop. 27.1] show that the superadditivity of a market game ν is inherited by its extension ν^*, so that ν^* is superlinear when ν is a market game. In particular, this implies that a game ν is a market game if and only if it has a superlinear na-continuous extension ν^*. We will also consider games having a concave na-continuous extension ν^*, and we will call them *pre-market games.*

In the sequel we will make use of the following result, due to [22, Prop. 4], which shows that convex and pre-market games have trivial overlapping.[5]

Lemma 2 *A bounded convex game is a pre-market game if and only if it is a non-atomic measure.*

3 Representation of Market Games

Before moving to our main results, we devote this section to state some useful representation results for market and pre-market games that show how they can be viewed as generalized vector measure games. This was first observed by Milchtaich [25], and our next result slightly refines his main finding on this issue (see [25, Lm. 2]).

Specifically, consider a vector measure $\boldsymbol{\mu} : \Sigma \to X$, where X is a Banach space. For all

[3] Of course, any subset of $B(\Sigma)$ can be endowed with the na-topology.

[4] When (Ω, Σ) is a standard Borel space, the class of games having na-continuous extensions coincides with Aumann and Shapley's class pNA', while market games are the subclass they denote by H' (see [2, p. 273] and [15]).

[5] See the discussion of Proposition 31 in the Appendix for a proof.

$f \in B(\Sigma)$, let $\int f d\boldsymbol{\mu}$ be its integral ([9, p. 6]). The extended range

$$R(\boldsymbol{\mu}) = \left\{ \int f d\boldsymbol{\mu} : f \in B_1(\Sigma) \right\}$$

of $\boldsymbol{\mu}$ is a weakly compact and convex subset of X ([9, p. 14]).

Proposition 3 *A game ν is a pre-market game (a market game, resp.) if there exists a non-atomic vector measure $\boldsymbol{\mu} : \Sigma \to X$ of bounded variation with values on a suitable Banach space X, and a weakly continuous and concave (superlinear, resp.) function $\varphi : R(\boldsymbol{\mu}) \to \mathbb{R}$, with $\varphi(0) = 0$, such that*

$$\nu(A) = \varphi(\boldsymbol{\mu}(A)), \qquad \forall A \in \Sigma. \tag{3}$$

In this case, the na-continuous extension of ν is given by

$$\nu^*(f) = \varphi\left(\int f d\boldsymbol{\mu}\right), \qquad \forall f \in B_1(\Sigma).$$

The converse is true provided (Ω, Σ) is a standard Borel space.

If X is finite dimensional, the game $\varphi \circ \boldsymbol{\mu}$ given by (3) is called a *vector measure game.* This is an important class of games (see, e.g., [17]) and Proposition 3 is the announced result that shows how pre-market games can be regarded as generalized vector measure games.

Using the representation established in Proposition 3, next we provide a differential characterization of cores of pre-market games (see [12, Thm. C] and [20]).

Proposition 4 *Let X be a Banach space, $\boldsymbol{\mu} : \Sigma \to X$ a non-atomic vector measure of bounded variation, and $\varphi : R(\boldsymbol{\mu}) \to \mathbb{R}$ a weakly continuous and concave function with $\varphi(0) = 0$. Then,*

$$\text{core}(\varphi \circ \boldsymbol{\mu}) = \{x^* \circ \boldsymbol{\mu} : x^* \in \partial\varphi(0) \text{ and } x^*(\boldsymbol{\mu}(\Omega)) = \varphi(\boldsymbol{\mu}(\Omega))\},$$

provided it is not empty.

Consider now a transferable utility exchange economy, consisting of: a standard Borel space $(\Omega, \Sigma, \lambda)$ of traders with a non-atomic probability measure λ; a commodity space $\mathbb{R}^n_+$; a utility function $u(\cdot, \omega) : \mathbb{R}^n_+ \to \mathbb{R}_+$ for each agent $\omega \in \Omega$ and an initial allocation $a : \Omega \to \mathbb{R}^n_+$, which is λ-integrable.

The standard assumptions made are:

(a.1) $u(\cdot, \omega)$ is monotone non-decreasing and $u(0, \omega) = 0$ for all $\omega \in \Omega$,

(a.2) $u(\cdot, \omega)$ is continuous for all $\omega \in \Omega$,

(a.3) $u(\cdot, \cdot)$ is jointly Borel measurable,

(a.4) $u(x, \omega) = o(\|x\|)$ as $\|x\| \to \infty$ integrably in ω (see [2, Ch. 6]),

(a.5) $\int a d\lambda \gg 0$.

This economy induces the game $\nu : \Sigma \to \mathbb{R}$ defined by

$$\nu(E) = \sup\left\{\int_E u(x(\omega), \omega)\ d\lambda : \int_E x(\omega)\, d\lambda = \int_E a(\omega)\ d\lambda\right\}, \tag{4}$$

where the sup is taken over all λ-integrable allocations $x : \Omega \to \mathbb{R}^N_+$. Under (a.3) and (a.4) this game is well defined, with the supremum attained (see [2, Ch. 6] and [16]).

By considering the vector measure $\boldsymbol{\mu} : \Sigma \to L_1(\lambda)$ given by $\boldsymbol{\mu}(A) = 1_A$, where 1_A is viewed as an element of $L_1(\lambda)$, it easily follows from the representation of Proposition 3 that the game (4) is a market game. This and other properties of this game are stated in the next result. Tightness of the bound in point (iii) is the only novelty, the rest is due to Hart [16] (see Proposition 3.4, Theorem C, and Corollary 2.16 of his paper, respectively).

Proposition 5 *Let (a.1)-(a.5) hold and ν be defined by (4). Then,*

(i) ν is a market game;

(ii) core (ν) is generically a singleton, i.e., this is true for all total initial allocations $\mathbf{a} = \int a d\lambda$ outside a set of Lebesgue measure zero in $\mathbb{R}^n_{++}$;

(iii) core $(\nu) \subseteq na(\Sigma)$, $\dim$ core $(\nu) \leq n$, and this bound is tight.

4 Main Results

4.1 Cores

We are now ready to state our main result.

Theorem 6 *The core of a pre-market game is a norm compact set of non-atomic measures.*

In other words, cores of pre-market and, *a fortiori*, of market games are norm compact subsets of $na(\Sigma)$. Notice that while market games are necessarily balanced, pre-market games are not. Example 1 in Section 5 shows that the concavity of the na-continuous extension is crucial for this result.

As a corollary of Theorem 6 we obtain a known result on the games in pNA, reported here for sake of completeness.[6] In reading it recall that market games in pNA arise, for instance, in the exchange economy described in Section 3 under smoothness assumptions on the utility function $u(\cdot, \omega)$ (see [2, Thm. J]).

Corollary 7 *If (Ω, Σ) is a standard Borel space, then the core of any balanced pre-market game in pNA is a singleton and it coincides with the Aumann-Shapley value.*

The next sum rule is a consequence of Theorem 6 and it shows that cores of pre-market games are stable under summation.

[6] pNA is the closure in the variation norm of the linear space of games that is generated by all powers of non-atomic probability measures.

Corollary 8 *If ν_1 and ν_2 are pre-market games, then*

$$core(\nu_1 + \nu_2) = core(\nu_1) + core(\nu_2). \tag{5}$$

Note that (5) implies that $core(\nu_1 + \nu_2) = \varnothing$ if $core(\nu_i) = \varnothing$ for some i. The sum rule (5) does not hold in general, and bounded convex games are the only other class of games for which it is known to hold (see [7] and [20, Cor. 9]). However, by Lemma 2 bounded convex games and pre-market games are essentially disjoint classes of games, and so Corollary 8 provides a new important class of games for which the sum rule holds.

4.2 Linear Production Games

Using the extension properties of ν^* we can provide a classification of market games. The simplest class of market games is given by the linear production games of Owen [28] and Billera and Raanan [4]. They are games of the form $\varphi \circ \boldsymbol{\mu}$, where $\boldsymbol{\mu} : \Sigma \to \mathbb{R}^n$ is a non-atomic vector measure and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is given by $\varphi(x) = \min_{t=1,\ldots,T} a^t \cdot x$, with each $a^t \in \mathbb{R}^n$.

Marinacci and Montrucchio [20] and [22] have generalized this notion by considering games of the form

$$\nu(A) = \min_{\mu \in \Gamma} \mu(A), \tag{6}$$

where Γ is a finite dimensional compact subset of $na(\Sigma)$. These games are called *generalized linear production games*, and *linear production games* are the special case corresponding to a finite set Γ.

Motivated by Theorem 6, here we further generalize these notions by considering

$$\nu(A) = \min_{\mu \in \Gamma} \mu(A)$$

for any norm compact $\Gamma \subseteq na(\Sigma)$. We call such a game an *abstract linear production game.*

We can now state the announced classification of market games based on the extension properties of ν^*.

Theorem 9 *The following properties are equivalent for a game ν:*

(i) ν is an abstract (generalized, resp.) linear production game,

(ii) ν is a market game such that its na-extension ν^* admits a further superlinear extension to the unit ball of $B(\Sigma)$ (to the entire $B(\Sigma)$, resp.) that is na-lower semicontinuous at 0.

In other words, abstract linear production games are the market games whose *na*-extension can be extended from $B_1(\Sigma)$ to the unit ball of $B(\Sigma)$, while generalized linear production games are the market games whose *na*-extension can be further extended to the entire space $B(\Sigma)$.

In view of Theorem 9, generalized and abstract linear production games are natural subclasses of market games. Interestingly, their cores admit a neat representation.

Proposition 10 *If* $\nu(A) = \min_{\mu\in\Gamma} \mu(A)$ *is an abstract linear production game, then*

$$core(\nu) = \overline{co}\{\mu \in \Gamma : \mu(\Omega) = \nu(\Omega)\}. \tag{7}$$

Proposition 10 extends to abstract linear production games the results of Billera and Raanan [4] and Marinacci and Montrucchio [20] on linear production games and generalized linear production games, respectively.

Along with Lemma 2, Theorem 9 implies that abstract production games are not convex, unless they are additive.

Corollary 11 *An abstract production game* $\nu(A) = \min_{\mu\in\Gamma} \mu(A)$ *is convex if and only if* ν *is additive, that is, if and only if* Γ *is the singleton consisting of* ν *itself.*

Corollary 11 shows, *inter alia*, that neither the core nor the set of the extreme points of the core of a non-additive and bounded convex game ν can be a compact (e.g., finite) subset of $na(\Sigma)$. In fact, denote by Γ either one of these two sets; if Γ is a compact subset of $na(\Sigma)$, then $\nu(\cdot) = \min_{\mu\in\Gamma} \mu(\cdot)$ is an abstract linear production game, something impossible by Corollary 11.

We conclude this subsection by presenting a condition under which a market game is an abstract linear production game. This condition is based on the vector measure representation provided by Proposition 3. Recall that a Banach space X has the *Radon-Nikodym property* (RNP) if, given any measure space (Ω, Σ) and any vector measure $\boldsymbol{\mu} : \Sigma \to X$ of bounded variation, there is a Bochner-integrable function $f : \Omega \to X$ such that $\boldsymbol{\mu}(A) = \int_A f d|\boldsymbol{\mu}|$ for each $A \in \Sigma$ (see, e.g., [5, p. 16]). For example, reflexive Banach spaces and separable dual spaces have the RNP (see [9, Sect. III]).

Here we say that a vector measure $\boldsymbol{\mu} : \Sigma \to X$ of bounded variation, defined on a measurable space (Ω, Σ), has the RNP if there is a Bochner-integrable function $f : \Omega \to X$ such that $\boldsymbol{\mu}(A) = \int_A f d|\boldsymbol{\mu}|$ for each $A \in \Sigma$. Clearly, all vector measures of bounded variation with values in X have the RNP if and only if X itself has the RNP.

