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A GENERAL, GAUGE-INVARIANT ANALYSIS OF THE COSMIC MICROWAVE ANISOTROPY

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ABSTRACT

We present a general, gauge-invariant analysis of the large-scale anisotropies in the cosmic background radiation produced by arbitrary scalar, vector, or tensor perturbations in open, closed, or flat Robertson-Walker spacetimes (with no cosmological constant). We contrast the multipole moment predictions for the scale invariant spectrum and $\Omega = 1$ universe predicted by inflationary cosmologies with those for other spectra and for open and closed universes. In all cases, we assume that the universe today is dominated by cold dark matter. Using the measured value of the dipole moment we set limits on the expected value of the quadrupole moment as a function of Ω for various spectra. We also compare the anisotropies produced by scalar fluctuations with those from tensor perturbations. Our analysis includes the statistical uncertainties associated with a Gaussian distribution of initial fluctuations.

Subject heading: cosmic background radiation

I. INTRODUCTION

The microwave background radiation provides us with our earliest glimpse of the universe. It reveals the remarkable homogeneity and isotropy of the universe at the time of recombination and determines our motion relative to this uniform background through measurements of the dipole anisotropy. However, it has the potential to do much more. The higher multipole moments (which have not yet been detected), as well as the dipole moment, contain information about the long-wavelength portion of the spectrum of energy density perturbations which produced the large-scale galactic structure of the universe. They are also sensitive to the presence of long-wavelength gravitational waves.

Our main interest in considering the microwave anisotropy is to test the predictions of inflationary cosmology (Guth 1980). In a class of inflationary models (Linde 1982; Albrecht and Steinhardt 1982) the spectrum of energy-density perturbations can be determined (Guth and Pi 1982; Bardeen, Steinhardt, and Turner 1983; Starobinskii 1982; Hawking 1982). In addition, the spectrum of gravitational waves produced during inflation can be predicted (Starobinskii 1979; Rubakov, Sazhin, and Veryaskin 1982; Fabbri and Pollock 1983; Abbott and Wise 1984*d*). These predictions can be used to compute the nondipole anisotropy of the microwave background coming from energy-density fluctuations (Peebles 1982; Abbott and Wise 1984*a*; Starobinskii 1984) or from gravitational waves (Starobinskii 1979; Rubakov, Sazhin, and Veryaskin 1982; Fabbri and Pollock 1983; Abbott and Wise 1984*d*) and to set a limit on perturbation amplitudes from the dipole anisotropy (Abbott and Wise 1984*c*). Inflation gives a scale-invariant spectrum and requires a critical energy density. In §§ V and VI of this paper we compare the predictions of inflationary cosmology with those for other spectra and for open or closed universes. We also compare the anisotropy coming from energy-density perturbations with that produced by gravitational waves.

A key element in our analysis is the inclusion of an error analysis for the anisotropy predictions. (However, in other respects our analysis is not new. The multipole moments have previously been calculated for a critical density universe [Peebles 1982; Silk and Wilson 1981; Wilson and Silk 1981], and for open universes [Wilson 1983]. Certain properties of the multipole moments have also been studied by other authors [Fabbri, Guidi, and Natale 1983; Tomita and Tanabe 1983].) The inflationary cosmology does not determine exact perturbation amplitudes but rather gives the width of a Gaussian probability distribution for the perturbations. We include in our analysis the effects of this uncertainty on the final moment predictions to see under what conditions the inflationary predictions can be distinguished from those of different spectra and noncritical energy densities. We will assume that the noninflationary predictions which we use for comparison are also Gaussian-distributed random variables. It is interesting to note that this assumption can, in principle, be tested by examining correlations of rich galactic clusters (Kaiser 1984; Politzer and Wise 1984).

