STIFFNESS SYNTHESIS OF A VARIABLE GEOMETRY PLANAR ROBOT

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Abstract   This paper addresses the problem of task-based stiffness synthesis of a variable geometry three DOF (Degrees Of Freedom) planar robot. The synthesis considers the case where the robot has a limited number of free geometric parameters and constant actuator stiffness coefficients. This defines twenty problems of stiffness synthesis, in which, three parameters of the stiffness matrix are controlled according to task requirements. These problems are modeled as systems of polynomials in the free geometric parameters of the robot's base platform. Using Gröbner bases, the solubility of these polynomial systems is characterized. It is shown that arbitrary desired values of the Cartesian stiffness elements (k_{xx} and k_{yy}) are unattainable when only the geometry of the base platform is variable. An example of synthesizing three stiffness elements of the planar robot is solved and shown to have at most 48 solutions in the complex plane. In a numerical case study, sixteen real solutions are obtained, of which only eight are non-singular.

1.   Introduction

Robots are used to perform various tasks involving complex manipulations and interactions with their environment. Consequently, there are inevitable compromises when using a fixed-geometry robot for some tasks. To overcome this problem, the use of variable geometry robots is considered. In particular we concentrate here on variable geometry parallel robots. These robots can change the geometry of their base/moving platforms to accommodate the required characteristics, e.g. stiffness, specific to each task.

Researchers used various methods to enhance parallel robots' capabilities for better fitting task requirements in terms of stiffness, singularity avoidance, and inclusion of specific paths in the workspace.

Actuation redundancy was used by Yi and Freeman (1993), Kim et al. (2000) for stiffness modulation. Kinematic redundancy of the robot for a given task was used by Merlet, et al. (2000) for path inclusion in the workspace and singularity avoidance. Stiffness/compliance synthesis algorithms were presented by Huang and Schimmels, (1998), and Roberts, (1999) for a system of springs supporting a rigid body.

Works directly addressing variable geometry parallel robots are limited in number. Zhiming and Song (1998) investigated the design aspects of modular Stewart-Gough platforms with workspace and joint limits considerations. Zhiming and Zhenqun (1999) presented an algorithm for identifying the parameters of the joint locations on the base in a modular Stewart-Gough platform. Merlet, (1997, 2000) presented a design algorithm for achieving a constant-orientation workspace of Stewart-Gough robots, which can be adapted for workspace modification of variable geometry robots[♠].

In the present investigation, a case study of stiffness synthesis for a point in a given path of a planar 3 DOF robot with a variable geometry of its base platform is presented. The aim of the synthesis is to obtain a specific stiffness in a given position/orientation of the robot's moving platform. Under a simplifying assumption that the stiffness coefficients of the redundant actuators that change the base geometry are considerably larger than the coefficients of the other actuators, this work may be viewed as an algorithm for changing the geometry of the base platform of a variable geometry 3 DOF planar robot for obtaining a required stiffness in a point along a path specified by the given task.

2.   Variable Geometry Planar Robot

The planar robot of Fig. 1 has an equilateral triangular moving platform connected to a circular base by three kinematic chains composed of a slider on the circular base, a revolute joint, a prismatic joint, and a revolute joint on its moving platform. The sliders on the circular base control the geometry of the base platform and the prismatic actuators manipulate the moving

= prismatic ○ = revolute

Figure 1. Planar Robot with variable geometry base platform

platform. This introduces a kinematic redundancy in this three DOF planar robot. The objective of this paper is to determine the geometry of

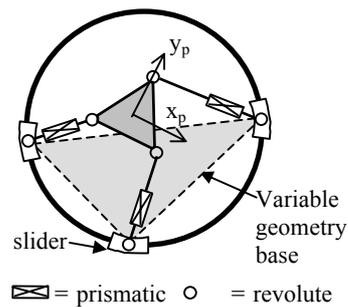the base platform by computing the locations of these three redundant sliders for achieving a desired stiffness goal in a point of a path defined by the task.

3.   Stiffness Polynomial Formulation

In this section, the stiffness matrix of the robot of Fig. 1 is formulated as a function of the positions of the sliders on the circular base and is described in a Platform–attached Coordinate System (PCS). The location and orientation of the moving platform are given by the task while the orientations of the prismatic actuators are given by the stiffness synthesis solution and are easily transformed to desired positions of the redundant sliders on the circular base.