Proposition 12 *Let* X *be a Banach space,* $\boldsymbol{\mu} : \Sigma \to X$ *a non-atomic vector measure of bounded variation, and* $\varphi : X \to \mathbb{R}$ *a superlinear norm continuous function. If* $\boldsymbol{\mu}$ *has the RNP, then*

$$\nu(A) = \varphi(\boldsymbol{\mu}(A)), \qquad \forall A \in \Sigma, \tag{8}$$

is an abstract linear production game.

For example, Proposition 12 implies that vector measure games generated by Lipschitz functions are market games if and only if they are abstract linear production games.[7] In fact, $\mathbb{R}^n$ has the RNP and it is easy to check that any superlinear Lipschitz function defined on a

[7]Specific games of the form (8), closely related to the so-called "newsvendor games," are studied in [26]. Example 2 in Section 5 shows that the Lipschizianity assumption cannot be dispensed with.

convex subset of $\mathbb{R}^n$ admits a continuous superlinear extension to the whole space. On the other hand, the vector measure $\boldsymbol{\mu} : \Sigma \to L^1(\lambda)$ given by $\boldsymbol{\mu}(A) = 1_A \in L_1(\lambda)$, which we used for the exchange economy market game (4), does not have the RNP property (see, e.g., [5, p. 15]). Hence, Proposition 12 cannot be used to determine when this market game is an abstract linear production game.

4.3 Exact Market Games

By Lemma 2, market games are never convex, unless they are additive. Next we show that, instead, there are plenty of examples of non trivial exact market games, so that exactness is for market games a much more relevant notion than convexity. In particular, we show that exact market games are a special case of abstract linear production games, and we fully characterize them and their cores. Observe that the lower probabilities considered in Section 6 are games of this type.

Theorem 13 *The following conditions are equivalent for a game ν:*

(i) ν is an exact market game;

(ii) ν is superadditive, and it admits an na-continuous extension ν^* to $B_1(\Sigma)$ such that

$$\nu^*(\alpha g + (1-\alpha)\beta 1_\Omega) = \alpha\nu^*(g) + (1-\alpha)\beta\nu^*(1_\Omega) \tag{9}$$

for all $\alpha, \beta \in [0,1]$ and all $g \in B_1(\Sigma)$;

(iii) ν is exact and ν_e is na-continuous on $B_1(\Sigma)$;

(iv) ν is exact, continuous, non-atomic, and its core is norm compact;

(v) ν is the lower envelope of some norm relatively compact subset K of $na_\alpha(\Sigma)$;

(vi) ν is a uniform limit of exact linear production games.

In this case, for any K such that (v) holds and for any sequence ν_n of exact linear production games that uniformly converges to ν, we have:

$$\overline{co}(K) = \overline{co}^*(K) = core(\nu) = \lim_n core(\nu_n), \tag{10}$$

where the limit is taken in the Hausdorff metric.

The following stronger version of Theorem 6 for the exact case is an immediate but noteworthy consequence of Theorem 13.

Corollary 14 *An exact game is a market game if and only if its core is a norm compact set of non-atomic measures.*

By Marinacci and Montrucchio [22], the core is also the unique von Neumann-Morgenstern stable set of an exact market game.

5 Examples

In this section we illustrate our results by means of some examples.

Example 1 Consider the bounded convex game $\nu(A) = \lambda^2(A)$, where λ is the Lebesgue measure on $[0,1]$. This game has the (convex) na-continuous extension $\nu^*(f) = \left(\int f d\lambda\right)^2$. Then its core is a weak compact subset of non-atomic probability measures (see Lemma 23 in Appendix). Since ν is non-additive, its core cannot be norm compact (see also Example 5). ▲

Example 2 Not all market games are abstract linear production games. Consider for example the vector measure game $\nu = \varphi(\lambda, \mu)$, where $\varphi(x,y) = \sqrt{xy}$ for all $(x,y) \in \mathbb{R}^2_+$ and λ, μ are two non atomic probability measures on a standard Borel space. The function φ is not Lipschitz and the game ν is a market game in pNA. Its extension ν^* cannot be extended as a superlinear function to the unit ball of $B(\Sigma)$. Thus ν is not an abstract linear production game. Notice that, however, this game can be represented as the minimum over a family of non-atomic measures. In fact,

$$\sqrt{\lambda(A)\mu(A)} = \min\left\{a\lambda(A) + b\mu(A) : (a,b) \in \mathbb{R}^2_+,\ ab \geq 1\right\}, \qquad \forall A \in \Sigma.$$

The set $\left\{a\lambda + b\mu : (a,b) \in \mathbb{R}^2_+,\ ab \geq 1\right\}$ is not norm compact. ▲

Example 3 Let $\varphi(x,y) = -\left(x + y^+\right)^+$ for all $(x,y) \in \mathbb{R}^2$. The measure game $\nu = \varphi(\lambda, \mu)$, with $\lambda, \mu \in na(\Sigma)$, admits a natural extension to $B(\Sigma)$. Hence, ν is a linear production game. It is actually a glove market game in that it is easy to see that it has the representation $\nu = \min\{0, -\lambda, -\lambda - \mu\}$. ▲

Example 4 We provide a simple example of an abstract linear production game. Let $\mathcal{B}$ be the Borel σ-algebra of the unit interval I, and let λ be the Lebesgue measure. Denote by π_n the uniform partition of I with cardinality n, and by $\mathcal{B}_n$ the finite σ-algebra generated by π_n. Fix an element $f \in L^1(\lambda)$ and define in $L^1(\lambda)$ the sequence:

$$f_n = \mathbb{E}\left[f \mid \mathcal{B}_n\right] = \sum_{A \in \pi_n} \frac{\int_A f d\lambda}{\lambda(A)} 1_A.$$

By the classic Martingale Convergence Theorem (see, e.g., [9, p. 67]), $f_n \to f$ in the L^1 norm. Denoting $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, consider the collection $\{\mu_n\}_{n \in \mathbb{N}^*}$ of non-atomic measures such that $d\mu_n / d\lambda = f_n$ and $d\mu_\infty / d\lambda = f$. The game $\nu(A) = \min_{n \in \mathbb{N}^*} \mu_n(A)$ is clearly an abstract linear production game. If, in addition, f is not $\mathcal{B}_n$-measurable for any n, this game is not a linear production game. By Theorem 13, ν is an exact market game and $core(\nu) = \overline{co}\{\mu_n : n \in \mathbb{N}^*\}$. A slight modification of this example delivers a non-exact abstract linear production game. It suffices to define $\nu(A) = \min_{n \in \mathbb{N}^*} \alpha_n \mu_n(A)$, where α_n is a scalar sequence approaching 1. ▲

Example 5 Let $\nu_\Gamma(A) = \min_{\mu\in\Gamma}\mu(A)$, where the set $\Gamma \subseteq na_\alpha(\Sigma)$ is weakly compact but not norm compact. By Proposition 1, ν is an exact, continuous and non-atomic game. However, in view of Corollary 14 ν is not a market game. Note that $\Gamma \subseteq core(\nu)$ and the inclusion may be strict. ▲

By Theorem 13, any norm compact and convex subset K of $na(\Sigma)$ such that $\mu_1(\Omega) = \mu_2(\Omega)$ for all $\mu_1, \mu_2 \in K$ is the core of an exact market game. The next examples show that this is not true for compact and convex sets not contained in $na(\Sigma)$. More generally, these examples show that (10) may fail altogether if the non-atomicity assumption is removed.

Example 6 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\Sigma = 2^\Omega$. Set

$$K = \{\mu \in ba(\Sigma) : \mu \text{ is a probability measure and } \mu(\omega_1) \le \mu(\omega_2)\}.$$

The probability measure μ such that $\mu(\omega_1) = \mu(\omega_3) = 1/2$ belongs to $core(\nu_K)$, but it does not belong to $co(K)(= \overline{co}(K) = \overline{co}^*(K))$. ▲

Example 7 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\Sigma = 2^\Omega$. Set $K = \{\mu, \phi\}$, where μ and ϕ are probability measures such that $\mu(\omega_1) = \mu(\omega_2) = 1/2$ and $\phi(\omega_3) = 1$, respectively. We have:

$$core(\nu_K) = \{(\xi_1, \xi_2, \xi_3) \in [0, 1/2] \times [0, 1/2] \times [0, 1] : \xi_1 + \xi_2 + \xi_3 = 1\},$$

where (ξ_1, ξ_2, ξ_3) is the measure taking value ξ_i on ω_i. Hence, $core(\nu_K)$ is larger than $co(K)$ $(= \overline{co}(K) = \overline{co}^*(K))$. Since ν_K is convex, this example, based on [19] and [31], shows that (10) fails even for sets of measures generating convex games. ▲

Example 8 Let $\Omega = [0, 1]$ and Σ is its Borel σ-algebra. Set $K = \{\delta_0, \lambda\}$, where δ_0 is the Dirac measure concentrated at 0 and λ is the Lebesgue measure on $[0, 1]$. Then,

$$\nu_K(A) = \begin{cases} \lambda(A) & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A, \end{cases}$$

is a continuous and convex game. A probability measure μ belongs to $core(\nu_K)$ if and only if $\mu(A) \ge \lambda(A)$ for all A containing 0, and

$$core(\nu_K) = \left\{ \left(1 - \int f d\lambda\right) \delta_0 + f d\lambda : 0 \le f \le 1 \ \lambda\text{-a.e.} \right\},$$

where $fd\lambda$ is the measure taking value $\int_A f d\lambda$ on A. Clearly, $core(\nu_K)$ is not finite dimensional and it is therefore much larger than the finite dimensional set $co(K)$ $(= \overline{co}(K) = \overline{co}^*(K))$. ▲

6 Multiple Priors Interpretation

Some of our results can be interpreted in the Multiple Priors (MP) model axiomatized by Gilboa and Schmeidler [14]. Recall that in the MP model beliefs are represented by a set C

of priors, and the non-singleton nature of this set reflects the limited information on which decision makers base the quantification of their beliefs.

Formally, *priors* are probability charges and *payoff prospects* are elements of $B(\Sigma)$. Gilboa and Schmeidler [14] give necessary and sufficient conditions on a preference relation $\succsim$ on $B(\Sigma)$ that guarantee the existence of a weak* compact set C of priors such that

$$f \succsim g \Leftrightarrow \min_{p\in C} p(f) \geq \min_{p\in C} p(g). \tag{11}$$

In the recent finance literature this model has been quite successful and the functional

$$V_C(f) = \min_{p\in C} p(f), \qquad \forall f \in B(\Sigma)$$

is sometimes called *coherent utility function* (see Delbaen [8]). Finally, the *lower probability*

$$\nu_C(A) = \min_{p\in C} p(A), \qquad \forall A \in \Sigma$$

can be viewed as the willingness to bet of the decision maker (see [13]).

The next result extends [6, Thm. 2], which give necessary and sufficient conditions on the preference $\succsim$ that guarantee that C is a subset of $na(\Sigma)$. Here we show that a simple technical condition on $\succsim$ ensures that C is a norm compact (finite dimensional, resp.) subset of $na(\Sigma)$.

Proposition 15 *Let $\succsim$ be a MP preference with a weak* compact set of priors C. The following conditions are equivalent:*

(i) C is a norm compact (finite dimensional, resp.) subset of $na(\Sigma)$;

(ii) if f_d, g_d are na-convergent bounded nets (na-convergent nets, resp.) in $B(\Sigma)$ such that $f_d \succsim g_d$ for all d, then $\lim_d f_d \succsim \lim_d g_d$.