The most elegant method for treating cosmological perturbations is the gauge-invariant approach of Bardeen (1980). In §§ II, III, and IV we apply this formalism to the Sachs and Wolfe (1967) result for the induced microwave anisotropy. The formalism of § II can be applied to any spectrum in either open, closed, or flat universes involving scalar, energy-density perturbations. The results of § IV apply to tensor perturbations, that is, to gravitational waves. Although we will not use it in this paper, for completeness we present the analysis for vector perturbations in § III. Predictions for the dipole and quadrupole moments arising from this analysis are presented and discussed in § V, and the higher multipole moments of the microwave background are considered in § VI.

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Fluctuations δT_o in the observed blackbody temperature of the microwave background arise both from fluctuations in the source temperature δT_e and in the redshift factor $1 + z$. That is,

$$T_o + \delta T_o = \frac{T_e + \delta T_e}{1 + z} , \tag{1}$$

where a subscript o refers to observation and a subscript e refers to emission. We will use coordinates for which

$$ds^2 = S^2(\tau)(-d\tau^2 + \gamma^{-2}|d\mathbf{x}|^2 + h_{\mu\nu} dx^\mu dx^\nu) , \tag{2}$$

where $h_{\mu\nu}$ represents fluctuations in the metric and

$$\gamma = 1 + \tfrac{1}{4}Kr^2 , \tag{3}$$

$K = -1, +1$, and 0 corresponding to an open, closed, and critical universe, respectively. The scale factor $S(\tau)$ is determined by the usual Friedmann equations

$$(\dot{S}/S)^2 = \frac{8\pi G}{3} S^2 \rho - K , \tag{4a}$$

$$(\dot{S}/S)^{\cdot} = -\frac{4\pi G}{3} S^2(\rho + 3p) , \tag{4b}$$

where ρ and p are the background energy density and pressure. Since we will be concerned with times since the time of recombination, which is close to or later than the time of matter domination, we will need the solution to these equations for $p = 0$:

$$\frac{\dot{S}}{S} = \begin{cases} \coth(\tau/2) & \text{for} \quad K = -1 , \\ 2/\tau & \text{for} \quad K = 0 , \\ \cot(\tau/2) & \text{for} \quad K = +1 . \end{cases} \tag{5}$$

We choose a gauge for which

$$h_{00} = T_0^{\ i} = 0 . \tag{6}$$

Then (Sachs and Wolfe 1967; Anile and Motta 1976) the redshift factor $1 + z$ is given by

$$1 + z = \frac{S(\tau_o)}{S(\tau_e)} \left[1 + \frac{1}{2} \int_0^{\tau_o - \tau_e} dy (\gamma^2 \dot{h}_{ij} e^i e^j - 2\gamma \dot{h}_{0i} e^i) \right] . \tag{7}$$

In this formula τ_o is the present value of τ, τ_e the value of τ at the time of emission of the microwave radiation, and the dots denote τ derivatives. The vector e^i is a unit vector pointing in the direction along which the microwave background is being viewed, and $\dot{h}_{ij}$ and $\dot{h}_{oi}$ are to be evaluated at points (x^i, τ) satisfying

$$\tau = \tau_o - y \quad \text{and} \quad \frac{dx^i}{dy} = \gamma e^i , \tag{8}$$

which lie along a null geodesic extending from the observer to the emission point. Equation (5) gives

$$x^i = e^i \times \begin{cases} 2 \tanh(y/2) & \text{for} \quad K = -1 , \\ y & \text{for} \quad K = 0 , \\ 2 \tan(y/2) & \text{for} \quad K = +1 . \end{cases} \tag{9}$$