The transformation of the desired stiffness from World Coordinate System (WCS) to PCS is given by, Tsai (1999):

$$^{p}\mathbf{K}_{des} = \mathbf{A}^{T}{}^{w}\mathbf{K}_{des}\mathbf{A} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{1}$$

where $\mathbf{R}$ is the 3x3 rotation matrix from the PCS to WCS, $^{p}\mathbf{K}_{des}$ and $^{w}\mathbf{K}_{des}$ are the desired stiffness matrix in PCS and WCS, respectively. Hereafter, all vectors and matrices are represented in PCS.

The only controllable geometric variables by the sliders' locations are the unit vectors of the prismatic actuators' axes, ($\hat{\mathbf{l}}_i$, i=1,2,3), Figs. 1-2.

$$\hat{\mathbf{l}}_i = a_i\,\hat{\mathbf{e}}_{i1} + b_i\,\hat{\mathbf{e}}_{i2} \quad i = 1,2,3 \tag{2}$$

where the symbol ^ indicates a unit vector, $\hat{\mathbf{e}}_{i1}$ and $\hat{\mathbf{e}}_{i2}$ are unit vectors indicated in Fig. 2, $\hat{\mathbf{l}}_i$ is a unit vector along the ith prismatic actuator, and a_i, b_i are the projections of $\hat{\mathbf{l}}_i$ on $\hat{\mathbf{e}}_{i1}$ and $\hat{\mathbf{e}}_{i2}$. In order for $\hat{\mathbf{l}}_i$ to be a unit vector, a_i and b_i (i=1,2,3) must obey:

$$a_i^2 + b_i^2 + 2a_i\,b_i\left(\hat{\mathbf{e}}_{i1}\cdot\hat{\mathbf{e}}_{i2}\right) - 1 = 0 \tag{3}$$

For an equilateral platform, Eq. (3) is an ellipse in a_i-b_i plane:

$$a_i^2 + b_i^2 + a_i\,b_i - 1 = 0 \qquad i = 1,2,3 \tag{4}$$

A simplifying assumption is made that the sliders on the circular platform have a mechanical means to lock rigidly on the circular base once the desired geometry of the base is obtained or that the stiffness coefficients of the sliders are considerably larger than the stiffness coefficients of the prismatic actuators. With this simplifying assumption, the stiffness matrix depends only on the stiffness coefficients of the three remaining active prismatic
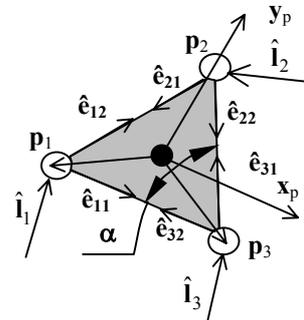

Figure 2. Geometric notations for the planar robot

actuators. These three active actuators are assumed identical having stiffness coefficient k_d. The rows of the Jacobian matrix of the robot of Fig. 1, with its sliders locked on the circular base, are the Plücker line coordinates of the three axes of the prismatic actuators, Eq. (5).

$$\mathbf{J} = \begin{bmatrix} a_1 + \frac{b_1}{2} & \frac{b_1\sqrt{3}}{2} & 0 & 0 & 0 & \frac{r}{2}\left(a_1 + \frac{b_1}{2}\right) - \frac{3}{4}rb_1 \\ -\frac{a_2}{2} + \frac{b_2}{2} & -\frac{a_2\sqrt{3}}{2} - \frac{b_2\sqrt{3}}{2} & 0 & 0 & 0 & -r\left(-\frac{a_2}{2} + \frac{b_2}{2}\right) \\ -b_3 - \frac{a_3}{2} & \frac{a_3\sqrt{3}}{2} & 0 & 0 & 0 & \frac{r}{2}\left(-\frac{a_3}{2} - b_3\right) + \frac{3}{4}ra_3 \end{bmatrix} \quad (5)$$

The 6×6 Stiffness matrix of the planar robot with three identical prismatic actuators is given by $\mathbf{K} = k_d\,\mathbf{J}^T\mathbf{J}$, Tsai (1999), and the reduced planar 3×3 stiffness matrix is then, obtained. Notations of the elements of this symmetric stiffness matrix are given in Eq. (6).