The first important property of this kind of MP preferences is that they are uniquely determined by their willingness to bet:

Corollary 16 *Let $\succsim_1$ and $\succsim_2$ be two MP preferences with norm compact sets of priors C_1 and C_2 contained in $na(\Sigma)$. Then the following conditions are equivalent:*

(i) $f \succsim_1 k \Rightarrow f \succsim_2 k$ for all $f \in B(\Sigma)$ and $k \in \mathbb{R}$.

(ii) $\nu_{C_1} \leq \nu_{C_2}$.

In particular, $\succsim_1 = \succsim_2$ if and only if $\nu_{C_1} = \nu_{C_2}$.

Notice that point (i) amounts to say that $\succsim_1$ is more ambiguity averse than $\succsim_2$ in the sense of Ghirardato and Marinacci [13]. Moreover, by Proposition 10 we have:

Corollary 17 *If a norm compact set of priors C is contained in $na(\Sigma)$, then $core(\nu_C) = \overline{co}(C)$.*

As a result, it is easy to derive the core of the lower probability ν_C when C is a norm compact set of priors consisting of non-atomic probability measures. In particular, $core(\nu_C) = C$ if C itself is convex, so that convex and norm compact subsets of non-atomic probability measures are always the cores of the lower probabilities they generate.

All these considerations are false for general sets of priors, as Examples 6, 7, and 8 show.

By Corollary 8, lower probabilities generated by norm compact subsets of $na(\Sigma)$ form a convex set, and the map $\nu \mapsto core(\nu)$ is affine.

Corollary 18 *Let ν_{C_1} and ν_{C_2} be lower probabilities generated by norm compact sets of priors C_1 and C_2 contained $na(\Sigma)$. Then, for each $t \in [0,1]$, the set function $t\nu_{C_1} + (1-t)\nu_{C_2}$ is the lower probability generated by $tC_1 + (1-t)C_2$, and*

$$core(t\nu_{C_1} + (1-t)\nu_{C_2}) = t\,core(\nu_{C_1}) + (1-t)\,core(\nu_{C_2}).$$

Corollary 11 takes the following form in the MP context.

Corollary 19 *The lower probability ν_C generated by a norm compact set C of priors contained in $na(\Sigma)$ is convex if and only if it is a non-atomic measure.*

As observed after Corollary 11, this implies that if the core of a convex lower probability consists of non-atomic measures (that is, if the lower probability is continuous and non-atomic), then it cannot have finitely many extreme points (unless the lower probability is a non-atomic probability measure itself).

Another important consequence of Corollaries 16 and 19 is that MP preferences featuring sets of priors that are norm compact subsets of $na(\Sigma)$ are never Choquet Expected Utility preferences (see Schmeidler [30]), unless they are Subjective Expected Utility preferences.[8]

A Proofs and Related Material

A.1 Some Preliminary Lemmas on ν^*

In this section we report some essentially known results on the na-extensions that we will need in what follows.

If λ is a positive measure, we denote by $I^\infty(\lambda)$ the positive unit ball of $L^\infty(\lambda)$, i.e., $I^\infty(\lambda) = \{g \in L^\infty(\lambda) : 0 \leqslant g \leqslant 1\ \lambda - a.e.\}$, and by $ca(\Sigma, \lambda)$ the λ-absolutely continuous measures, i.e., $ca(\Sigma, \lambda) = \{\mu \in ca(\Sigma, \lambda) : \mu \ll \lambda\}$. Clearly, $g \in I^\infty(\lambda)$ iff there is $f \in B_1(\Sigma)$ such that $g(\omega) = f(\omega)$ for λ-almost every ω in Ω; to be precise, $I^\infty(\lambda)$ is the image of $B_1(\Sigma)$ through the projection π_λ that associates to each $f \in B(\Sigma)$ the equivalence class $[f] \in L^\infty(\lambda)$ of all measurable functions that are λ-almost everywhere equal to f.[9] The set

[8] In other words, coherent utility functions generated by norm compact subsets of $na(\Sigma)$ are comonotonic additive if and only if they are additive (see Schmeidler [29]).

[9] With a a little abuse, we will often neglect distinguishing f from $[f]$.

$ca(\Sigma, \lambda)$ is a closed subspace of $ca(\Sigma)$ and it is isometrically isomorphic to $L^1(\lambda)$. We will use repeatedly the fact that $L^\infty(\lambda)$ is (isometrically isomorphic to) the norm dual of $L^1(\lambda)$.

Given a subset F of $L^\infty(\lambda)$, the relative $\sigma\left(L^\infty(\lambda), L^1(\lambda)\right)$-topology (or $\sigma\left(L^\infty(\lambda), ca(\Sigma, \lambda)\right)$) has as neighborhood base at $f \in F$ the sets $V_f(\varepsilon; \mu_1, ..., \mu_n)$ of the form:

$$V_f(\varepsilon; \mu_1, ..., \mu_n) = \{g \in F : |\langle g, \mu_i \rangle - \langle f, \mu_i \rangle| < \varepsilon \ \forall i = 1, ..., n\}$$

where each $\mu_i \ll \lambda$ and $\varepsilon > 0$. The relative $\sigma\left(L^1(\lambda), L^\infty(\lambda)\right)$-topology has as neighborhood base at $f \in F$ the sets $V_f(\varepsilon; g_1, ..., g_n)$ of the form:

$$V_f(\varepsilon; g_1, ..., g_n) = \left\{g \in F : \left| \int g_i g d\lambda - \int g_i f d\lambda \right| < \varepsilon \ \forall i = 1, ..., n\right\}$$

where each $g_i \in L^\infty(\lambda)$ and $\varepsilon > 0$. Clearly, the $\sigma\left(L^\infty(\lambda), L^1(\lambda)\right)$-topology is a relative weak* topology, while $\sigma\left(L^1(\lambda), L^\infty(\lambda)\right)$-topology is a relative weak topology. These relative topologies coincide if F is bounded in $L^\infty(\lambda)$.

Lemma 20 *The relative $\sigma\left(L^\infty(\lambda), L^1(\lambda)\right)$ and $\sigma\left(L^1(\lambda), L^\infty(\lambda)\right)$ topologies coincide on $\|\cdot\|_\infty$-bounded subsets of $L^\infty(\lambda)$.*

If λ belongs to $na(\Sigma)$, then $ca(\Sigma, \lambda)$ is the set of all λ-absolutely continuous non-atomic measures. In this case, the topology induced on a subset G of $B(\Sigma)$ by the projection π_λ on $L^\infty(\lambda)$ with the weak* topology is called *na_λ-topology*. Clearly the na_λ-topology is the coarsest topology that makes continuous all the functionals $g \mapsto \mu(g)$ with $\mu \in ca(\Sigma, \lambda)$, and hence it is weaker than the na-topology.

The following result is essentially due to [2] and [24].

Proposition 21 *Let (Ω, Σ) be a standard Borel space. The following conditions are equivalent for a game $\nu : \Sigma \to \mathbb{R}$.*

(i) ν admits an na-continuous extension to $B_1(\Sigma)$;

(ii) ν admits an na_λ-continuous extension to $B_1(\Sigma)$ for some non-atomic probability measure λ;

(iii) ν admits an $\sigma\left(L^\infty(\lambda), L^1(\lambda)\right)$-continuous extension to $I^\infty(\lambda)$ for some non-atomic probability measure λ;

(iv) ν belongs to the closure pNA' in the supnorm of the linear space of games that is generated by all powers of non-atomic probability measures.

The proof is based on the following useful result due to [24].

Lemma 22 *Let (Ω, Σ) be a standard Borel space and $\upsilon : B_1(\Sigma) \to \mathbb{R}$ a na-continuous function. Then there is a non-atomic probability measure λ such that*

$$\upsilon(f) = \upsilon(g) \ \text{ for all } f, g \in B_1(\Sigma) \ \text{ s.t. } g = f \ \lambda - a.e. \tag{12}$$

The next result is essentially [2, Prop. 44.27].

Lemma 23 *If a game ν admits an extension ν^* to $B_1(\Sigma)$ which is na-continuous at* 0 *and at* 1_Ω*, then core* $(\nu) \subseteq na(\Sigma)$.

The functional $\nu_e : B(\Sigma) \to \mathbb{R}$ given by (1) is defined for any balanced game ν (on the other hand, it is an extension of ν iff ν is exact).

Lemma 24 *If a balanced game ν admits a na-continuous extension ν^* to $B_1(\Sigma)$, then $\nu^* \leq \nu_e$ on $B_1(\Sigma)$.*

A.2 Norm Compactness and Support Functionals

Lemma 25 *A non-empty subset K of a Banach space X is norm relatively compact iff it is bounded and its support functional $\sigma_K : X^* \to \mathbb{R}$ ($x^* \mapsto \sup_{x \in K} \langle x, x^* \rangle$) is continuous in the bounded weak* topology.*[10]

Notice that this lemma completes Theorems 6 and 7 of Hormander [18].

Proof. Let $B_{X^*} = \{x^* \in X^* : \|x^*\| \leq 1\}$ be the unit ball in X^*. Since σ_K is positively homogeneous, it is continuous in the bounded weak* topology iff its restriction to B_{X^*} is continuous for the relative weak* topology.

Let K be relatively norm compact. Given $\varepsilon > 0$, choose $x_1, ..., x_n \in K$ such that for all $x \in K$, there exists i such that $\|x - x_i\| \leq \varepsilon$. For all $x^* \in B_{X^*}$, consider the relative weak* neighborhood of x^* given by

$$U_{x^*}(x_1, ..., x_n; \varepsilon) = \{y^* \in B_{X^*} : |\langle x_j, x^* - y^* \rangle| < \varepsilon \ \forall j = 1, ..., n\}.$$

If $y^* \in U_{x^*}$, for all $x \in K$,

$$\begin{aligned} \langle x, x^* - y^* \rangle - \langle x_i, x^* - y^* \rangle &= \langle x - x_i, x^* - y^* \rangle \\ &\leq \|x - x_i\| \|x^* - y^*\| \leq 2\varepsilon, \text{ i.e.,} \\ \langle x, x^* - y^* \rangle &\leq \langle x_i, x^* - y^* \rangle + 2\varepsilon \\ &\leq \max_{j=1,...,n} \langle x_j, x^* - y^* \rangle + 2\varepsilon \leq 3\varepsilon. \end{aligned}$$

Whence $\sigma_K(x^* - y^*) = \sup_{x \in K} \langle x, x^* - y^* \rangle \leq 3\varepsilon$; analogous considerations yield $\sigma_K(y^* - x^*) \leq 3\varepsilon$ (just switch x^* and y^*). By the subadditivity of the support function, we can conclude that

$$|\sigma_K(x^*) - \sigma_K(y^*)| \leq \max(\sigma_K(x^* - y^*), \sigma_K(y^* - x^*)) \leq 3\varepsilon,$$

for all $y^* \in U_{x^*}$, and so the restriction of σ_K to B_{X^*} is continuous in the relative weak* topology.

[10] See [10, Ch. V.5] or [23, Ch. 2.7] for the definition and properties of this topology.