In §§ II, III, and IV we will evaluate the expressions (1) and (7) for scalar, vector, and tensor perturbations. The resulting expression for $\delta T_o/T_o$ will be written in terms of gauge-invariant variables. The various fluctuation variables appearing in this expression will be expanded in normal modes $\hat{a}_{lm}(\beta) f(\tau) \Phi_\beta^{\ l}(r) Y_{lm}(\theta, \phi)$ in a usual spherical expansion. The function $f(\tau)$ which describes the temporal evolution of the perturbation variable and $\Phi_\beta^{\ l}(r)$ describing the radial dependence of the mode depend on the specific fluctuation being considered. However, a universal feature which we describe here is the presence of the Gaussian random variable $\hat{a}_{lm}(\beta)$ reflecting the uncertainty in the prediction for the amplitude of the fluctuation mode discussed above. Following the prediction of inflationary cosmology we take $\hat{a}_{lm}(\beta)$ to be a random variable with zero expectation value satisfying

$$\langle \hat{a}^*_{lm}(\beta) \hat{a}_{l'm'}(\beta') \rangle = \frac{1}{k^3} \delta_{ll'} \delta_{mm'} \frac{1}{\beta^2} \times \begin{cases} \delta(\beta - \beta') & \text{for} \quad K = -1 , \\ \delta(\beta - \beta') & \text{for} \quad K = 0 , \\ \delta_{\beta\beta'} & \text{for} \quad K = +1 , \end{cases} \tag{10}$$

where $\beta^2 = k^2 + (R + 1)K$, where R is the rank of the type of perturbation under consideration ($R = 0$ for scalar, 1 for vector, and 2 for tensor). The quantity β is an integer for $K = +1$, since this universe is spatially closed. Note that β is essentially the wavenumber for wavelengths which are smaller than the radius of curvature of the universe. The angle brackets in equation (10) denote a statistical (ensemble) average. The normalization in equation (10) assures that we will reproduce the results of inflationary cosmology for a critical energy-density and scale invariant spectrum, and that we will preserve isotropy in the mean (Wilson 1983).

The final results of our analysis will be multipole coefficients a_l of the microwave background defined by

$$a_l \equiv \left(\left\langle \sum_{m=-l}^{l} |a_{lm}^2| \right\rangle\right)^{1/2} , \tag{11}$$

when the blackbody temperature is expressed in a multipole expansion

$$\frac{\delta T_o(e)}{T_o} = \sum_{l,m} a_{lm} Y_{lm}(e) , \tag{12}$$

with coefficients a_{lm}. Because the individual fluctuation modes which contribute to $\delta T_o/T_o$ are parameterized by Gaussian random variables $\hat{a}_{lm}(\beta)$ the resulting multipole coefficients a_{lm} will also be random with a Gaussian distribution. However, using equation (7), we can predict the expectation values $\langle a_l^2 \rangle$, and on the basis of these we can determine a probability distribution of these a_l values (Abbott and Wise 1984*c*), namely

$$\Pr(a_l) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{1\cdot 3 \cdot 5 \cdot \cdots \cdot (2l-1)} \frac{(a_l)^{2l}}{(\sigma_l)^{2l+1}} \exp\left(-\frac{a_l^2}{2\sigma_l^2}\right) , \tag{13}$$

where

$$\sigma_l^2 = \frac{\langle a_l^2 \rangle}{2l+1} . \tag{14}$$

In §§ V and VI we use this probability distribution to predict the microwave moment coefficients a_l^2 which are the final results of this paper. Our entire analysis is based on a linear treatment of the fluctuations of cold (dark) matter, which of course is not valid in the present universe for wavelengths of galactic cluster and supercluster size and shorter. However, linear analysis of longer wavelength perturbations is still valid today, and we will restrict ourselves to quantities which are dominated by such long-wavelength fluctuations. For this reason we only consider the first nine moments of the microwave background temperature.

