

LARGE-SCALE ANISOTROPY OF THE MICROWAVE BACKGROUND AND THE AMPLITUDE OF ENERGY DENSITY FLUCTUATIONS IN THE EARLY UNIVERSE¹

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ABSTRACT

The spectrum of energy density fluctuations predicted by the inflationary cosmology is used to compute multipole moments of the microwave background temperature and the probability distributions for these moments. Comparison with the observed dipole moment gives a 90% confidence level upper bound of 3.4×10^{-6} on the amplitude of these fluctuations when they cross the horizon.

Subject headings: cosmic background radiation — cosmology

The large-scale anisotropy of the microwave background can provide important information about the amplitude and spectrum of long-wavelength energy density fluctuations in the early universe. In the inflationary cosmology (Guth 1981; Linde 1982; Albrecht and Steinhardt 1982), the spectrum of energy density perturbations is predicted (Guth and Pi 1982; Bardeen, Steinhardt, and Turner 1983; Starobinskii 1982; Hawking 1982) to be scale invariant (Harrison 1970; Zeldovich 1972), i.e., the amplitude of a fluctuation when it crosses the horizon is equal to a constant ϵ_H , independent of the wavelength of that fluctuation. The value of ϵ_H is determined by parameters of the scalar potential used to produce the phase transition that gives rise to the inflationary epoch. Thus, a knowledge of ϵ_H is of interest in both astrophysics and particle physics. Here we discuss what measurements of the microwave background imply about the value of ϵ_H .

In the inflationary cosmology, the universe is very near critical density so, in first approximation, it is described by a spatially flat Robertson-Walker metric with scale factor R and Hubble constant $H \equiv \dot{R}/R$. We use Bardeen's (1980) gauge-invariant variable, ϵ , to characterize energy density fluctuations. For a fluctuation of coordinate wavenumber k ,

$$\epsilon(\mathbf{k}, t) = \frac{\delta\rho}{\rho} + 3\left(\frac{\rho + p}{\rho}\right)\left(\frac{RH}{k}\right)(v - B), \quad (1)$$

where ρ and p are the unperturbed energy density and pressure and $\delta\rho$, v , and B are the amplitudes of fluctuations in the energy density, velocity, and off-diagonal metric respectively (see Bardeen 1980 for details). The variable ϵ reduces to $\delta\rho/\rho$ when the physical wavelength of the fluctuation is much less than the horizon size, $1/H$.

The prediction of the inflationary cosmology for wavelengths which enter the horizon during the matter-dominated era is

$$\epsilon(\mathbf{k}, t) = \frac{1}{4}\hat{a}(\mathbf{k})\epsilon_H k^2 t_0^{4/3} t^{2/3}, \quad (2)$$

where t_0 is the present time and $\hat{a}(\mathbf{k})$ is a random variable satisfying

$$\langle \hat{a}^*(\mathbf{k}) \hat{a}(\mathbf{q}) \rangle = \frac{1}{k^3} \delta^3(\mathbf{k} - \mathbf{q}). \quad (3)$$

Equation (2) can also be written in terms of the conformal time variable $\tau = 3t_0^{2/3}t^{1/3}$ as

$$\epsilon(\mathbf{k}, \tau) = \frac{1}{4}\hat{a}(\mathbf{k})\epsilon_H k^2 \tau^2. \quad (4)$$

The time of horizon crossing is $\tau = 2/k$, so the average value of ϵ^2 , defined as

$$k^3 \int d^3\mathbf{q} \langle \epsilon^*(\mathbf{k}, t) \epsilon(\mathbf{q}, t) \rangle, \quad (5)$$

is equal to ϵ_H^2 at the time of horizon crossing. The random nature of the prediction for ϵ reflects the underlying quantum mechanical origin of the fluctuations in the inflationary cosmology.