$$\mathbf{K} = \begin{bmatrix} k_{xx} & k_{xy} & k_{x\theta} \\ k_{xy} & k_{yy} & k_{y\theta} \\ k_{x\theta} & k_{y\theta} & k_{\theta\theta} \end{bmatrix} \quad (6)$$

4.   Stiffness synthesis with limited number of variable geometry parameters

Given a desired triplet of stiffness elements from the upper triangular part of $\mathbf{K}$ in Eq. (6), the problem of stiffness synthesis with limited number of variable geometry parameters deals with finding the geometry of the base platform (i.e., finding a_i, b_i, i=1,2,3) of the robot in Fig. 1.

To fully depict the 3×3 stiffness matrix, all the six equations in Eq. (7) given below must be fulfilled together with the three equations in Eq (4). Since the planar robot of Fig. 1 has a kinematic redundancy of order three, only three stiffness equations from Eq. (7) can be simultaneously fulfilled.

$$k_{n_i} - k_{n_{i_{desired}}} = 0, \ i = 1, 2, 3 \quad | \quad n_1 \neq n_2 \neq n_3 \in \{xx, xy, x\theta, yy, y\theta, \theta\theta\} \quad (7)$$

Note that this results in $\binom{6}{3}$=20 systems of six polynomials with each having a total degree of 2 in a_i, b_i (i=1,2,3).

Generally, we ask whether it is possible to solve this problem for any such triplet and, if so, is changing the directions of the lines in Fig. 2 enough to allow controlling all such triplets?

To solve the polynomial systems derived from Eq. (4) and Eq. (7) ,the method of multiplication tables' eigenvalues (see Stetter (1993)) is used. The following sub-section briefly describes this method. Further details can be found in Möller and Stetter (1995) and Cox et al. (1998).

5. The method of eigenvalues for solving polynomial systems

Let $\mathscr{C}[x_1..x_m]$ represent the *ring* of polynomials with variables $x_1..x_m$, and coefficients over the complex field, $\mathscr{C}$. Let also $S=\{p_1,p_2,..p_n \mid p_1,p_2..p_n \in \mathscr{C}[x_1..x_m]\}$ be a system of n polynomials with a corresponding zero-dimensional Ideal $\mathscr{I} = <p_1,p_2,..p_n>$, $\mathscr{I} \subset \mathscr{C}[x_1..x_m]$. The *variety* $V(\mathscr{I})$ of solution is defined by all the m-tuples of $x_1..x_m$ such that $p_1=p_2=..p_n=0$, i.e., $V(\mathscr{I})=\{[x_1..x_m] \in \mathscr{C}^m \mid p_1=p_2=..p_n=0\}$. We seek all the solutions of S.

The original system of polynomial equations, S, can be replaced by another minimal set of polynomials, $G=\{g_1...g_t\}$, called *standard basis* (or *Gröbner basis*) of the ideal $\mathscr{I}$ using Buchberger's algorithm, Buchberger (1965), which is not reviewed here for lack of space. Questions regarding ideal-membership of a given polynomial to $\mathscr{I}$, solubility of S, and finiteness of the dimension of $V(\mathscr{I})$ are readily answered when using this basis, Heck (1997). Also, for lexographic ordering G is a system of polynomials with successively eliminated number of variables as in the result of Gauss-Jordan elimination method for linear equations. However, this elimination method is unfavorable for large systems due to the computation effort associated with this ordering, Cox et al. (1998).

Two polynomials $f, g \in \mathscr{C}[x_1..x_m]$, are said to be *congruent* modulo $\mathscr{I}$, $f \equiv g \mod \mathscr{I}$, if $f-g \in \mathscr{I}$. Consequently, f and g have the same *normal form* with respect to G and equal cosets $[g]=[f]$. A coset $[f]$ of $f \in \mathscr{C}[x_1..x_m]$ is the sub-group of $\mathscr{C}[x_1..x_m]$ in which all its elements have the same normal form with respect to G, $[f]=f+\mathscr{I}=\{f+h \mid h \in \mathscr{I}\}$. The totality of cosets of the polynomials in $\mathscr{C}[x_1..x_m]$ is the quotient ring of $\mathscr{C}[x_1..x_m]$ modulo $\mathscr{I}$ indicated by $\mathscr{C}[x_1..x_m]/\mathscr{I}$, i.e., $\mathscr{C}[x_1..x_m]/\mathscr{I}=\{f+\mathscr{I} \mid f \in \mathscr{C}[x_1..x_m]\}$.