II. SCALAR PERTURBATIONS

a) Sachs Wolfe Formula

Scalar perturbations are characterized by scalar harmonics $Q(x)$ satisfying

$$D^2 Q + k^2 Q = 0 , \tag{15}$$

where $D^2 = D_i D^i$ and D^i is the three-dimensional covariant derivative in the spaces of equation (2). We will write $Q(x)$ in the form

$$Q(x) = \Phi_\beta^l(r) Y_{lm}(\theta, \phi) \tag{16}$$

for each mode of oscillation, and so will label individual modes by β, l, and m. For each mode we define perturbations in the metric using Bardeen's (1980) notation:

$$h_{00} = -2A(\tau)Q(x) , \quad h_{0i} = -B(\tau)Q_i(x) , \quad h_{ij} = 2[H_L(\tau)Q(x)g_{ij} + H_T(\tau)Q_{ij}(x)] , \tag{17a}$$

and perturbations in the energy-momentum tensor

$$\delta T_0{}^0 = -\rho(\tau)\delta(\tau)Q(x) , \quad \delta T_i^0 = [\rho(\tau) + p(\tau)][v(\tau) - B(\tau)]Q_i(x) , \quad \delta T_j^i = \rho \frac{dp}{d\rho} \delta(\tau)Q(x)\delta_j{}^i , \tag{17b}$$

with

$$Q_i(x) = -\frac{1}{k} D_i Q(x) , \tag{18a}$$

and

$$Q_{ij} = \frac{1}{k^2} D_i D_j Q(x) + \frac{1}{3} g_{ij} Q(x) , \tag{18b}$$

where g_{ij} is the three-space metric in equation (2). Note that the perturbations in the energy-momentum tensor here assume a universe filled with perfect fluids undergoing adiabatic perturbations. In terms of the variables A, B, H_T, H_L, v, and δ, Bardeen (1980) defines the gauge invariant combinations:

$$\Phi_A = A + \frac{1}{k}\dot{B} + \frac{1}{k}\frac{\dot{S}}{S}B - \frac{1}{k^2}\left(\ddot{H}_T + \frac{\dot{S}}{S}\dot{H}_T\right) , \quad \Phi_H = H_L + \frac{1}{3}H_T + \frac{1}{k}\frac{\dot{S}}{S}B - \frac{1}{k^2}\frac{\dot{S}}{S}\dot{H}_T ,$$

$$\epsilon = \delta + 3\frac{(\rho + p)}{\rho}\frac{1}{k}\frac{\dot{S}}{S}(v - B) , \quad v^S = v - \frac{1}{k}\dot{H}_T . \tag{19}$$

Our goal here is to rewrite expressions (1) and (7) in terms of these gauge-invariant variables. We first note that the temperature fluctuation in the emitting plasma can be written for each mode as

$$\delta T_e = \tfrac{1}{4}\delta_\gamma(\tau)Q(\boldsymbol{x})\,, \tag{20}$$

where δ_γ is the value of δ for photons. Equation (20) follows from the fact that ρ_γ is proportional to T^4. Using this and the definitions of equation (18) in the equation (7), we find

$$\frac{\delta T_o}{T_o} = \frac{1}{4}\,\delta_\gamma Q\bigg|_{y=\tau_o-\tau_e} - \int_0^{\tau_o-\tau_e} dy\left\{\gamma^2\left[\left(\dot{H}_L + \frac{1}{3}\,\dot{H}_T\right) + \frac{1}{k^2}\,\dot{H}_T e^i e^j D_i \partial_j\right]Q - \gamma\,\frac{1}{k}\,\dot{B}e^i\partial_i Q\right\}, \tag{21}$$

where the perturbations are evaluated at values τ and $\boldsymbol{x}$ is given in terms of y by equation (8).