The fluctuation in the observed cosmic blackbody temperature (Sachs and Wolfe 1967; Abbott and Wise 1984c) viewed along the direction unit vector \mathbf{e} , produced by an energy density fluctuation of coordinate wave vector \mathbf{k} , is

$$\frac{\delta T_0}{T_0}(\mathbf{e}, \mathbf{k}) = \frac{1}{k^2} \int_0^{\tau_0 - \tau_E} e^{i\mathbf{k} \cdot \mathbf{e} y} (\mathbf{e} \cdot \mathbf{k})^2 \dot{\epsilon}(\mathbf{k}, \tau_0 - y). \quad (6)$$

Here the dot represents a τ derivative, $\tau_0 = 3t_0$ and τ_E is the conformal time of recombination, $\tau_E = 3t_0^{2/3}t_E^{1/3}$. The micro-

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wave background temperature can be expanded in multipoles

$$\frac{\delta T_0}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\mathbf{e}). \quad (7)$$

The quantities

$$a_l^2 = \sum_{m=-l}^l |a_{lm}|^2 \quad (8)$$

are rotationally invariant. Predictions for the multipole moments of the background temperature have been obtained from equations (3), (4), and (6) by projecting out a given multipole in equation (6), integrating over wave vectors, k , squaring, and taking an expectation value. The result, for $l \geq 2$ is given to a good approximation by (Peebles 1982; Abbott and Wise 1984a)

$$\langle a_l^2 \rangle = \frac{2\pi^2 \epsilon_H^2 (2l+1)}{l(l+1)}. \quad (9)$$

For $l \leq 20$, this prediction depends only on very long wavelength fluctuations and so is insensitive to fluctuations which have gone nonlinear and to the details of the process of recombination.

Equation (9) is the prediction for an expectation value averaged over many universes. Measurements of the microwave background are of course restricted to a single universe. Fortunately, the inflationary cosmology (in the weak-coupling approximation for the scalar field in the inflationary epoch) also predicts the higher correlation functions of $\hat{a}(\mathbf{k})$, and these can be used to construct a probability distribution for $a_l \equiv (a_l^2)^{1/2}$. In the inflationary cosmology, the multipole coefficients a_{lm} are Gaussian random variables, so a_l has the probability distribution

$$P_l(a_l) = \sqrt{\frac{2}{\pi}} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2l-1)} \frac{(a_l)^{2l}}{\sigma_l^{2(l+1)}} e^{-[1/2(a_l/\sigma_l)^2]} \quad (10)$$

with σ_l given by

$$\sigma_l = \frac{2\pi^2 \epsilon_H^2}{l(l+1)}. \quad (11)$$

This determines the statistical uncertainties in the predictions of equation (9). The results for the first 10 moments are shown in Figure 1 where 68% confidence level error bars are given. Even with the relatively large statistical uncertainties, these predictions for a_l^2 are significantly different from those of some non-scale-invariant spectra (Abbott and Wise 1984c).

A limit on the value of ϵ_H can be obtained by comparing the prediction for $l=2$ with the present limit on the quadrupole anisotropy (Lubin, Epstein, and Smoot 1983; Fixen, Cheng, and Wilkinson 1983)

$$a_2^2 < 5.2 \times 10^{-8}. \quad (12)$$

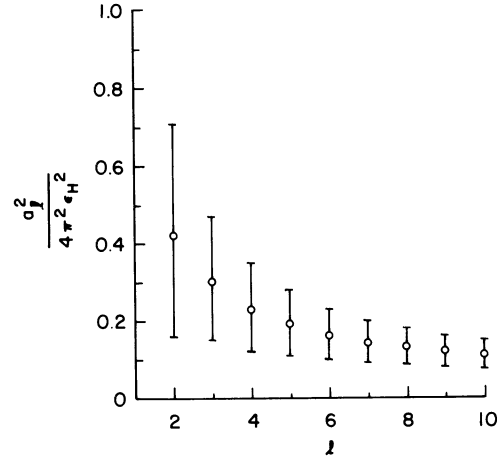


FIG. 1.—Predictions for moments 2 through 10 with error bars showing 68% confidence level limits.

