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CAUSALITY CONSTRAINTS ON COSMOLOGICAL PERTURBATIONS

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ABSTRACT

Using recently derived integral constraints, we give a general-relativistic proof of the k^4 power spectum for perturbations generated by local, causal processes in a Robertson-Walker universe. If galactic clusters formed from such fluctuations and if light traces mass, then in order to reconcile the integral constraints with observed cluster-cluster correlations, these perturbations must have been created when the age of the universe was greater than a few thousand years. Inflation provides a way out of this bound.

Subject headings: cosmology — galaxies: clustering — relativity

I. INTRODUCTION

It has long been recognized that the limited horizon size in the standard big bang cosmology causes severe problems for understanding how galactic structure arose from primordial fluctuations in the early universe. Recently, general-relativistic integral constraints were derived which state precisely what restrictions causality imposes on cosmological perturbations (Traschen 1984, 1985). Here, we use these constraints to provide a general relativistic proof of the k^4 power spectrum for perturbations formed by local causal processes in a Robertson-Walker universe. In addition, we set a limit on the time at which causal perturbations could have been generated if they are to account for large-scale galactic clustering.

The constraints we use are integral conditions in the spirit of Gauss's law. Let G be a spatial volume with boundary ∂G in a constant-time hypersurface of a general Robertson-Walker spacetime. Then, there exist (Traschen 1984, 1985) four vector fields V^μ such that any stress-energy perturbation δT^μ_ν must satisfy.

$$\int_G dv \delta T^0_\mu V^\mu = \int_{\partial G} da_l B^l , \qquad (1.1)$$

where B_l is a quantity defined on the boundary ∂G which is linear in V^μ and in the perturbation quantities.

The constraints (1.1) imply, for example, that certain volume integrals of the two-point mass correlation function $\xi(r)$ are related to surface integrals and are thus determined by the long-range behavior of the correlation function. For simplicity, we will consider a spatially flat Robertson-Walker space-time

$$d\tau^2 = dt^2 - a^2(t)|d\boldsymbol{x}|^2 , \qquad (1.2)$$

and we will assume that scalar perturbations dominate. In this case, we can choose a gauge in which $\delta T^0_k = 0$. In this gauge δT^0_0 is proportional to Bardeen's (1980) gauge-invariant variable ϵ_m. [In general, we can neglect the term $V^k \delta T^0_k$ in the integrand of eq. (1.1) when $|\lambda(t)v/t| \ll |\delta\rho/\rho|$, where $\lambda(t)$ is the proper length scale associated with the density perturbation $\delta\rho$, and v is a typical perturbation velocity. For causal perturbations, $\lambda/t < 1$]. We will also assume that the universe is pressureless, in which case we can simultaneously choose a synchronous gauge. For scalar perturbations in this gauge, equation (1.1) implies that

$$\int_G d^3 r \xi(r) V^0 = \left\langle \frac{\delta\rho(0)}{\rho} \int_G d^3 r \frac{\delta\rho(r)}{\rho} V^0 \right\rangle = \int_{\partial G} da_l \left\langle \frac{\delta\rho(0)}{\rho} B^l \right\rangle . \qquad (1.3)$$

Thus, to compute this integral all one needs is the long-range correlation between $\delta\rho/\rho$ at the origin and on the surface ∂G. The brackets in equation (1.3) denote averages over the observed universe. Equations (1.1) and (1.3) will provide the basis for the results of the next two sections.

II. A PROOF OF THE k^4 POWER SPECTRUM

It is a commonly accepted statement that the power spectrum for perturbations generated by local causal processes must vanish at least as fast as k^4 for small wavenumber k. The usual proofs (see, for example, Peebles 1980; Carr and Silk 1983) require energy and momentum conservation and thus are only valid in the Newtonian limit. In general relativity, energy and momentum are not always conserved. The integral constraints (1.1) allow us to extend the proof to the general-relativistic case.

Assume that the density perturbation $\delta\rho/\rho$ in some large volume V is the sum of N randomly distributed and uncorrelated individual perturbations, each of which was generated by a local causal process,

$$\frac{\delta\rho}{\rho}(\boldsymbol{x}) = \sum_{a=1}^{N} F_a(\boldsymbol{x}\text{-}\boldsymbol{x}_a) . \qquad (2.1)$$

Both the functions F_a and the sites $\boldsymbol{x}_a$ are randomly choosen. By causality, $F_a(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}|$ greater than some horizon size r_c. In addition, the individual F_a's must satisfy integral constraints (1.1) with zero boundary term, if G is taken to be a

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sphere of radius r_c or larger. For the flat Robertson-Walker universe, these reduce to the special relativistic statements that the monopole and dipole moments of the F_a must vanish,

$$\int_G d^3x F_a(\boldsymbol{x}) = 0 ,$$
$$\int_G d^3x \boldsymbol{x} F_a(\boldsymbol{x}) = 0 , \tag{2.2}$$

for G a sphere of radius r_c or greater. These are the same as the special relativistic statements that mass and momentum are conserved if the perturbation vanishes at some initial time.