Using the identity for an arbitrary vector V_i along a geodesic

$$[(\boldsymbol{e}\cdot\partial)\gamma V_i]e^i = [\gamma(\boldsymbol{e}\cdot\boldsymbol{D})V_i]e^i\,, \tag{22}$$

and the fact that

$$\frac{d}{dy} = -\frac{d}{d\tau} + \gamma\boldsymbol{e}\cdot\partial\,, \tag{23}$$

from equation (8) we can rewrite equation (21) as

$$\frac{\delta T_o}{T_o} = -\frac{\dot{B}}{k}\,Q\bigg|_{y=0} + \left(\frac{1}{4}\,\delta_\gamma + \frac{\dot{B}}{k}\right)Q\bigg|_{y=\tau_o-\tau_e} - \frac{1}{k^2}\,(\ddot{H}_T + \gamma\dot{H}_T\boldsymbol{e}\cdot\partial)Q\bigg|_{y=0}^{y=\tau_o-\tau_e} - \int_0^{\tau_o-\tau_e} dy\left(\dot{H}_L + \frac{1}{3}\,\dot{H}_T - \frac{\ddot{B}}{k} + \frac{1}{k^2}\,\dddot{H}_T\right)Q\,. \tag{24}$$

The first term in equation (24) only contributes to the monopole moment of $\delta T_o/T_o$, so we will absorb it into our definition of T_o. The next step is to reexpress equation (24) in terms of the gauge-invariant variables of equation (19). Here a slight complication arises. In equation (24) the photon energy-density fluctuation, δ_γ, appears. Therefore we must introduce gauge-invariant variables ϵ_γ and $v_\gamma{}^s$ in analogy with equation (19) (Abbott and Wise 1984*b*),

$$\epsilon_\gamma = \delta_\gamma + 3\,\frac{(\rho_\gamma + p_\gamma)}{\rho_\gamma}\,\frac{\dot{S}}{S}\,(v_\gamma - B)\,, \qquad v_\gamma^s = v_\gamma - \frac{1}{k}\,\dot{H}_T\,. \tag{25}$$

Equation (24) is valid in the gauge $A = v_\gamma = 0$. From this we can rewrite $\delta T_o/T_o$ using gauge-invariant variables as

$$\frac{\delta T_o}{T_o} = \left[\frac{1}{4}\,\epsilon_\gamma + \Phi_A - \frac{1}{k}\left(\dot{v}_\gamma^s + \frac{\dot{S}}{S}\,v_\gamma^s\right)\right]Q\bigg|^{y=\tau_o-\tau_e} + \frac{1}{k}\,(\dot{v}_\gamma^s + \gamma v_\gamma^s\boldsymbol{e}\cdot\partial)Q\bigg|_{y=0}^{y=\tau_o-\tau_e} + \int_0^{\tau_o-\tau_e} dy(\dot{\Phi}_A - \dot{\Phi}_H)Q \tag{26}$$

for each mode β, l, and m. We now use the evolution equations for the gauge-invariant variables (Bardeen 1980; Abbott and Wise 1984*b*) which come from energy and momentum conservation and from the Einstein equations. For a perfect fluid in a matter-dominated universe these give

$$\frac{1}{4}\,\epsilon_\gamma + \Phi_A - \frac{1}{k}\left(\dot{v}_\gamma^s + \frac{\dot{S}}{S}\,v_\gamma^s\right) = 0\,, \qquad \Phi_A = -\Phi_H\,, \qquad \dot{\Phi}_H = \frac{3}{2}\left[\frac{(\dot{S}/S)^2 + K}{k^2 - 3K}\right]\left(\dot{\epsilon} - \frac{\dot{S}}{S}\,\epsilon\right), \qquad \frac{1}{k}\,\dot{v}_\gamma^s = -\left(\frac{1}{k^2 - 3K}\right)\dot{\epsilon}_\gamma\,. \tag{27}$$