At 90% confidence level, the probability distribution (10) gives

$$a_2^2 > 0.3 \frac{5\pi^2}{3} \epsilon_H^2, \quad (13)$$

so we find that

$$\epsilon_H < 1.0 \times 10^{-4}. \quad (14)$$

A more stringent bound on ϵ_H can be obtained from the observed dipole anisotropy. The dipole moment has been considered in a similar way by Silk and Wilson (1981a, b) and by Kaiser (1983). The prediction for $l=1$ is obtained by combining equations (3), (4), and (6). We find that wavelengths with $k < k_{\max}$, which entered the horizon during the matter-dominated era, give a contribution to the dipole moment of

$$\langle a_1^2 \rangle = 6\pi^2 \epsilon_H^2 k_{\max}^2 t_0^2. \quad (15)$$

Unlike the higher multipoles, the dipole moment is sensitive to short-wavelength fluctuations. This is because the dipole moment is dominated by the peculiar velocity of the observer. However, the observed dipole moment can still be used to obtain a bound on ϵ_H without having to consider modes which have gone nonlinear. If we assume that the short-wavelength modes that have gone nonlinear are uncorrelated with the long-wavelength modes that are evolving linearly, then the contributions of both of these sets of modes to $\langle a_1^2 \rangle$ is positive. The full probability distribution for the dipole moment is a convolution of probability distributions for the linear and nonlinear contributions. With the distribution for the linear modes given by equation (10), it is straightforward to show that the nonlinear modes can only make the full probability distribution broader than that of the linear modes alone. We thus obtain a bound on ϵ_H by demanding that the computable long-wavelength contribution to the dipole anisotropy is no larger than the observed dipole anisotropy (Lubin, Epstein, and Smoot 1983; Fixen, Cheng, and Wilkinson 1983)

$$a_1^2 = 5.5 \times 10^{-6}, \quad (16)$$

at 90% confidence level. (The statistical error in our prediction dominates over the experimental error in the measurement of the dipole moment.)

To derive the best possible bound, we must consider wavelengths which entered the horizon before the time of matter domination, but which are still evolving linearly today. In keeping with the prediction of inflation that we are at critical density, we assume that the universe is dominated by an uncoupled form of dark matter such as neutrinos (Cowsik and McClelland 1972; DeRujula and Glashow 1980), axions (Preskill, Wise, and Wilczek 1983; Abbott and Sikivie 1983; Dine and Fischler 1983), photinos (Cabibbo, Farrar, and Maiani 1983), or gravitinos (Pagels and Primack 1982), and that the universe became matter dominated by this dark matter at about 1,000 years. Before the time of matter domination, the inflationary cosmology predicts that fluctuations in the radiation density satisfy

$$\epsilon_{\text{rad}}(\mathbf{k}, t) = \hat{a}(\mathbf{k}) A_{\text{rad}} \left[\frac{\sin x - x \cos x}{x} \right], \quad (17)$$

where

$$x = \frac{k}{\sqrt{3} RH}, \quad (18)$$

A_{rad} is a constant, and \hat{a} is a random variable satisfying equation (3). To analyze how these fluctuations are transferred to the dark matter, we use an extension of Bardeen's (1980) gauge-invariant formalism to the case of multiple, uncoupled fluids (Abbott and Wise 1984b).

The amplitude of fluctuations which entered the horizon before the matter-dominated era is determined by numerically integrating the coupled dark matter–radiation evolution equations for ϵ_{rad} and ϵ_{dm} . We find that the amplitude of the dark matter fluctuations during the matter-dominated era is

$$\epsilon_{\text{dm}}(\mathbf{k}, t) = \frac{9}{4} \hat{a}(\mathbf{k}) A_{\text{dm}}(k) k^2 t_0^{4/3} t^{2/3}, \quad (19)$$

where A_{dm} is given approximately by

$$A_{\text{dm}} \approx \frac{\epsilon_{\text{H}}}{1 + 0.04 x_m^2 + 0.0005 x_m^4} \quad (20)$$

with

$$x_m = \frac{2}{\sqrt{3}} (3/4)^{2/3} k t_0^{2/3} t_m^{1/3}. \quad (21)$$

Here t_m is the time the universe becomes matter dominated. Equation (20) is a good approximation to our numerical results for $x_m \leq 8$. For wavelengths which enter the horizon after the time of matter domination, x_m is small so equations (19) and (20) reproduce equation (2).