Given two polynomials $f, g \in \mathscr{C}[x_1..x_m]$ then a normal form arithmetic similar to number arithmetic exists for addition and multiplication: $n_f(f+g)=n_f(f)+n_f(g)$ and $n_f(f\,g)=n_f(n_f(f)\,n_f(g))$. Since every normal form is associated with a coset and vise-versa, this arithmetic is also translated to an associated coset arithmetic in the ring $\mathscr{C}[x_1..x_m]/\mathscr{I}$ resulting in the fact that $\mathscr{C}[x_1..x_m]/\mathscr{I}$ is a vector space in $\mathscr{C}^n$. Let B be a basis of monomials for this space $B=\{b_i, i=1..n\}$. This means that the remainder (or normal form) of any $f \in \mathscr{C}[x_1..x_m]$ is given by:

$$n_f(f)=n_f\left(\sum_{i=1..n}c_i b_i\right) \mid c_i \in \mathscr{C}, b_i \in B \qquad (8)$$

or in congruence terms:

$$f \equiv \sum_{i=1..n}c_i b_i \mod \mathscr{I} \qquad (9)$$

Define the monomial basis vector $\mathbf{b}=[b_1, b_2, ...b_n]^t$, $b_i \in B$, $i=1..n$, then each polynomial $f \in \mathscr{C}[x_1..x_m]$ has a multiplication table $\mathbf{M}_f$ such that:

$$f\,\mathbf{b} \equiv \mathbf{M}_f \mathbf{b} \mod \mathscr{I} \qquad (10)$$

Since the congruence relation in Eq. (10) indicates that $f\mathbf{b} - \mathbf{M}_f\mathbf{b} \in \mathscr{I}$ then $f\mathbf{b} - \mathbf{M}_f\mathbf{b} = \mathbf{0}$ for all the points of $V(\mathscr{I})$. Consequently, the eigenvalues method is based on Eq. (11):

$$(\mathbf{M}_f - f\mathbf{I})\mathbf{b} = \mathbf{0} \qquad\qquad (11)$$

Equation (11) shows that the eigenvalues of $\mathbf{M}_f$ are the values of f for all the points of $V(\mathscr{I})$. If f is taken as $f=x_1$ then Eq. (11) gives all the coordinates x_1 of the points of $V(\mathscr{I})$. Thus, by constructing the multiplication tables $\mathbf{M}_{x1}...\mathbf{M}_{xm}$ and solving for their eigenvalues, all the values of the coordinates $x_1...x_m$ for the points of $V(\mathscr{I})$ are obtained. The minimal polynomials of $\mathbf{M}_{xi}$, i=1..m, when written with x_i as its variable, give the monic generators of the elimination ideals $\mathscr{I} \cap \mathscr{C}[x_i]$.

There are both symbolic and numerical advantages of this method compared to standard sequential elimination of variables by resultants, Raghavan and Roth (1995), Neilsen and Roth (1999). Since this method is based on Gröbner basis construction, solvability of the system of polynomial equations is readily determined. Moreover, this method is unaffected by the term order used for the computation of G, which reduces the computation time when using more efficient term orders such as total degree order, Cox et al. (1998). Compared to sequential elimination methods, in this method the numerical computation is kept to a minimum since numerical values are used only in the computation procedure of eigenvalues and the solution of each coordinate x_i is independent of the numerical solutions of the other variables and, thus, it is unaffected by computation errors in the other variables x_j.

6.   Application to the parallel planar robot

To answer the questions of section 4, the reduced Gröbner bases associated with all the 20 possible systems of equations in the form of Eq. (7) were computed. A total-degree ordering with $a_1>b_1>a_2>b_2>a_3>b_3$ was used for reducing the computation effort of these bases.