This leads to an expression for $\delta T_o/T_o$ involving ϵ_γ as well as the total energy-density fluctuation variable ϵ. However, for the long-wavelength modes which dominate the multipole moments we will evaluate, ϵ_γ is proportional to ϵ. This can be verified using the multifluid extension of Bardeen's formalism (Abbott and Wise 1984*b*; Kodama and Sasaki 1984), and is a result of the fact that these long-wavelength fluctuations are more massive than the maximum Jeans mass. Therefore, we can write equation (24) purely in terms of ϵ as

$$\frac{\delta T_o}{T_o} = \left(\frac{-1}{k^2 - 3K}\right)\left\{\gamma\dot{\epsilon}\boldsymbol{e}\cdot\partial - \left(\frac{\dot{S}}{S}\right)\dot{\epsilon} + \frac{3}{2}\left[\left(\frac{\dot{S}}{S}\right)^2 + K\right]\epsilon\right\}Q\bigg|_{y=0}^{y=\tau_o-\tau_e} + 3\int_0^{\tau_o-\tau_e} dy\left[\left(\frac{\dot{S}}{S}\right)^2 + K\right]\left[\dot{\epsilon} - \left(\frac{\dot{S}}{S}\right)\epsilon\right]Q\bigg\}\,. \tag{28}$$

To obtain our final results we must sum and/or integrate equation (28) over the mode variables β, l, and m and project out the appropriate multipole moment. We do this in §§ V and VI.

b) Evolution of $\epsilon(\tau)$

The gauge-invariant variable ϵ obeys the evolution equation for cold matter (Bardeen 1980)

$$\ddot{\epsilon} + (\dot{S}/S)\dot{\epsilon} + 3(\dot{S}/S)^{\cdot}\epsilon = 0\,. \tag{29}$$

This can most easily be solved by writing ϵ as a function of $\dot{S}/S$ which in turn is a function of τ. Defining

$$W = \frac{\dot{S}}{S}\,, \tag{30}$$

equation (29) has the solutions

$$f_s(W) = W(W^2 + K)\,, \tag{31}$$

which is a shrinking mode and can be ignored, and the growing mode

$$f(W) = \begin{cases} 1 - \frac{3}{2}W^2 + \frac{3}{2}W(W^2 - 1)\,\mathrm{arctanh}\,(1/W)\,, & K = -1\,, \\ 1/W^2\,, & K = 0\,, \\ 1 + \frac{3}{2}W^2 + \frac{3}{2}W(W^2 + 1)\arctan(1/W)\,, & K = +1\,. \end{cases}$$

We will characterize the amplitude of ϵ by its value at the time of horizon crossing when $\dot{S}/S = k$ (i.e., when the physical wavelength equals the size of the horizon). Thus, we write

$$\epsilon(\tau) = 4\pi\epsilon_H \frac{f(W)}{f(k)}\,\hat{a}_{lm}(\beta)\,, \tag{33}$$

where $\hat{a}_{lm}(k)$ is the Gaussian random variable satisfying equation (10). The factor of 4π is introduced so that ϵ_H will agree with that commonly used in inflationary cosmology (Guth and Pi 1982; Abbott and Wise 1984*a*, *c*). In the case of inflation ϵ_H is a constant, but for nonscale invariant spectra it is a function of k or equivalently β. The expression (30) is to be substituted into equation (28) to obtain $\delta T_o/T_o$.

In order to facilitate comparison of our work with that of other researchers (especially those in the area of galaxy formation) we would like to describe how our notation relates to theirs. In the astrophysics literature the spectrum of density perturbations is often characterized by the function $|\delta_k{}^2|$ (see, e.g., Peebles 1980). In this notation a Harrison-Zel'dovich spectrum is characterized by $|\delta_k{}^2| \propto k$, whereas we use $\epsilon_H{}^2 \propto$ a constant. In flat space we have the relation between the two notations

$$|\delta_k{}^2| = \pi^3 \epsilon_H{}^2 k\tau^4/2\,,$$

and so we see that for $\epsilon_H =$ constant, $|\delta_k{}^2| \propto k$.