The bound we derive from the dipole anisotropy will be expressed in terms of ϵ_{H} . However, ϵ_{H} and A_{rad} are proportional to each other, so our bound also applies to A_{rad} . In the radiation-dominated era, for small x , the dark matter fluctua-

tions satisfy (Abbott and Wise 1984b)

$$\epsilon_{\text{dm}} = \frac{9}{16} \epsilon_{\text{rad}} \quad (22)$$

and for large x , ϵ_{dm} grows logarithmically, eventually growing by a factor of 4–5. To relate ϵ_{H} and A_{rad} we take equation (22) and match it onto the growing and shrinking matter-dominated modes of ϵ_{dm} at the time when matter first dominates. Then comparing the coefficient of the growing mode with the prediction (2), we find³

$$A_{\text{rad}} = 8 \left(\frac{4}{3} \right)^{4/3} \epsilon_{\text{H}}. \quad (23)$$

To determine the contribution of modes that are evolving linearly to the dipole moment, we use equations (19) and (20) in equation (6), integrate over k , project out the dipole term, square, and take the expectation value. The k integration must be cut off at the point where linear perturbation theory breaks down, i.e., when the average value of ϵ^2 , as defined in equation (5), is of order unity. With a Hubble constant, $H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$, the average value of ϵ^2 for a 60 Mpc wavelength is at most 10^{-1} , so we feel confident that linear perturbation theory is valid for wavelengths longer than this. We find that the contribution of modes with a wavelength longer than 60 Mpc to a_1^2 is greater than or equal to $4.8 \times 10^5 \epsilon_{\text{H}}^2$ at the 90% confidence level. Comparison of this result with the observed dipole moment (16) gives the bound

$$\epsilon_{\text{H}} < 3.4 \times 10^{-6} \quad (24)$$

at the 90% confidence level. If we are less conservative and use a 30 Mpc cutoff, then the bound improves to $\epsilon_{\text{H}} < 1.9 \times 10^{-6}$. The bound is not very sensitive to the present value of the Hubble constant. If a Hubble constant $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is used, then we find $\epsilon_{\text{H}} < 3.8 \times 10^{-6}$ for a 120 Mpc cutoff and $\epsilon_{\text{H}} < 2.7 \times 10^{-6}$ for a 60 Mpc cutoff.

The bound of equation (24) is valid for any form of cold dark matter. For neutrinos (hot dark matter), our bound is modified somewhat due to free streaming. Using the results of Bond and Szalay (1983), we find that

$$\epsilon_{\text{H}} < 7 \times 10^{-6} \quad (25)$$

for neutrinos using a 60 Mpc cutoff.

The bound (24) implies that the contribution of energy density fluctuations to the quadrupole moment satisfies $a_2^2 <$

³To obtain this result, we must determine the value of the scale factor R in the radiation-dominated period and use this result in equation (18). If we write

$$R = \left[\frac{t + (1/3)t_m}{t_0} \right]^{2/3}$$

in the matter-dominated era, then by continuity of R and \dot{R} ,

$$R = \left(\frac{4}{3} \right)^{2/3} \left(\frac{t_m}{t_0} \right)^{2/3} \left(\frac{t}{t_m} \right)^{1/2}$$

in the radiation-dominated era.

3.2×10^{-10} at the 90% confidence level. If the scale of the symmetry breaking responsible for inflation is of order 10^{17} GeV, then long-wavelength gravitational waves generated during the inflationary era will contribute significantly to the quadrupole and higher moments (Starobinskii 1979; Rubakov, Sazhin, and Veryaskin 1982; Fabbri and Pollock 1983). Thus, in this case, a_2^2 could be bigger than 3.2×10^{-10} without violating our dipole bound (gravitational waves do not contribute to the dipole moment). If the scale of inflation is below 10^{17} GeV, then no appreciable gravitational contribution exists. The bound on ϵ_H in equation (24) is close to the value of ϵ_H determined by Peebles from an analysis of the galaxy-galaxy correlation function (Peebles 1982). This analysis does not

involve any large theoretical statistical uncertainties; however, the correlation function is not well measured at large distances. Also, the correlation function approach must assume that galaxies are a good indicator of the distribution of mass in the universe, even though the total mass is dominated by dark matter.

Throughout this *Letter* we have assumed an inflationary cosmology. However, our results are valid in any cosmology with $\Omega = 1$ and a scale-invariant spectrum of energy density fluctuations that has a Gaussian probability distribution.

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