The non-solubility of a polynomial system is determined by checking whether its reduced Gröbner basis is {1}, Adams and Loustanau (1994). Performing this task using Maple® shows that all the polynomial systems including equations for simultaneously fulfilling the desired values of k_{xx} and k_{yy} are unsolvable. Consequently, changing the directions of the prismatic actuators relative to the moving platform is not sufficient for simultaneously achieving these stiffness elements.

Next, the problem of Eq. (7) for k_{xx}, k_{xy}, and $k_{x\theta}$ is solved, i.e., all the stiffness elements in the x direction of PCS are prescribed based on task requirements. The reduced Gröbner basis for this problem, hereafter called G, with total degree ordering $a_1>b_1>a_2>b_2>a_3>b_3$ has 28 generators of degrees ranging from 1 to 5 in the variables. The ith column in table 1

presents the degrees of the ith basis polynomial in the variables corresponding to a_1,b_1,a_2,b_2,a_3,b_3. It can be seen that the total degree ranges from 4 to 8. This basis is not presented here due to lack of space.

Table 1. Degrees of the 28 polynomials of G in the variables

a_1	0,0,0,0,1,2,1,1,0,1,1,0,0,1,1,0,1,1,1,0,1,0,0,0,0,1,1,1
b_1	0,0,0,2,1,0,1,1,0,1,1,0,0,1,1,0,1,1,1,0,1,0,0,0,0,1,1,1
a_2	0,1,2,0,0,0,1,1,1,0,0,1,1,0,0,0,1,1,1,0,1,0,0,1,1,0,0,1
b_2	0,2,2,2,2,2,1,1,3,2,2,1,1,0,0,2,1,1,1,2,1,2,2,1,1,0,0,1
a_3	2,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,1,1,1,1,1,1,1,1,1
b_3	2,2,2,2,2,1,1,1,2,2,2,3,3,3,3,4,2,2,2,4,2,5,4,4,4,4,4,3

The leading terms of G are given by:

$$LT = [a_3^2, a_2 b_2, a_2^2, b_1^2, a_1 b_1, a_1^2, a_3 a_2 b_1, a_3 a_1 a_2, b_2^3, b_2^2 b_1, b_2^2 a_1,$$

$$b_3^2 a_3 b_2, b_3^2 a_3 a_2, b_3^2 b_1 a_3, b_3^2 a_3 a_1, b_2^2 b_3^2, b_3^2 b_2 a_1, b_3^2 a_2 b_1, b_3^2 a_1 a_2, \quad (12)$$

$$b_2^2 a_3 b_3, a_3 b_2 b_3 a_1, b_3^5, b_3^4 a_3, b_3^4 b_2, b_3^4 a_2, b_3^4 b_1, a_1 b_3^4, b_3^3 b_2 b_1]$$

Based on the finiteness theorem, Adams and Loustaunau (1994), the system of polynomials corresponding to G is solvable and has a zero-dimensional variety. This is established by examining the group of leading terms in Eq. (12) which shows that each variable among $\{a_1,b_1,a_2,b_2,a_3,b_3\}$ appears alone as a leading term in G with the corresponding degrees of $\{2, 2, 2, 3, 2, 5\}$. Consequently, the group of all the remainders in $\mathscr{C}[a_1,b_1,a_2,b_2,a_3,b_3]/\mathscr{I}$, denoted by D, has terms with maximal degrees of $\{1, 1, 1, 2, 1, 4\}$ in $\{a_1, b_1, a_2, b_2, a_3, b_3\}$, respectively.

The monomial basis of $\mathscr{C}[a_1,b_1,a_2,b_2,a_3,b_3]/\mathscr{I}$, denoted by B, is found from D by extracting all the monomials in D that are equal to their own normal forms, Cox et al. (1998). This leads to the 48-dimensional monomial basis in Eq. (13):

$$B = [1, b_3, a_3, b_2, a_2, b_1, a1, b_3^2, a_3 b_3, b_2 b_3, a_2 b_3, b_1 b_3, a1 b_3, a_3 b_2, a_3 a_2, a_3 b_1, a_3 a1,$$

$$b_2^2, b_2 b_1, b_2 a1, a_2 b_1, a1 a_2, b_3^3, b_3^2 a_3, b_2 b_3^2, b_3^2 a_2, b_3^2 b_1, b_3^2 a1, b_3 a_3 b_2,$$