If we denote the Fourier transform of $F_a(\boldsymbol{x})$ by $\hat{F}_a(\boldsymbol{k})$, then equations (2.2) imply that

$$\hat{F}_a(0) = \nabla_k \hat{F}_a(0) = 0 . \tag{2.3}$$

Thus, $\hat{F}_a(\boldsymbol{k})$ must vanish at least as fast as k^2 near $\boldsymbol{k} = 0$. (For an extension of this argument see the Appendix). Because the F_a's are uncorrelated,

$$\left\langle \left| \frac{\delta \hat{\rho}(\boldsymbol{k})}{\rho} \right|^2 \right\rangle = \sum_{a=1}^{N} \langle | \hat{F}_a(\boldsymbol{k}) |^2 \rangle . \tag{2.4}$$

Thus, the power spectrum which is proportional to $\langle | \delta \hat{\rho}(\boldsymbol{k})/\rho |^2 \rangle$ must vanish at least as fast as k^4.

In the simplest situation, where the F_a are functions of distance only, they can be written as $F_a(r) = r_c^2 \nabla^2 f_a(r)$, for same function f_a. For example, suppose that

$$f_a(r) = \frac{1}{\sqrt{8\pi^5}\, r_c^5} r^2 (r_c - r)^3 \theta(r_c - r) . \tag{2.5}$$

Then f_a and F_a are continuous everywhere, and the F_a satisfy equation (2.2) for any volume G with radius greater than r_c. In this case,

$$\xi(k) \propto \begin{cases} (kr_c)^4 , & kr_c \ll 1 , \\ \dfrac{\sin^2 kr_c}{(kr_c)^6} , & kr_c \gg 1 . \end{cases} \tag{2.6}$$

Finally, we note that in configuration space the correlation function for equation (2.1) is $\xi(r) = \Sigma \langle \int d^3x F_a(\boldsymbol{x} + \boldsymbol{r}) F_a(\boldsymbol{x}) \rangle$. Therefore, $\xi(r)$ is zero for $r > 2r_c$. Also, the right-hand side of equation (1.3) is zero for large volumes G, since it depends on correlations between points on ∂G and the center of G.

III. A BOUND ON THE TIME OF FORMATION OF COSMOLOGICAL PERTURBATIONS

We now compare observed galaxy-galaxy and cluster-cluster correlation functions with the constraints discussed in §§ I and II to derive a bound on the time at which cosmological perturbations arose. As in § II, we assume that the observed structure in the universe resulted from randomly scattered perturbations which were formed between times $t_f - \Delta t$ and t_f by local, causal processes. By local and causal we mean that at time t_f all perturbations were uncorrelated over distances greater than the distance which light could travel between times $t_f - \Delta t$ and t_f. As an upper limit on Δt we can use t_f, in which case this distance becomes the horizon size $r_c(t_f)$. (For cosmological perturbations this is an adequate approximation, but for models in which perturbations formed recently we retain an arbitrary Δt.) Of course, this lack of correlations outside the horizon depends both on causal evolution and on an assumption about the absence of long-range correlations in the initial-value data. This assumption is crucial to our results.

At time t_f, the perturbation is given by equation (2.1), so the power spectrum goes like k^4 for small k, and the two-point correlation function $\xi(r)$ vanishes for $r \geq 2r_c(t_f)$. From these two facts it is easy to prove that at time t_f the correlation function must satisfy

$$\int_0^R dr\, r^2 \xi(r) = \int_0^R dr\, r^4 \xi(r) = 0 , \tag{3.1}$$

for $R \geq 2r_c(t_f)$.