Conversely, assume K is bounded and σ_K is continuous in the bounded weak* topology. Then, the restriction of σ_K to $2B_{X^*}$ is continuous in the relative weak* topology. For all $\varepsilon > 0$ there exists a relative weak* neighborhood of 0 (in $2B_{X^*}$)

$$U_0(x_1, ..., x_n; \delta) = \{z^* \in 2B_{X^*} : |\langle x_j, z^* \rangle| < \delta \; \forall j = 1, ..., n\}$$

such that $\sigma_K(z^*) \leq \varepsilon$ for all $z^* \in U_0$. For all $x^* \in B_{X^*}$ consider the relative weak* neighborhood $U_{x^*}(x_1, ..., x_n; \delta)$. For all $y^* \in U_{x^*}$ we have $x^* - y^*, y^* - x^* \in U_0$. Therefore, $\sigma_K(x^* - y^*) \leq \varepsilon$ and $\sigma_K(y^* - x^*) \leq \varepsilon$, so that

$$\sup_{y^* \in U_{x^*}} \sup_{x \in K} |\langle x, x^* - y^* \rangle| \leq \varepsilon.$$

When the elements of X are regarded as weak* continuous functions on the weak* compact set B_{X^*}, what we proved amounts to the fact that: For all $\varepsilon > 0$ and all $x^* \in B_{X^*}$, there exists a neighborhood U_{x^*} of x^* such that $\sup_{x \in K} \sup_{y^* \in U_{x^*}} |\langle x, x^* \rangle - \langle x, y^* \rangle| \leq \varepsilon$. That is, K is equicontinuous. Being bounded ($\sup_{x^* \in B_{X^*}} |\langle x, x^* \rangle| = \|x\| \leq \sup_{y \in K} \|y\|$ for all $x \in K$) and equicontinuous, K is relatively compact by the Ascoli-Arzelà Theorem in the space $C(B_{X^*})$ of all weak* continuous functions on B_{X^*} endowed with the supnorm. Notice that

$$d_X(x, y) = \|x - y\| = \max_{x^* \in B_{X^*}} |\langle x, x^* \rangle - \langle y, x^* \rangle| = d_{C(B_{X^*})}(\langle x, \cdot \rangle, \langle y, \cdot \rangle).$$

Since every sequence in K admits a Cauchy subsequence in $C(B_{X^*})$, it admits a Cauchy subsequence in X. We conclude that K is norm relatively compact in X (since X is complete). ■

For future reference we state the following immediate reformulation of Lemma 25.

Lemma 26 *A subset K of a Banach space X is norm relatively compact iff it is bounded and the functional $x^* \mapsto \inf_{x \in K} \langle x, x^* \rangle$ is continuous on B_{X^*} in the relative weak* topology.*

We close with a result on superdifferentials.

Lemma 27 *Let X be a Banach space and $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ a concave function. If $\bar{x} \in X$ is such that $\varphi(\alpha \bar{x}) = \alpha \varphi(\bar{x})$ for all $\alpha \in [0, 1]$, then*

$$\partial \varphi(\alpha \bar{x}) = \{x^* \in \partial \varphi(0) : x^*(\bar{x}) = \varphi(\bar{x})\} = \{x^* \in \partial \varphi(\bar{x}) : x^*(\bar{x}) = \varphi(\bar{x})\}$$

for all $\alpha \in (0, 1)$.

Proof. Let $\alpha \in (0, 1)$ and $x^* \in \partial \varphi(\alpha \bar{x})$. By definition,

$$\varphi(y) - x^*(y) \leq \varphi(\alpha \bar{x}) - x^*(\alpha \bar{x}) = \alpha [\varphi(\bar{x}) - x^*(\bar{x})] \tag{13}$$

for all $y \in X$. Setting $y = \beta \bar{x}$ with $\beta \in (0, 1) \setminus \{\alpha\}$, we get

$$(\beta - \alpha) [\varphi(\bar{x}) - x^*(\bar{x})] \leq 0$$

whence $\varphi(\bar{x}) = x^*(\bar{x})$. Hence, getting back to (13), we conclude that $\varphi(y) - x^*(y) \leq 0 = \varphi(\bar{x}) - x^*(\bar{x})$. This implies both $x^* \in \partial\varphi(0)$ and $x^* \in \partial\varphi(\bar{x})$. That is

$$\partial\varphi(\alpha\bar{x}) \subseteq \{x^* \in \partial\varphi(0) : x^*(\bar{x}) = \varphi(\bar{x})\} \cap \{x^* \in \partial\varphi(\bar{x}) : x^*(\bar{x}) = \varphi(\bar{x})\}.$$

Conversely:

- if $x^* \in \partial\varphi(0)$ and $x^*(\bar{x}) = \varphi(\bar{x})$, then for all $y \in X$

$$\varphi(y) - x^*(y) \leq 0 = \varphi(\bar{x}) - x^*(\bar{x}) = \varphi(\alpha\bar{x}) - x^*(\alpha\bar{x});$$

- if $x^* \in \partial\varphi(\bar{x})$ and $x^*(\bar{x}) = \varphi(\bar{x})$, then for all $y \in X$

$$\varphi(y) - x^*(y) \leq \varphi(\bar{x}) - x^*(\bar{x}) = 0 = \varphi(\alpha\bar{x}) - x^*(\alpha\bar{x});$$

  in any case $x^* \in \partial\varphi(\alpha\bar{x})$. ■

Lemma 28 *Let X be a Banach space, K^* a weak* compact subset of X^*, and*

$$\varphi(x) \equiv \min_{x^* \in K^*} \langle x, x^* \rangle, \quad \forall x \in X.$$

Then $\partial\varphi(x) = \overline{co}^{w^*}(\{x^* \in K^* : \langle x, x^* \rangle = \varphi(x)\})$ for each $x \in X$.

Proof. Clearly,

$$\varphi(x) = \min_{x^* \in \overline{co}^{w^*}(K^*)} \langle x, x^* \rangle, \quad \forall x \in X,$$

and so $\partial\varphi(0) = \overline{co}^{w^*}(K^*)$ as $\overline{co}^{w^*}(K^*)$ is convex and weak* compact. Moreover, $\partial\varphi(x) = \{x^* \in \partial\varphi(0) : \langle x, x^* \rangle = \varphi(x)\}$ for all $x \in X$. Since K^* is weak* compact $ext(\partial\varphi(0)) = ext\left(\overline{co}^{w^*}(K^*)\right) \subseteq K^*$ (see [23, Cor. 2.10.18]).

Arbitrarily choose $x \in X$. $\partial\varphi(x)$ is a weak* compact and convex subset of X^*. Next we show that $\partial\varphi(x)$ is extremal in $\partial\varphi(0)$. In fact, if $x^*, y^* \in \partial\varphi(0)$, $t \in (0,1)$, and $tx^* + (1-t)y^* \in \partial\varphi(x)$, then

$$\begin{aligned} \langle x, x^* \rangle &\geq \varphi(x), \\ \langle x, y^* \rangle &\geq \varphi(x), \text{ and} \\ t\langle x, x^* \rangle + (1-t)\langle x, y^* \rangle &= \langle x, tx^* + (1-t)y^* \rangle = \varphi(x); \end{aligned}$$

therefore $\langle x, x^* \rangle = \langle x, y^* \rangle = \varphi(x)$ and $x^*, y^* \in \partial\varphi(x)$.

Since $\partial\varphi(x)$ is extremal in $\partial\varphi(0)$, if u^* is an extreme point of $\partial\varphi(x)$, then u^* is an extreme point of $\partial\varphi(0)$ and it belongs to K^*. That is

$$ext(\partial\varphi(x)) = ext(\partial\varphi(0)) \cap \partial\varphi(x) \subseteq K^* \cap \partial\varphi(x) = \{x^* \in K^* : \langle x, x^* \rangle = \varphi(x)\}.$$

By the Krein-Milman Theorem

$$\begin{aligned} \partial\varphi(x) &= \overline{co}^{w^*}(ext(\partial\varphi(x))) \subseteq \overline{co}^{w^*}(\{x^* \in K^* : \langle x, x^* \rangle = \varphi(x)\}) \\ &\subseteq \overline{co}^{w^*}\left(\left\{x^* \in \overline{co}^{w^*}(K^*) : \langle x, x^* \rangle = \varphi(x)\right\}\right) = \left\{x^* \in \overline{co}^{w^*}(K^*) : \langle x, x^* \rangle = \varphi(x)\right\} \\ &= \partial\varphi(x). \end{aligned}$$

■

A.3 A Separation Result

In the sequel we will need the following separation lemma, which is of some independent interest.

Lemma 29 *Let K_1 and K_2 be nonempty, disjoint, norm compact, and convex subsets of $na_\alpha(\Sigma)$. Then, there exists A in Σ such that*

$$\min_{\phi_1 \in K_1} \phi_1(A) > \max_{\phi_2 \in K_2} \phi_2(A). \tag{14}$$

Proof. Since K_i is weakly compact and it consists of non-atomic measures, there exists a non-atomic probability measure λ_i such that $\phi_i \ll \lambda_i$ for all ϕ_i in K_i, $i = 1, 2$ (see [10, Thm IV.9.2]). Therefore, all measures in $K_1 \cup K_2$ are absolutely continuous w.r.t. $\lambda = 2^{-1}(\lambda_1 + \lambda_2)$. Therefore, K_1 and K_2 are norm compact subsets of $ca(\Sigma, \lambda)$.

The Separation Hyperplane Theorem guarantees that there exist $f \in L^\infty(\lambda) \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that

$$\min_{\phi_1 \in K_1} \phi_1(f) > \beta > \max_{\phi_2 \in K_2} \phi_2(f). \tag{15}$$

Since $K_1, K_2 \subseteq na_\alpha(\Sigma)$, w.l.o.g. we can choose $f \in I^\infty(\lambda)$. In other words, setting $F(g) = \min_{\phi_1 \in K_1} \phi_1(g) - \max_{\phi_2 \in K_2} \phi_2(g)$, there exists $f \in I^\infty(\lambda)$ such that $F(f) > 0$. By Lemma 25, F is continuous w.r.t. the relative weak* topology on $I^\infty(\lambda)$. Hence, since the set $\{1_A\}_{A \in \Sigma}$ is dense in $I^\infty(\lambda)$ by the Lyapunov Theorem, we conclude that (14) holds. ■

Corollary 30 *Let K_1 and K_2 be nonempty subsets of $na_\alpha(\Sigma)$. If K_2 is norm relatively compact, then*

$$\inf_{\phi_1 \in K_1} \phi_1(A) \geq \inf_{\phi_2 \in K_2} \phi_2(A) \quad \forall A \in \Sigma \Rightarrow K_1 \subseteq \overline{co}(K_2).$$

Proof. Assume, by contradiction that $\inf_{\phi_1 \in K_1} \phi_1(A) \geq \inf_{\phi_2 \in K_2} \phi_2(A)$ for all $A \in \Sigma$ and there exists $\phi \in K_1 - \overline{co}(K_2)$. Since $\overline{co}(K_2) = \overline{co}(\overline{K_2})$ and $\overline{K_2}$ is norm compact, by the Mazur Compactness Theorem (see, [23, Thm 2.8.15]) $\overline{co}(K_2)$ is norm compact. Direct application Lemma 29 to $\overline{co}(K_2)$ and $\{\phi\}$, yields that there exists $B \in \Sigma$ such that

$$\inf_{\phi_2 \in K_2} \phi_2(B) \geq \min_{\mu \in \overline{co}(K_2)} \mu(B) > \phi(B) > \inf_{\phi_1 \in K_1} \phi_1(B),$$

which is absurd. ■

A.4 Proofs of the Results

Proposition 3. Let $\nu = \varphi \circ \boldsymbol{\mu}$. The variation $\lambda = |\boldsymbol{\mu}|$ of $\boldsymbol{\mu}$ belongs to $na(\Sigma)$. Consider the linear map $T_{\boldsymbol{\mu}} : B(\Sigma) \to X$ defined by $T_{\boldsymbol{\mu}}(f) = \int f d\boldsymbol{\mu}$. As $\boldsymbol{\mu} \ll \lambda$, we can regard $T_{\boldsymbol{\mu}}$ as a map $T_{\boldsymbol{\mu}} : L^\infty(\lambda) \to X$. It is easy to check that this map is a weak*- to-weak continuous linear operator. Moreover,

$$R(\boldsymbol{\mu}) = \left\{ \int f d\boldsymbol{\mu} : f \in B_1(\Sigma) \right\} = \left\{ \int f d\boldsymbol{\mu} : f \in I^\infty(\lambda) \right\},$$

hence, $\varphi \circ T_{\boldsymbol{\mu}} : I^{\infty}(\lambda) \to \mathbb{R}$ is concave and weak* continuous. As λ is non-atomic and the projection $\pi_\lambda : B_1(\Sigma) \to I^{\infty}(\lambda)$ is (by definition) na_λ-to-weak* continuous, we conclude that $\varphi \circ T_{\boldsymbol{\mu}} \circ \pi_\lambda : B_1(\Sigma) \to \mathbb{R}$ is the na_λ-continuous extension of ν to $B_1(\Sigma)$.