$$a_3 a_2 b_3, b_3 a_3 b_1, a_3 a1 b_3, b_3 b_2^2, b_3 b_2 b_1, b_3 b_2 a1, b_1 a_2 b_3, a1 a_2 b_3, a_3 b_2^2, a_3 b_2 b_1, \quad (13)$$

$$a_3 b_2 a1, b_3^4, a_3 b_3^3, b_3^3 b_2, a_2 b_3^3, b_3^3 b_1, a1 b_3^3, b_3^2 b_2 b_1, b_1 b_2 a_3 b_3]$$

Next, three 48×48 multiplication tables, $\mathbf{M}_{f1}$, $\mathbf{M}_{f2}$ and $\mathbf{M}_{f3}$ for $f_1 = a_1 + b_1$, $f_2 = a_2 + b_2$, $f_3 = a_3 + b_3$ are computed together with their corresponding minimal polynomials mp$_{f1}$, mp$_{f2}$, and mp$_{f3}$. These minimal polynomials have only even degrees, so there are at most 24 pairs of complex solutions and their conjugate solutions (48 in total). These solutions give the values of $f_1=a_1+b_1$, $f_2=a_2+b_2$, and $f_3=a_3+b_3$. The next step is solving for the values of a_1, b_1, a_2, b_2, a_3, b_3. These values establish the locations of the sliders on the circular base. The solution algorithms for obtaining the

values of (a_1, b_1), (a_2, b_2), and (a_3, b_3) are identical; hence, only the algorithm for obtaining (a_1, b_1) is presented herein.

Let $\pm C$ be one of the 24 solution pairs of mp_{f1}. The matching solutions for (a_1, b_1) are the intersections of the line and the ellipse of Eq. (14).

$$a_1 + b_1 = \pm C \quad \cap \quad a_1{}^2 + b_1{}^2 + a_1 b_1 - 1 = 0 \tag{14}$$

The solutions for $+C$ and for $-C$ are:

$$\text{for } +C : (a1, b1) = \left(\frac{1}{2}C \pm \frac{1}{2}\Delta, \frac{1}{2}C \mp \frac{1}{2}\Delta \right) \qquad \Delta \equiv \sqrt{4 - 3C^2} \tag{15}$$
$$\text{for } -C : (a1, b1) = -\left(\frac{1}{2}C \mp \frac{1}{2}\Delta, \frac{1}{2}C \pm \frac{1}{2}\Delta \right)$$

Since only real solutions for (a_1, b_1) are of interest, only the *admissible* real solution pairs of mp_{f1} satisfying $|C| \leq 2/\sqrt{3}$ are used in Eq. (15). Note that the two solutions for $+C$ (and $-C$) represent a mirror image about the bisector of the angle α in Fig. 2 and that the two solutions for $+C$ are mirror images of the two solutions for $-C$ about the normal to the bisector of the angle α in Fig. 2.

Once this procedure is repeated for the roots of mp_{f2} and mp_{f3}, sets of solutions for (a_1, b_1), (a_2, b_2) and (a_3, b_3) are obtained and all the sextuplets $(a1, b_1, a_2, b_2, a_3, b_3)$ satisfying Eqs. (4, 7) are found.

7.   Numerical Example

To verify the solution procedure, a predefined geometry of the planar robot was selected with $[a_1, b_1, a_2, b_2, a_3, b_3] = [\sqrt{3}/3, \sqrt{3}/3, 1, 0, 1, 0]$. This corresponds to $[\theta_1, \theta_2, \theta_3] = [30°, 240°, 120°]$, where θ_1, θ_2, and θ_3 are the angles of $\hat{\mathbf{i}}_1$, $\hat{\mathbf{i}}_2$, and $\hat{\mathbf{i}}_3$ relative to $\mathbf{x}_p$ in Fig. 2. The corresponding stiffness matrix, using a platform radius of 0.1[m] and $k_d = 1e+5$ [N/m], is:

$$K = \begin{bmatrix} 125000.000\,00 & 43301.270\,189 & -5000.00000 \\ 43301.2701 & 174999.999\,99 & 0. \\ -5000.00000 & 0. & 500.00000 \end{bmatrix} \tag{16}$$

The aim of the following example is finding all the solutions for (a_i, b_i), $i=1,2,3$, for obtaining the stiffness elements k_{xx}, k_{xy}, and $k_{x\theta}$ of Eq. (16) at a given manipulation point of the path. The solution method is validated if one of the solutions gives the values of the predefined geometry.