After the perturbations are generated at time t_f, they presumably move with the Hubble flow, and their subsequent evolution is described by the usual linear perturbation analysis, at least as long as they are small in amplitude. In a pressureless universe, the characteristic length scale associated with a fluctuation grows like the scale factor $a(t)$. When background pressure is still important, perturbations which are inside the horizon are wave packets which propagate at the speed of sound of the fluid (Sachs and Wolfe 1967). The length scale of the packet likewise scales as $a(t)$ up to dispersion. Thus, at a later time $t > t_f$, equation (3.1) should hold for $R \geq 2r_c(t_f)a(t)/a(t_f)$. In particular, at the present time we get the constraint (3.1) for $R \geq 2r_c(t_f)a(t_0)/a(t_f)$.

Equation (3.1) obviously requires that the correlation function vanish for distances larger than $2r_c(t_f)a(t_o)/a(t_f)$. However, it is difficult to verify experimentally whether a correlation function is strictly zero. Fortunately, equation (3.1) makes a stronger statement. Any function which satisfies equation (3.1) must have at least two zeros over the range of integration. Observationally, it is much easier to identify where a correlation crosses zero than where it vanishes identically. If we define r_2 to be the location of the second zero of the mass correlation function, then we obtain the constraint.

$$2r_c(t_f)a(t_0)/a(t) > r_2 . \tag{3.2}$$

Observations indicate that the cluster-cluster correlation function is well fitted by (Bahcall and Soneira 1983; Klypin and Kopylov 1983).

$$\xi(r) \approx \left(\frac{r}{25h^{-1}\,\text{Mpc}} \right)^{-1.8} , \quad \text{out to } r = 50\text{–}100h^{-1}\ \text{Mpc} . \tag{3.3}$$

If we accept Kaiser's (1984) interpretation, this is proportional to the mass correlation function, so r_2 must be greater than $50h^{-1}$ Mpc. If we use the galaxy-galaxy correlation function $\xi(r) = (r/6h^{-1}\ \text{Mpc})^{-1.8}$ out to $r \approx 25h^{-1}$ Mpc, then we find $r_2 > 25h^{-1}$ Mpc (Davis and Peebles 1983; Shanks *et al.* 1983).

Let t_m be the time at which the universe becomes matter-dominated. Then for $t_f < t_m$,

$$r_c(t_f) = 2t_f , \tag{3.4}$$

and for $t_f > t_m$,

$$r_c(t_f) = 3(t_f + (t_f + \tfrac{1}{3}t_m) - 2(\tfrac{3}{4})^{2/3} t_m^{1/3} (t_f + \tfrac{1}{3}t_m)^{2/3} . \tag{3.5}$$

We now scale up these distances by the expansion factor $a(t_o)/a(t_f)$ and require the result to be less than $\frac{1}{2}r_2$. This implies that the formation process must have concluded, and pregalactic perturbations must have joined the Hubble flow, at a time t_f satisfying the bound

$$t_f > \frac{1}{12} \left(\frac{4}{3} \right)^{1/3} \left(\frac{t_m^{1/3} r_2^2}{t_0^{4/3}} \right) , \quad \text{for } t_f < t_m , \tag{3.6}$$

or

$$t_f > \left[\frac{r_2}{6t_0^{2/3}} + \left(\frac{t_m}{6}\right)^{1/3}\right]^3 - \frac{1}{3}t_m\,, \quad \text{for } t_f > t_m\,. \tag{3.7}$$

If we take r_2 to be 50 Mpc, use a Hubble constant of 100 km s^{-1} Mpc^{-1}, and take $t_m = 10^5$ yr, then

$$t_f > 8.8 \times 10^3 \text{ yr}\,. \tag{3.8}$$

If instead we take $t_m = 10^3$ yr, then

$$t_f > 1.9 \times 10^3 \text{ yr}\,. \tag{3.9}$$

Equations (3.8) and (3.9) give relatively recent times for galaxy formation processes. Bardeen (1980) has shown that anisotropic pressure stresses or nonadiabatic pressure perturbations must be present to generate density perturbations if the universe was originally homogeneous. Therefore, it is interesting that the bound requires the formation process to occur very close to or later than the time of matter domination. Particular models for the formation of fluctuations typically generate them at very early times. Carr and Silk (1983) list a number of possibilities ranging from X-boson grains or magnetic monopoles at $t_f \approx 10^9$ s. Peebles (1980) also summarizes a variety of mechanisms which have been suggested, all of which occur at early times.