As to the converse, by Lemma 20 and Proposition 21 there is a non-atomic probability measure λ such that the map $1_A \mapsto \nu(A)$ admits a weakly continuous extension ν^* to $I^{\infty}(\lambda)$. Consider the measure $\boldsymbol{\mu} : \Sigma \to L^1(\lambda)$ given by $\boldsymbol{\mu}(A) = 1_A$ and $\varphi = \nu^*$. Then, $\nu = \varphi \circ \boldsymbol{\mu}$. ■

Proposition 4. Denote $T_{\boldsymbol{\mu}} : B(\Sigma) \to X$ the linear operator defined by

$$T_{\boldsymbol{\mu}}(f) = \int f d\boldsymbol{\mu}, \quad \forall f \in B(\Sigma).$$

By [9, p. 6], $T_{\boldsymbol{\mu}}$ is a bounded linear operator. Set $\varphi(x) = -\infty$ if $x \notin R(\boldsymbol{\mu})$. Notice that, by Proposition 3, $(\varphi \circ \boldsymbol{\mu})^* = (\varphi \circ T_{\boldsymbol{\mu}})_{|B_1(\Sigma)}$, and since $2^{-1}1_\Omega$ is in the supnorm interior of $B_1(\Sigma)$

$$\partial (\varphi \circ \boldsymbol{\mu})^* \left(2^{-1}1_\Omega\right) = \partial (\varphi \circ T_{\boldsymbol{\mu}}) \left(2^{-1}1_\Omega\right).$$

Assume $core(\varphi \circ \boldsymbol{\mu}) \neq \varnothing$, then by Claim 1 in the proof of Theorem 6 $(\varphi \circ \boldsymbol{\mu})^*$ is positively homogeneous along the diagonal, whence

$$\begin{aligned} \varphi(\alpha \boldsymbol{\mu}(\Omega)) &= \varphi(T_{\boldsymbol{\mu}}(\alpha 1_\Omega)) = (\varphi \circ \boldsymbol{\mu})^* (\alpha 1_\Omega) \\ &= \alpha (\varphi \circ \boldsymbol{\mu})^* (1_\Omega) = \alpha \varphi(\boldsymbol{\mu}(\Omega)), \quad \forall \alpha \in [0,1]. \end{aligned}$$

Proposition 31, the chain rule for superdifferentials (see, e.g. [11, p. 119]), and Lemma 27 guarantee that

$$\begin{aligned} core(\varphi \circ \boldsymbol{\mu}) &= (\varphi \circ \boldsymbol{\mu})^* \left(2^{-1}1_\Omega\right) = \partial (\varphi \circ T_{\boldsymbol{\mu}}) \left(2^{-1}1_\Omega\right) \\ &= \left\{x^* \circ T_{\boldsymbol{\mu}} : x^* \in \partial\varphi\left(T_{\boldsymbol{\mu}}\left(2^{-1}1_\Omega\right)\right)\right\} \\ &= \left\{x^* \circ \boldsymbol{\mu} : x^* \in \partial\varphi\left(2^{-1}\boldsymbol{\mu}(\Omega)\right)\right\} \\ &= \left\{x^* \circ \boldsymbol{\mu} : x^* \in \partial\varphi(0) \text{ and } x^*(\boldsymbol{\mu}(\Omega)) = \varphi(\boldsymbol{\mu}(\Omega))\right\} \\ &= \left\{x^* \circ \boldsymbol{\mu} : x^* \in \partial\varphi(\boldsymbol{\mu}(\Omega)) \text{ and } x^*(\boldsymbol{\mu}(\Omega)) = \varphi(\boldsymbol{\mu}(\Omega))\right\}. \end{aligned}$$

■

Proposition 5. Let us prove point (iii). Assume that the game ν is generated by an economy with n goods. Namely, $u(\cdot, \omega) : \mathbb{R}^n_+ \to \mathbb{R}$. Let P be the set of competitive prices. By [16, Proposition 2.20], $P = \partial u_\Omega \left(\int a d\lambda\right)$ where $u_\Omega(\overline{a})$ is the value function as a function of the total initial allocation $\overline{a} = \int a d\lambda$. Therefore, $\dim P \leq n$. By [16, Proposition 2.10], if $x(\omega)$ is such that $\int u(x(\omega), \omega) d\lambda = \nu(\Omega)$, then the set of all competitive payoff densities is given by $u(x(\omega), \omega) - p \cdot (x(\omega) - a(\omega))$ with $p \in P$. Such a set of L^1 functions belongs to a finitely dimensional space having dimension not greater than n. Hence $\dim core(\nu) \leq n$.

To conclude the proof we show that the bound n is reached. Consider a continuous and concave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ and assume that the $u0^+ = 0$. Here $u0^+$ is the recession function given by

$$\left(u0^+\right)(y) = \lim_{\lambda \to +\infty} \lambda^{-1} \left[u(x + \lambda y) - u(x)\right].$$

We then consider the transferable utility exchange economy with utilities $U(x,\omega) = (u\omega)(x) = \omega u\left(\omega^{-1}x\right)$ for $\omega \in I = [0,1]$ and where the players's space is the standard Borel space $(I, \mathcal{B}, \lambda)$. The transferable utility competitive equilibria $(p, x(\omega))$ are characterized by the two conditions:

$$p \in \partial U(x(\omega), \omega), \quad \int_I x d\lambda = \int_I a d\lambda.$$

A dual characterization can be formulated by the Fenchel conjugate $U^*(p,\omega)$ defined as

$$U^*(p,\omega) = \inf_{x \geq 0} p \cdot x - U(x,\omega).$$

More precisely, $U^*(p,\omega)$ is the so-called concave monotone Fenchel conjugate to U. Clearly, $p \in \partial U(x(\omega),\omega) \Longleftrightarrow x(\omega) \in \partial U^*(p,\omega)$. On the other hand, $U^*(p,\omega) = \omega u^*(p)$, where $u^*(p)$ is the Fenchel conjugate to $u(x)$. Therefore, $\omega^{-1}x(\omega) \in \partial u^*(p)$. It is easy to deduce that

$$x(\omega) = 2\omega \int_I a, \ \ 2\int_I a \in \partial u^*(p) \Longleftrightarrow p \in \partial u\left(2\int_I a\right).$$

If we pick an utility function $u(x)$ such that at the point $\overline{x} = 2\int_I a$ its superdifferential is of dimension n we are done. Actually, by the Fenchel's equality $U(x(\omega),\omega) + U^*(p,\omega) = p \cdot x(\omega)$ we get easily that the competitive payoff densities are $p \cdot a(\omega) - \omega u^*(p)$ with $p \in \partial u\left(2\int_I a\right)$. By a suitable choice of the vector $a(\omega)$, the set of the payoff distribution is of dimension n. Of course the set of the competitive payoff distributions coincides with the core. ■

Theorem 6. If $core(\nu) = \varnothing$ the result is trivial. Assume that $core(\nu) \neq \varnothing$, and denote by ν^* the concave and *na*-continuous extension of ν to $B_1(\Sigma)$. Lemma 23 guarantees that $core(\nu) \subseteq na(\Sigma)$, it remains to show that $core(\nu)$ is norm compact.

Claim 1. $core(\nu) = \partial\nu^*\left(2^{-1}1_\Omega\right)$, where $\partial\nu^*$ is the superdifferential of ν^*.

The proof of this claim follows the same lines of [12, Thm. A]. First we prove that $\nu^*(\alpha 1_\Omega) = \alpha\nu^*(1_\Omega)$ for all $\alpha \in [0,1]$, namely, ν^* is positively homogeneous along the diagonal. If $\mu \in core(\nu)$, then $\mu(f) \geq \nu^*(f)$ for all $f \in B_1(\Sigma)$ (see Lemma 24). In particular, $\alpha\nu(\Omega) = \alpha\mu(\Omega) = \mu(\alpha 1_\Omega) \geq \nu^*(\alpha 1_\Omega)$. Concavity of ν^* implies $\nu^*(\alpha 1_\Omega) \geq \alpha\nu^*(1_\Omega)$. The claim immediately follows. In fact, if $\mu \in core(\nu)$, for all $f \in B_1(\Sigma)$ we have $\mu(f) \geq \nu^*(f)$ and $\mu\left(2^{-1}1_\Omega\right) = 2^{-1}\mu(1_\Omega) = 2^{-1}\nu^*(1_\Omega) = \nu^*\left(2^{-1}1_\Omega\right)$, whence

$$\mu(f) - \mu\left(2^{-1}1_\Omega\right) \geq \nu^*(f) - \nu^*\left(2^{-1}1_\Omega\right), \quad \forall f \in B_1(\Sigma), \tag{16}$$

that is $\mu \in \partial\nu^*\left(2^{-1}1_\Omega\right)$. Conversely, if $\mu \in \partial\nu^*\left(2^{-1}1_\Omega\right)$, μ satisfies (16), setting $f = 0$ we obtain $\mu\left(2^{-1}1_\Omega\right) \leq \nu^*\left(2^{-1}1_\Omega\right)$, setting $f = 1_\Omega$ we obtain $\mu\left(2^{-1}1_\Omega\right) \geq \nu^*\left(2^{-1}1_\Omega\right)$, whence

$$\mu(\Omega) = \nu(\Omega) \ \text{ and } \mu(f) \geq \nu^*(f), \quad \forall f \in B_1(\Sigma),$$

a fortiori, $\mu \in core(\nu)$. □

Given the concave function $\nu^* : B_1(\Sigma) \to \mathbb{R}$, let f_0 be an interior point of $B_1(\Sigma)$, and choose a positive scalar η such that $\|f - f_0\| \leq \eta$ implies $f \in B_1(\Sigma)$. Denote by $D\nu^*(f_0; h)$, with $h \in B(\Sigma)$ the directional derivative, i.e.,

$$\begin{aligned} D\nu^*(f_0; h) &= \lim_{t \to 0^+} \frac{\nu^*(f_0 + th) - \nu^*(f_0)}{t} \\ &= \sup_{t>0} \frac{\nu^*(f_0 + th) - \nu^*(f_0)}{t}. \end{aligned}$$

Since ν^* is supnorm continuous, we have (see [3, Prop. 2.3]) that $\partial\nu^*(f_0)$ is a non-empty, convex, and weak* compact subset of $ba(\Sigma)$. Moreover,

$$D\nu^*(f_0; h) = \min_{\phi \in \partial\nu^*(f_0)} \phi(h), \quad \forall h \in B(\Sigma). \tag{17}$$

Claim 2. If $\partial\nu^*(f_0) \subseteq na(\Sigma)$, the function $D\nu^*(f_0; \cdot)$ is continuous on the unit ball $B_{B(\Sigma)}$ in the na-topology.