Three minimal polynomials, mp_{f1}, mp_{f2}, and mp_{f3} are obtained with their solutions. Table 2 lists only the admissible distinct solution pairs C_1, C_2, and C_3, of mp_{f1}, mp_{f2}, and mp_{f3}, respectively. These solutions are distinct up to 1e-3 resolution from other close solutions. Table 3 lists the distinct 16 solutions for a_i, b_i, $i=1,2,3$. Note that, as expected, Table 3 contains a solution corresponding to the exact values of $[\sqrt{3}/3, \sqrt{3}/3, 1, 0, 1, 0]$ of the pre-defined example. These 16 solutions are presented in Fig. 3. Note also that only the last eight solutions, (i) through (p), are non-singular.

Table 2. Admissible real distinct solutions of mp_{f1}, mp_{f2}, and mp_{f3}

C_1	±0.577350	±(1-4.4e-15)*	±1.145112	±1.154700
C_2	±0.38207e-13	±0.376135	±0.967869	±0.999999
C_3	±0.514087	±0.968432	±(1+12e-30)*	±1.154700

* All numerical computations were made with 32 digits, but results are truncated to 6 decimal digits for presesntation purposes

Table 3. 16 real solutions to the problem in the numerical example

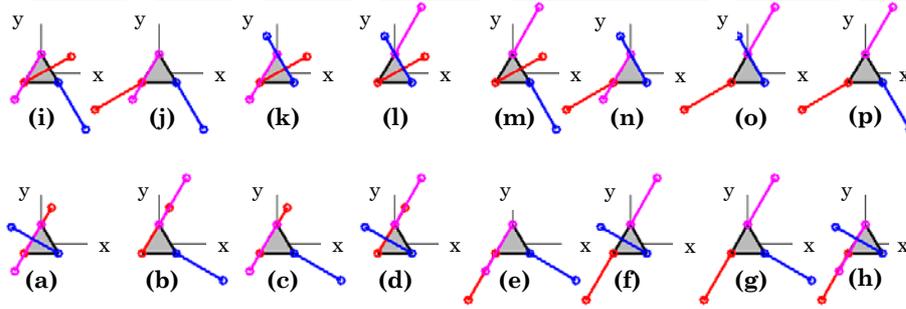| a1	b1	a2	b2		a3	b3
$0.8869987582\,^{-14}$	-1	-1	0.33502100	10^{-9}	±0.57735027	±0.57735027
$0.8869987582\,^{-14}$	-1	1	-0.33502100	10^{-9}	±0.57735027	±0.57735027
$-0.8869987582\,^{-14}$	1	-1	0.33502100	10^{-9}	±0.57735027	±0.57735027
$-0.8869987582\,^{-14}$	1	1	-0.33502100	10^{-9}	±0.57735027	±0.57735027
0.57735027	0.57735027	-1	0.33502100	10^{-9}	±1	$\pm0.2240\,10^{-28}$
0.57735027	0.57735027	1	-0.33502100	10^{-9}	±1	$\pm0.2240\,10^{-28}$
-0.57735027	-0.57735027	-1	0.33502100	10^{-9}	±1	$\pm0.2240\,10^{-28}$
-0.57735027	-0.57735027	1	-0.33502100	10^{-9}	±1	$\pm0.2240\,10^{-28}$

Figure 3. 16 solutions with only eight non-singular solutions (i, through p)

8. Conclusions

This investigation addresses the stiffness synthesis problem of a planar kinematically redundant 3-DOF robot by controlling a limited number of its free geometric parameters. It is shown that it is impossible to control both Cartesian stiffness elements k_{xx} and k_{yy}, by only changing the locations of the sliders on its circular base. In an example of controlling k_{xx}, k_{xy}, and $k_{x\theta}$, it is shown that, at most, there are 48 solutions in the complex plane. The numerical example solved shows only eight real non-singular solutions.

9. Acknowledgements

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