An alternative scenario for galaxy formation is an Ostriker-Cowie (1981)–type model. In this model, explosions occurred in the interstellar medium at recent times ($z < 100$). The resulting shock waves lead to the formation of shells, which fragment to galaxies. Cowie and Ostriker find that the explosions must have happened at $z_1 \leq 5$, and fragmentation at $z_2 \approx 4$, to get galactic-size masses. For such late galaxy formation, causality bounds the duration of the process rather than the time at which it occurred. It must last for time Δt such that

$$\Delta t > \left(\frac{1}{1+z_2}\right)\left(\frac{r_2}{50h^{-1}\text{ Mpc}}\right) \times 10^8 \text{ yr}\,. \tag{3.10}$$

The shell formation time depends on model parameters, but one way of satisfying the bound is to require that the seed explosions heat up the gas enough so that the cooling time is sufficiently long.

The easiest way to avoid the late times for generating perturbations imposed by causality is to use an inflationary cosmology (Guth 1981). Then, although the bound of equation (3.2) still applies, the factor $a(t_0)/a(t_f)$ is so enormous that t_f can be very small, provided that it precedes the end of inflation. Indeed, these bounds provide a strong indication that some type of inflationary expansion did in fact occur.

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APPENDIX

The following theorem relates constraints on functions in configuration space to constraints in Fourier space. It is more general than what is needed to derive the k^4 spectrum but includes this when one chooses the harmonic functions $\phi(\boldsymbol{x})$ to be the particular functions 1, x, y, and z. Further, this theorem describes an easy way to construct density perturbations which satisfy the integral constraints (2.2).

The following definitions are used: S is a sphere of radius R, and

$$\frac{\partial f}{\partial r^+} = \lim_{\boldsymbol{x}\to\partial S} \frac{\partial f}{\partial r}\,.$$

THEOREM. *Let F be a function which vanishes outside S and is continuous, except for a possible step discontinuity at r = R. Suppose there exists a harmonic function* $\phi(\nabla^2\phi = 0$ *everywhere), such that*

$$\int_S d^3x F\,\phi = 0\,. \tag{A.1}$$

Then there exists a function f such that F is given by

$$F(\boldsymbol{x}) = \nabla^2 f(\boldsymbol{x}) + \frac{\partial f}{\partial r^+}\,\delta(R - r)\,, \tag{A.2}$$

where f is continuous everywhere and satisfies

$$f(\boldsymbol{x}) = 0 \text{ for } r \geq R\,,$$

$$\int d\Omega\,\frac{\partial f}{\partial r^+}\,\phi = 0\,. \tag{A.3}$$

The converse is also true.

COROLLARY. *Let* ϕ *be the harmonic functions* 1 rY_{1m} *(where we have previously written* 1, x, y, z*), as in equation* (2.2). *Then the Fourier transforms of F and f are related by*

$$\hat{F}(\boldsymbol{k}) = -k^2\hat{f}(\boldsymbol{k}) + k^2R^2P(kR)\,, \tag{A.4}$$

where $\hat{f}(\boldsymbol{k})$ *is finite as* $k \to 0$, *and P is the polynomial*

$$P(kR) = \sum_{n=0}^{\infty} \frac{(iR)^{n+2}k^n}{(n+2)!} \int d\Omega\,\frac{\partial f}{\partial r^+}(\cos\theta)^{n+2}\,. \tag{A.5}$$

In particular, if F depends only on the distance r, then

$$\hat{F}(\boldsymbol{k}) = -k^2 \hat{f}(\boldsymbol{k}) . \quad \text{(A.6)}$$

Proof. Let G be the Green's function for the sphere S which vanishes on the boundary ∂S. (See, for example, Jackson 1975). Define $f(\boldsymbol{x})$ for all $\boldsymbol{x}$ by $f(\boldsymbol{x}) = \theta(R - r) \int_S d^3x' G(\boldsymbol{x}, \boldsymbol{x}')F(\boldsymbol{x}')$ for $\boldsymbol{x}$ inside S. Then f is continuous (since G vanishes on ∂S), and taking the Laplacian of f one gets equation (A.2), using the properties of the delta function.

To prove the converse, multiply equation (A.2) by ϕ and integrate over some big volume which contains S. Since $F = 0$ on the boundary of this big volume, using equations (A.3) one finds equation (A.1).

Note that if $f(\boldsymbol{x})$ is a continuous function which vanishes as $r \to \infty$, then its Fourier transform approaches a constant as $k \to 0$.

Causality and locality are needed to reduce the general integral constraints (1.1) to integrals with zero boundary term. From the proof it is clear that all four integral constraints are necessary to get a k^4 power spectrum. This can be understood by noting that the set of functions for which a Robertson-Walker universe has integral constraints is consistent with its symmetries: the geometry is invariant under spatial rotations and translations, and the functions 1, rY_{1m} mix among themselves under these transformations.

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