In view of (17), $D\nu^*(f_0; \cdot)$ is na-upper semicontinuous on $B_{B(\Sigma)}$, provided that $\partial\nu^*(f_0) \subseteq na(\Sigma)$. Consequently, it suffices to check that the function $D\nu^*(f_0; \cdot)$ is na-lower semicontinuous on $B_{B(\Sigma)}$. This is obtained with a technique inspired by Hart [15].

Fix $h_0 \in B_{B(\Sigma)}$ and $\varepsilon > 0$. We can find a scalar positive scalar $\bar{t} \leq \eta$ such that:

$$\frac{\nu^*\left(f_0 + \bar{t}h_0\right) - \nu^*(f_0)}{\bar{t}} \geq D\nu^*(f_0; h_0) - \varepsilon/2 \tag{18}$$

(the scalar η has been defined in before to guarantee that $f \in B_1(\Sigma)$ if $\|f - f_0\| \leq \eta$, hence $f_0 + \bar{t}h \in B_1(\Sigma)$ for all $h \in B_{B(\Sigma)}$ and $\nu^*\left(f_0 + \bar{t}h\right)$ is well defined).

As ν^* is na-continuous on $B_1(\Sigma)$, there exist a na-neighborhood

$$U_{f_0+\bar{t}h_0}(\mu_1, ..., \mu_n; \delta) = \left\{ f \in B_1(\Sigma) : \max_{j=1,..,n} \left|\mu_j\left(f - \left(f_0 + \bar{t}h_0\right)\right)\right| < \delta \right\} \tag{19}$$

of $f_0 + \bar{t}h_0$ in $B_1(\Sigma)$ such that

$$\left|\nu^*(f) - \nu^*\left(f_0 + \bar{t}h_0\right)\right| < \varepsilon\bar{t}/2, \quad \forall f \in U_{f_0+\bar{t}h_0}(\mu_1, ..., \mu_n; \delta).$$

Consider now the following na-neighborhood of h_0 in $B_{B(\Sigma)}$:

$$U_{h_0}\left(\mu_1, ..., \mu_n; \delta/\bar{t}\right) = \left\{ h \in B_{B(\Sigma)} : \max_{j=1,..,n} |\mu_j(h - h_0)| < \delta/\bar{t} \right\}.$$

It is immediately checked that if $h \in U_{h_0}$, then $f_0 + \bar{t}h \in U_{f_0+\bar{t}h_0}$. In fact, $f_0 + \bar{t}h \in B_1(\Sigma)$ and

$$\max_{j=1,..,n} \left|\mu_j\left(f_0 + \bar{t}h - \left(f_0 + \bar{t}h_0\right)\right)\right| = \bar{t} \max_{j=1,..,n} |\mu_j(h - h_0)| < \delta.$$

Hence, $h \in U_{h_0}$ implies $\left|\nu^*\left(f_0 + \bar{t}h\right) - \nu^*\left(f_0 + \bar{t}h_0\right)\right| < \varepsilon\bar{t}/2$ and

$$\nu^*\left(f_0 + \bar{t}h\right) > \nu^*\left(f_0 + \bar{t}h_0\right) - \varepsilon\bar{t}/2.$$

In view of (18), we have

$$\begin{aligned} D\nu^*(f_0;h) &\geq \frac{\nu^*\left(f_0+\bar{t}h\right)-\nu^*(f_0)}{\bar{t}} > \frac{\nu^*\left(f_0+\bar{t}h_0\right)-\varepsilon\bar{t}/2-\nu^*(f_0)}{\bar{t}} \\ &= \frac{\nu^*\left(f_0+\bar{t}h_0\right)-\nu^*(f_0)}{\bar{t}} - \varepsilon/2 \geq D\nu^*(f_0;h_0)-\varepsilon \end{aligned}$$

for all $h \in U_{h_0}$. This means that $D\nu^*(f_0;\cdot)$ is na-lower semicontinuous at h_0. Since h_0 was chosen arbitrarily, $D\nu^*(f_0;\cdot)$ is na-lower semicontinuous on $B_{B(\Sigma)}$. □

Claim 3. If Γ is a weak* compact subset of $ba(\Sigma)$ and $\varrho : B(\Sigma) \to \mathbb{R}$, defined by

$$\varrho(f) = \min_{\mu\in\Gamma} \mu(f), \quad \forall f \in B(\Sigma),$$

is na-continuous on the unit ball $B_{B(\Sigma)}$ of $B(\Sigma)$, then Γ is a norm compact subset of $na(\Sigma)$.

Clearly ϱ is na-continuous on every ball $tB_{B(\Sigma)}$ with $t>0$. For all $\mu \in \Gamma$, $\mu : B_1(\Sigma) \to \mathbb{R}$ is continuous in the na-topology, Lemma 23 implies $\mu \in core(\mu) \subseteq na(\Sigma)$. Then $\Gamma \subseteq na(\Sigma)$ is weak* compact in $ba(\Sigma)$, hence it is weak compact [21, p. 53], *a fortiori*, there exists a non-atomic probability measure $\lambda \in na(\Sigma)$ such that $\Gamma \subseteq ca(\Sigma,\lambda)$ (see [10, Thm. IV.9.2]). As a consequence, setting

$$\rho(f) = \min_{\mu\in\Gamma} \mu(f), \quad \forall f \in L^\infty(\lambda)$$

defines a function $\rho : L^\infty(\lambda) \to \mathbb{R}$. Next we show that under the identification of $L^\infty(\lambda)$ with the norm dual of $ca(\Sigma,\lambda)$, the functional ρ is continuous on $B_{L^\infty(\lambda)}$ in the relative weak* topology $\sigma(L^\infty(\lambda), ca(\Sigma,\lambda))$, that is Γ (being weak compact and - by Lemma 26 - norm relatively compact) is norm compact.

Notice that $\varrho(f) = \rho(f)$ for all $f \in B(\Sigma)$. Let $f' \in B_{L^\infty(\lambda)}$ and $\varepsilon > 0$. Choose $f \in B_{B(\Sigma)}$ such that $f' = f$ λ-a.e. and let

$$U_f(\mu_1,...,\mu_n;\delta) = \left\{ g \in B_{B(\Sigma)} : |\mu_i(g) - \mu_i(f)| < \delta \ \forall i = 1,...,n \right\}$$

be a na-neighborhood of f such that $\varrho(U_f) \subseteq (\varrho(f)-\varepsilon, \varrho(f)+\varepsilon)$. W.l.o.g., suppose each μ_i is a probability measure.

Let $\mu_i = \mu_i^a + \mu_i^s$ be the Lebesgue decomposition of each μ_i, with $\mu_i^a \ll \lambda$ and $\mu_i^s \perp \lambda$. Set

$$V_f = \left\{ h \in B_{B(\Sigma)} : |\mu_i^a(h) - \mu_i^a(f)| < \frac{\delta}{2} \text{ and } |\mu_i^s(h) - \mu_i^s(f)| < \frac{\delta}{2} \ \forall i = 1,...,n \right\}.$$

Clearly, $V_f \subseteq U_f$ and so $\varrho(V_f) \subseteq (\varrho(f)-\varepsilon, \varrho(f)+\varepsilon)$.

Consider the weak* neighborhood of f'

$$W_{f'} = \left\{ g' \in B_{L^\infty(\lambda)} : \left|\mu_i^a(g') - \mu_i^a(f')\right| < \frac{\delta}{2} \ \forall i = 1,...,n \right\}.$$

For each $g' \in W_{f'}$, let $g \in B_{B(\Sigma)}$ such that $g' = g$ λ-a.e.. Choose and $E \in \Sigma$ be such that $\lambda(E) = 0$ and $\mu_i^s(E^c) = 0$ for each $i = 1,...,n$.[11] Set $h = f1_E + g1_{E^c}$. Then $h = g = g'$ λ-a.e. and $\rho(g') = \rho(h) = \varrho(h)$. Moreover, for all $i = 1,...,n$

$$|\mu_i^a(h) - \mu_i^a(f)| = \left|\mu_i^a(g') - \mu_i^a(f')\right| < \frac{\delta}{2},$$

[11] For $i = 1,...n$ there exists A_i such that $\lambda(A_i) = \mu_i^s(A_i^c) = 0$ ($\mu_i^s \perp \lambda$), set $E = \bigcup_{i=1}^n A_i$.

while

$$\left|\mu_i^s(h) - \mu_i^s(f)\right| = \left|\mu_i^s(f1_E) + \mu_i^s(g1_{E^c}) - \mu_i^s(f1_E) - \mu_i^s(f1_{E^c})\right| = 0,$$

so that $h \in V_f$. In sum, for each $g' \in W_{f'}$ there exists $h \in V_f$ such that $\rho(g') = \varrho(h)$ and

$$\rho(g') = \varrho(h) \in \varrho(V_f) \subseteq (\varrho(f) - \varepsilon, \varrho(f) + \varepsilon) = \left(\rho(f') - \varepsilon, \rho(f') + \varepsilon\right).$$

This proves that ρ is continuous on $B_{L^\infty(\lambda)}$ in the relative weak* topology. □

Claim 3, when applied to the function $f_0 = 2^{-1}1_\Omega$, delivers the norm compactness of *core* (ν). ■

Inspection of the above proof (Claims 1 and 2) delivers the following:

Proposition 31 *If a game ν admits a concave and na-continuous extension ν^* to $B_1(\Sigma)$ and core $(\nu) \neq \varnothing$, then $\nu_e = D\nu^*\left(2^{-1}1_\Omega; \cdot\right)$ is continuous on the unit ball $B_{B(\Sigma)}$ in the na-topology, and*

$$core(\nu) = \partial\nu^*\left(2^{-1}1_\Omega; \cdot\right) = \partial\nu^*(t1_\Omega; \cdot), \quad \forall t \in (0,1).$$

In particular, if ν is exact, the uniqueness of the na-continuous extension to $B_1(\Sigma)$ implies that $\nu^* = \nu_{e|B_1(\Sigma)}$ and ν^* is positively homogeneous; therefore ν is a market game.

In turn this implies that a bounded convex (hence exact) pre-market game is a market game and [22, Prop. 4] is equivalent to Lemma 2.

Corollary 7. By Proposition 31, *core* $(\nu) = \partial\nu^*(t1_\Omega)$ for all $t \in (0,1)$, where $\partial\nu^*$ is the superdifferential of ν^*. By [2, Prop. 24.1], for almost every $t \in (0,1)$ the na-extension ν^* is Frechet differentiable at $t1_\Omega$, provided $\nu \in pNA$. Hence, *core* (ν) is a singleton and $\partial\nu^*(t1_\Omega)$ agrees with the derivative. Moreover, by the diagonal formula (see [2, Thm. H]) the game value is

$$(\varphi\nu)(S) = \int_0^1 \partial\nu^*(t1_\Omega)(S)\,dt = \mu(S)$$

where $\mu \in core(\nu)$. Accordingly, $\varphi\nu = \mu \in core(\nu)$. ■

Corollary 8. By Claim 1 of the proof of Theorem 6, if ν_i is balanced then ν_i^* is positively homogeneous along the diagonal and *core* $(\nu_i) = \partial\nu_i^*\left(2^{-1}1_\Omega\right) = \partial\nu_i^*(t1_\Omega)$ for all $t \in (0,1)$. Analogously, if $\nu_i^*(t1_\Omega) = t\nu_i^*(1_\Omega)$ for some $t \in (0,1)$, then $\emptyset \neq \partial\nu_i^*(t1_\Omega) = core(\nu_i)$. Suppose first that, for instance, *core* $(\nu_1) = \varnothing$. Then, $\nu_1^*\left(2^{-1}1_\Omega\right) > 2^{-1}\nu_1(\Omega)$. This implies

$$(\nu_1 + \nu_2)^*\left(2^{-1}1_\Omega\right) = (\nu_1^* + \nu_2^*)\left(2^{-1}1_\Omega\right) = \nu_1^*\left(2^{-1}1_\Omega\right) + \nu_2^*\left(2^{-1}1_\Omega\right) > 2^{-1}(\nu_1 + \nu_2)(\Omega),$$

and the game $\nu_1 + \nu_2$ has empty core. Suppose now that *core* $(\nu_i) \neq \varnothing$ for $i = 1, 2$. It follows that:

$$\begin{aligned} core(\nu_1 + \nu_2) &= \partial(\nu_1 + \nu_2)^*\left(2^{-1}1_\Omega\right) = \partial(\nu_1^* + \nu_2^*)\left(2^{-1}1_\Omega\right) \\ &= \partial\nu_1^*\left(2^{-1}1_\Omega\right) + \partial\nu_2^*\left(2^{-1}1_\Omega\right) = core(\nu_1) + core(\nu_2), \end{aligned}$$

where we are using the sum rule for superdifferentials (see, e.g., [3, Ch. 3, Thm. 2.6]). ■

Theorem 9. (i) $\Rightarrow$ (ii). Assume there exists a norm compact (and finite dimensional, resp.) subset Γ of $na(\Sigma)$ such that $\nu(A) = \min_{\mu\in\Gamma}\mu(A)$ for all $A \in \Sigma$. There exists a non-atomic probability measure $\lambda \in na(\Sigma)$ such that $\Gamma \subseteq ca(\Sigma, \lambda)$ (see [10, Thm. IV.9.2]). Lemma 26 guarantees that the functional

$$\begin{array}{rccl} \hat{\nu}: & L^{\infty}(\lambda) & \rightarrow & \mathbb{R} \\ & f & \mapsto & \min_{\mu\in\Gamma}\mu(f) \end{array}$$

is continuous in the relative weak* topology of $B_{L^{\infty}(\lambda)}$, *a fortiori*, the "restriction" of $\hat{\nu}$ to the unit ball of $B(\Sigma)$ is $\sigma\left(B_{B(\Sigma)}, ca(\Sigma,\lambda)\right)$-continuous and na-continuous. If, moreover, Γ is finite dimensional, [18, Thm. 6] guarantees that $\hat{\nu}$ is continuous on (the entire) $L^{\infty}(\lambda)$ in the weak* topology.

(ii) $\Rightarrow$ (i). Assume that a market game ν admits an extension ν^* to $B_{B(\Sigma)}$ which is superlinear and na-lower semicontinuous at 0. ν^* can be extended (by) positive homogeneity to a superlinear function on $B(\Sigma)$, which we still denote by ν^*. A standard argument guarantees that ν^* is na-continuous on every bounded subset of $B(\Sigma)$. *A fortiori*, ν^* is supnorm continuous on $B(\Sigma)$ and

$$\nu^*(f) = \min_{\mu\in\Gamma}\mu(f), \quad \forall f \in B(\Sigma) \tag{20}$$

where Γ is a weak* compact and convex subset of $ba(\Sigma)$. Claim 3 of the proof of Theorem 6 shows that Γ is a norm compact subset of $na(\Sigma)$. A similar argument, building on [18, Thm. 6] rather than Lemma 26, shows that if ν^* is na-lower semicontinuous at 0, as a function on $B(\Sigma)$ rather than on $B_{B(\Sigma)}$, then Γ is finite dimensional. ■

Proposition 10. If $\nu(A) = \min_{\mu\in\Gamma}\mu(A)$, with Γ a norm compact subset of $na(\Sigma)$, then, as observed in the proof of Theorem 9,

$$\nu^*(f) = \min_{\mu\in\Gamma}\mu(f), \quad \forall f \in B_1(\Sigma).$$

Consider the extension of ν^* to $B(\Sigma)$ defined by

$$\nu^{**}(f) = \min_{\mu\in\Gamma}\mu(f), \quad \forall f \in B(\Sigma).$$

Since Γ is weak* compact in $ba(\Sigma)$, by Lemma 28,

$$\partial\nu^{**}(f_0) = \overline{co}^{w^*}\{\mu \in \Gamma : \mu(f_0) = \nu^{**}(f_0)\}, \quad \forall f_0 \in B(\Sigma).$$

Finally,

$$\begin{aligned} core(\nu) &= \partial\nu^*\left(2^{-1}1_\Omega\right) = \partial\nu^{**}\left(2^{-1}1_\Omega\right) = \overline{co}^{w^*}\left\{\mu\in\Gamma : \mu\left(2^{-1}1_\Omega\right) = \nu^{**}\left(2^{-1}1_\Omega\right)\right\} \\ &= \overline{co}^{w^*}\{\mu\in\Gamma : \mu(\Omega) = \nu(\Omega)\} = \overline{co}\{\mu\in\Gamma : \mu(\Omega) = \nu(\Omega)\}, \end{aligned}$$

where the last equality holds since Mazur Compactness Theorem guarantees norm compactness of the set $\overline{co}\{\mu\in\Gamma : \mu(\Omega) = \nu(\Omega)\}$. ■

Proposition 12. W.l.o.g. $\boldsymbol{\mu} \neq \mathbf{0}$. Let $K^* = \partial\varphi(0)$. K^* is a weak* compact subset of X^*, $\varphi(x) = \min_{x^* \in K^*} \langle x, x^* \rangle$ for all $x \in X$, and, *a fortiori*,

$$\nu(A) = \min_{x^* \in K^*} \langle \boldsymbol{\mu}(A), x^* \rangle = \min_{x^* \in K^*} x^*(\boldsymbol{\mu}(A)), \quad \forall A \in \Sigma.$$

It suffices to prove that the set $M = \{x^* \circ \boldsymbol{\mu} : x^* \in K^*\}$ is a norm compact set of non-atomic measures. Let λ be the variation of $\boldsymbol{\mu}$. Since $\boldsymbol{\mu}$ has the RNP, there is a Bochner integrable (w.r.t. λ) function $f : \Omega \to X$ such that $\boldsymbol{\mu}(E) = \int_E f d\lambda$, for all $E \in \Sigma$.

For all $x^* \in X^*$, $x^* \circ f : \Omega \to \mathbb{R}$ is integrable (w.r.t. λ) and

$$\int_E (x^* \circ f)\, d\lambda = x^* \left(\int_E f d\lambda \right) = x^*(\boldsymbol{\mu}(E)), \quad \forall E \in \Sigma.$$

See [5, Thm. 1.1.8]. This implies that $x^* \circ \boldsymbol{\mu} \in ca(\Sigma, \lambda) \subseteq na(\Sigma)$ and its Radon-Nikodym derivative is $x^* \circ f$. Let $\{x^*_\alpha \circ \boldsymbol{\mu}\}$ with $x^*_\alpha \in K^*$ be a net in M. Since K^* is weak* compact, x^*_α admits a weak* convergent subnet $y^*_\beta \to y^* \in K^*$. In order to show that the subnet $\left\{y^*_\beta \circ \boldsymbol{\mu}\right\}$ of $\{x^*_\alpha \circ \boldsymbol{\mu}\}$ converges to $y^* \circ \boldsymbol{\mu}$ in variation norm, we show that $y^*_\beta \circ f$ converges to $y^* \circ f$ in $L^1(\lambda)$. Since f is Bochner integrable, there is a sequence $f_n : \Omega \to X$ of simple measurable functions such that $\lim_n \int \|f - f_n\| d\lambda = 0$.

Let $\varepsilon > 0$. Denote by L the diameter of K^*, there exists n_0 such that $\int \|f - f_{n_0}\| d\lambda \leq \varepsilon/(2L)$. Then

$$\begin{aligned}\int \left|y^*_\beta \circ f - y^* \circ f\right| d\lambda &= \int \left|\left\langle f, y^*_\beta - y^*\right\rangle\right| d\lambda \leq \int \left|\left\langle f - f_{n_0}, y^*_\beta - y^*\right\rangle\right| d\lambda + \int \left|\left\langle f_{n_0}, y^*_\beta - y^*\right\rangle\right| d\lambda \\ &\leq L \int \|f - f_{n_0}\| d\lambda + \int \left|\left\langle f_{n_0}, y^*_\beta - y^*\right\rangle\right| d\lambda \leq \varepsilon/2 + \int \left|\left\langle f_{n_0}, y^*_\beta - y^*\right\rangle\right| d\lambda.\end{aligned}$$

Let $f_{n_0} = \sum_{i=1}^m x_i 1_{A_i}$ with $x_1, ..., x_m \in X$ and $\{A_1, ..., A_m\}$ a partition of Ω in Σ, then

$$\begin{aligned}\int \left|\left\langle f_{n_0}, y^*_\beta - y^*\right\rangle\right| d\lambda &= \int \left| \left\langle \sum_{i=1}^m x_i 1_{A_i}, y^*_\beta - y^* \right\rangle \right| d\lambda = \int \left| \sum_{i=1}^m \left\langle x_i, y^*_\beta - y^* \right\rangle 1_{A_i} \right| d\lambda \\ &\leq \int \sum_{i=1}^m 1_{A_i} \left|\left\langle x_i, y^*_\beta - y^*\right\rangle\right| d\lambda.\end{aligned}$$

Take β_0 such that $\left|\left\langle x_i, y^*_\beta - y^* \right\rangle\right| \leq \varepsilon/(2\lambda(\Omega))$ for all $i = 1, ..., m$ and $\beta \succeq \beta_0$, to obtain

$$\int \left|y^*_\beta \circ f - y^* \circ f\right| d\lambda \leq \varepsilon/2 + \int \sum_{i=1}^m 1_{A_i} \left|\left\langle x_i, y^*_\beta - y^*\right\rangle\right| d\lambda \leq \varepsilon,$$

as wanted. We conclude that any net in M admits a norm convergent subnet (with limit in M), and M is norm compact. ■

Proof of Theorem 13. We first show that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i) and the first part of (10).

(i) ⇒ (ii). Being a market game, ν admits a concave and na-continuous extension ν^* to $B_1(\Sigma)$, exactness guarantees $core(\nu) \neq \varnothing$. Proposition 31 yields that ν_e is continuous on the unit ball $B_{B(\Sigma)}$ in the na-topology. A fortiori ν_e is continuous on $B_1(\Sigma)$, and exactness guarantees that $\nu_{e|B_1(\Sigma)}$ is an extension of ν to $B_1(\Sigma)$. It follows that $\nu_{e|B_1(\Sigma)} = \nu^*$ and (9) holds.

(ii) $\Rightarrow$ (iii). As ν is superadditive, by [2, Prop. 27.1] ν^* is superadditive. While setting $\beta = 0$ in (9) guarantees positive homogeneity. A routine exercise shows that ν^* admits an extension ν^{**} to the entire space $B(\Sigma)$ which is superlinear and such that $\nu^{**}(g + \beta 1_\Omega) = \nu^{**}(g) + \beta\nu(\Omega)$ for all $g \in B(\Sigma)$ and all $\beta \in \mathbb{R}$. Next we show that ν^{**} is supnorm continuous. In fact, if $g_n \to g$ in the supnorm, there exist $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$ such that $\alpha g_n + \beta 1_\Omega, \alpha g + \beta 1_\Omega \in B_1(\Sigma)$ and $\alpha g_n + \beta 1_\Omega \to \alpha g + \beta 1_\Omega$ in the supnorm. Therefore $\mu(\alpha g_n + \beta) \to \mu(\alpha g + \beta)$ for all $\mu \in na(\Sigma)$. Since ν^* is na-continuous on $B_1(\Sigma)$, then $\nu^*(\alpha g_n + \beta) \to \nu^*(\alpha g + \beta)$ and

$$\alpha\nu^{**}(g_n) + \beta = \nu^*(\alpha g_n + \beta) \to \nu^*(\alpha g + \beta) = \alpha\nu^{**}(g) + \beta.$$

Therefore,

$$\nu^{**}(g) = \min_{\mu \in \partial\nu^{**}(0)} \mu(g), \quad \forall g \in B(\Sigma), \tag{21}$$

where $\partial\nu^{**}(0) = \{\mu \in ba(\Sigma) : \mu(g) \geq \nu^{**}(g) \text{ for all } g \in B(\Sigma)\}$. For all $\mu \in \partial\nu^{**}(0)$ and all $A \in \Sigma$ we have $\mu(A) \geq \nu(A)$; moreover, $-\mu(\Omega) = \mu(-1_\Omega) \geq \nu^{**}(-1_\Omega) = -\nu(\Omega)$, whence $\mu(\Omega) = \nu(\Omega)$ and $\mu \in core(\nu)$. Therefore, $\partial\nu^{**}(0) \subseteq core(\nu)$ and for all $g \in B_1(\Sigma)$

$$\nu^*(g) = \min_{\mu \in \partial\nu^{**}(0)} \mu(g) \geq \min_{\mu \in core(\nu)} \mu(g) = \nu_e(g) \geq \nu^*(g).$$

The last inequality follows from Lemma 24. This proves that $\nu_e = \nu^*$ is na-continuous on $B_1(\Sigma)$ and that ν is exact.

(iii) implies (iv). Exactness guarantees that $\nu_{e|B_1(\Sigma)}$ is a concave extension of ν to $B_1(\Sigma)$. Then na-continuity of $\nu_{e|B_1(\Sigma)} = \nu^*$, together with Theorem 6, guarantees that $core(\nu)$ is a norm compact subset of $na(\Sigma)$. Continuity and non-atomicity of ν follow from Proposition 1.

(iv) $\Rightarrow$ (v). Continuity and non-atomicity of ν, together with Proposition 1, guarantee $core(\nu) \subseteq na_{\nu(\Omega)}(\Sigma)$. Just set $K = core(\nu)$.

(v) $\Rightarrow$ (i) and the first part of (10). Assume ν is the lower envelope of a norm relatively compact subset K of $na_\alpha(\Sigma)$. Obviously, $\overline{co}(K) \subseteq \overline{co}^*(K) \subseteq core(\nu)$. Proposition 1 guarantees that ν is continuous and non-atomic, a second application of Proposition 1, yields that $core(\nu) \subseteq na(\Sigma)$. Since

$$\min_{\mu \in core(\nu)} \mu(A) = \nu(A) = \nu_K(A) = \inf_{\phi \in K} \phi(A), \quad \forall A \in \Sigma,$$

Corollary 30 implies $core(\nu) \subseteq \overline{co}(K)$. This delivers the first part of (10). Since $core(\nu) \subseteq na(\Sigma)$, there exists a non-atomic probability measure $\lambda \in na(\Sigma)$ such that $core(\nu) \subseteq ca(\Sigma, \lambda)$ (see [10, Thm. IV.9.2]), while the Mazur Compactness Theorem implies that of $core(\nu)$ is norm compact. Lemma 26 guarantees that the functional

$$\begin{array}{rccl} \nu_{ee}: & L^\infty(\lambda) & \to & \mathbb{R} \\ & f & \mapsto & \min_{\mu \in core(\nu)} \mu(f) \end{array}$$

is continuous in the relative weak* topology of $B_{L^\infty(\lambda)}$, a fortiori ν_e is $\sigma(B_1(\Sigma), ca(\Sigma,\lambda))$-continuous and na-continuous. We conclude that ν_e is the required na-continuous and superlinear extension.

In sum, (i)-(v) are equivalent and the first part of (10) holds.

Next we show the equivalence between (i) and (vi) and that the second part of (10) holds.

(i) $\Rightarrow$ (vi). Since $core(\nu)$ is a norm compact subset of $na(\Sigma)$, for all $\varepsilon > 0$, there is a finite subset M of $core(\nu)$ such that for all $\mu \in core(\nu)$ there exists $\phi \in M$ with $\|\mu - \phi\| < \varepsilon$. Consider the exact linear production game ν_M. Clearly, $\nu_M \geq \nu$. Let A be any coalition and $\mu \in core(\nu)$ such that $\mu(A) = \nu(A)$. If $\phi \in M$ satisfies $\|\mu - \phi\| < \varepsilon$, we have

$$0 \leq \nu_M(A) - \nu(A) \leq \phi(A) - \mu(A) \leq \|\phi - \mu\| < \varepsilon.$$

So that $\sup_{A\in\Sigma} |\nu_M(A) - \nu(A)| \leq \varepsilon$, which implies that ν belongs to the supnorm closure of the set of exact linear production games.

(vi) $\Rightarrow$ (i) and the second part of (10). Let ν_n be a sequence of exact linear production games uniformly converging to ν. Notice that ν_n^* is a na-continuous and bounded function on $B_1(\Sigma)$ for all $n \in \mathbb{N}$. Next we show that ν_n^* is a Cauchy sequence (in the space $C_b(B_1(\Sigma))$ of all bounded and na-continuous functions on $B_1(\Sigma)$ endowed with the supnorm). For all $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $m, n \geq p$

$$\sup_{A\in\Sigma} |\nu_m(A) - \nu_n(A)| \leq \varepsilon,$$

but, the function $|\nu_m^* - \nu_n^*| : B_1(\Sigma) \to \mathbb{R}$ is na-continuous and the characteristic functions are na-dense in $B_1(\Sigma)$, therefore

$$\sup_{f\in B_1(\Sigma)} |\nu_m^*(f) - \nu_n^*(f)| \leq \varepsilon.$$

Let ν^* be the limit of ν_n^* in $C_b(B_1(\Sigma))$ with the supnorm. Checking that ν is superadditive, ν^* is an extension of ν to $B_1(\Sigma)$ (obviously na-continuous), and that ν^* satisfies condition (9) of (ii) is an easy exercise. Hence ν is an exact market game.

Now, notice that, by (iii) we have that $\nu^* = \nu_{e|B_1(\Sigma)}$ and $\nu_n^* = (\nu_n)_{e|B_1(\Sigma)}$. The uniform convergence of ν_n^* to ν^* amounts to say that

$$\lim_n \left(\sup_{f\in B_1(\Sigma)} \left| \min_{\mu_n\in core(\nu_n)} \mu_n(f) - \min_{\mu\in core(\nu)} \mu(f) \right| \right) = 0. \tag{22}$$

Let $\lambda_0 \in na(\Sigma)$ be a probability measure such that $core(\nu) \subseteq ca(\Sigma, \lambda_0)$ and $\lambda_n \in na(\Sigma)$ be a probability measure such that $core(\nu_n) \subseteq ca(\Sigma, \lambda_n)$ for all $n \geq 1$. The non-atomic probability measure

$$\lambda = \frac{1}{2}\lambda_0 + \frac{1}{2}\sum_{n\geq 1} \frac{1}{2^n}\lambda_n$$

is such that $core(\nu), core(\nu_n) \subseteq ca(\Sigma, \lambda)$ for all $n \geq 1$. Therefore, (22) implies

$$\lim_n \left(\sup_{f\in I^\infty(\lambda)} \left| \min_{\mu_n\in core(\nu_n)} \mu_n(f) - \min_{\mu\in core(\nu)} \mu(f) \right| \right) = 0.$$

thus

$$\begin{aligned}
0 &\leq \liminf_n \left(\sup_{g \in B_{L^\infty(\lambda)}} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(g) - \min_{\mu \in core(\nu)} \mu(g) \right| \right) \\
&\leq \limsup_n \left(\sup_{g \in B_{L^\infty(\lambda)}} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(g) - \min_{\mu \in core(\nu)} \mu(g) \right| \right) \\
&= \limsup_n \left(\sup_{f \in I^\infty(\lambda)} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(2f - 1_\Omega) - \min_{\mu \in core(\nu)} \mu(2f - 1_\Omega) \right| \right) \\
&= \limsup_n \left(\sup_{f \in I^\infty(\lambda)} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(2f) - \min_{\mu \in core(\nu)} \mu(2f) + \nu(\Omega) - \nu_n(\Omega) \right| \right) \\
&\leq \limsup_n \left(\sup_{f \in I^\infty(\lambda)} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(2f) - \min_{\mu \in core(\nu)} \mu(2f) \right| + \left| \nu(\Omega) - \nu_n(\Omega) \right| \right) \\
&= \lim_n \left(2 \sup_{f \in I^\infty(\lambda)} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(f) - \min_{\mu \in core(\nu)} \mu(f) \right| + \left| \nu(\Omega) - \nu_n(\Omega) \right| \right) = 0.
\end{aligned}$$

This concludes the proof since the Hausdorff distance between $core(\nu_n)$ and $core(\nu)$ in $ca(\Sigma, \lambda)$ (and hence in $ba(\Sigma)$) is given by (see, e.g., [1, Ch. 6.7]):

$$\begin{aligned}
d_{\mathcal{H}}(core(\nu_n), core(\nu)) &= \sup_{g \in B_{L^\infty(\lambda)}} \left| \sigma_{core(\nu_n)}(g) - \sigma_{core(\nu)}(g) \right| \\
&= \sup_{g \in B_{L^\infty(\lambda)}} \left| \min_{\mu_n \in core(\nu_n)} \mu_n(g) - \min_{\mu \in core(\nu)} \mu(g) \right|.
\end{aligned}$$

■

Proposition 15. (i) ⇒ (ii). If C is a norm compact (resp. finite dimensional) subset of $na(\Sigma)$, then (see the proof of Theorem 9) V_C is na-continuous on $B_{B(\Sigma)}$ (resp. $B(\Sigma)$), and (ii) follows.

(ii) ⇒ (i). For all $r \in \mathbb{R}$, $\{V_C \leq r\} \cap B_{B(\Sigma)}$ (resp. $\{V_C \leq r\}$) and $\{V_C \geq r\} \cap B_{B(\Sigma)}$ (resp. $\{V_C \geq r\}$) are na-closed. Therefore, V_C is a superlinear and na-continuous extension of ν_C to $B_{B(\Sigma)}$ (resp. $B(\Sigma)$). The technique used at the end of the proof of Theorem 9 delivers (i). ■

Corollary 16. (i) ⇒ (ii). For all $f \in B(\Sigma)$, $f \sim_i V_{C_i}(f) 1_\Omega$ for $i = 1, 2$, hence

$$V_{C_2}(f) 1_\Omega \sim_2 f \succsim_2 V_{C_1}(f) 1_\Omega \text{ and } V_{C_2}(f) \geq V_{C_1}(f),$$

in particular $\nu_{C_1} \leq \nu_{C_2}$.

(ii) ⇒ (i). By Corollary 30, $C_2 \subseteq \overline{co}(C_1)$. Therefore, $V_{C_2} \geq V_{\overline{co}(C_1)} = V_{C_1}$, and (i) immediately follows. ■

Corollaries 17, 18, and 19 are just restatements of Proposition 10, Corollary 8, and 11.

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