We should also point out that while for flat spaces a power-law spectrum $|\delta_k{}^2| \propto k^n$ is the same as a power law $\epsilon_H{}^2 \propto k^{n+1}$, the same is not exactly true for curved spaces, although they are not too different. The two parametrizations differ only when the wavelengths are comparable to the "radius" of curvature (see, e.g., Wilson 1983).

c) *The Harmonic Functions $Q(\boldsymbol{x})$*

The harmonic functions $Q(\boldsymbol{x})$ have been expressed in terms of spherical harmonics

$$Q(\boldsymbol{x}) = \Phi^l_\beta(r) Y_{lm}(\theta, \phi)\,, \tag{34}$$

where the index β and the wavenumber k are related by $\beta^2 = k^2 + K$. The radial functions $\Phi_\beta{}^l$ are given by the following expressions. For $K = 0$

$$\Phi^l_\beta(r) = j_l(kr)\,, \tag{35}$$

where $j_l(kr)$ are the spherical Bessel functions which satisfy the orthonormality relation (no sum on l)

$$\int_0^\infty r^2 dr j_l(kr) j_l(k'r) = \frac{\pi}{2}\frac{1}{k^2}\,\delta(k - k')\,. \tag{36}$$

The radial functions $\Phi_\beta{}^l$ for $K \neq 0$ have been given by Harrison (1967). We have changed his normalization slightly to make the orthogonality properties more like those of $j_l(kr)$. For $K = -1$

$$\Phi^l_\beta = \left(\frac{\pi N^l_\beta}{2\beta^2 \sinh y}\right)^{1/2} P^{-1/2-l}_{-1/2+i\beta}(\cosh y)\,, \tag{37}$$

where $P_\mu{}^\nu$ is an associated Legendre function,

$$\sinh y = \frac{r}{1/\frac{1}{4}r^2}\,, \qquad k^2 = \beta^2 + 1 \qquad (\beta \geq 0)\,, \tag{38}$$

and

$$N^l_\beta = \prod_{n=0}^{l} (n^2 + \beta^2)\,.$$

In analogy with equation (36) these functions satisfy the orthonormality condition (no sum on l)

$$\int_0^2 \frac{r^2 dr}{(1 + \frac{1}{4}Kr^2)^3}\,\Phi^l_\beta(r)\Phi^l_{\beta'}(r) = \frac{\pi}{2}\frac{1}{\beta^2}\,\delta(\beta - \beta')\,. \tag{39}$$

Further properties of $\Phi_\beta{}^l$ are given in the Appendix.

For $K = +1$,

$$\Phi^l_\beta = \left(\frac{\pi M^l_\beta}{2\beta^2 \sin y}\right)^{1/2} P^{-1/2-l}_{-1/2+\beta}(\cos y)\,, \tag{40}$$

with

$$\sin y = \frac{r}{1 + \frac{1}{4}r^2}\,, \qquad k^2 = \beta^2 - 1 \qquad (\beta = 3, 4, 5, \ldots)\,, \tag{41}$$

and

$$M_\beta^l = \prod_{n=0}^{l} (\beta^2 - n^2)\,.$$

These functions satisfy the orthogonality relation (no sum on l)

$$\int_0^\infty \frac{r^2 dr}{(1 + \frac{1}{4}r^2)^3}\, \Phi_\beta^l(r)\Phi_{\beta'}^l(r) = \frac{\pi}{2}\frac{1}{\beta^2}\,\delta_{\beta\beta'}\,. \tag{42}$$

Further properties are also given in the Appendix. Note that since β and k are related, we can summarize the orthogonality condition by

$$\int_{\text{all space}} \frac{r^2 dr}{(1 + \frac{1}{4}Kr^2)^3}\, \Phi_\beta^l(r)\Phi_{\beta'}^l(r) = \frac{\pi}{2}\frac{1}{\beta^2} \begin{cases} \delta(\beta - \beta') & K = 0, -1\,, \\ \delta_{\beta\beta'} & K = +1\,. \end{cases} \tag{43}$$

III. VECTOR FLUCTUATIONS

a) *Sachs-Wolfe Formula*

We now repeat the analysis of § II for vector fluctuations. These are characterized by a vector harmonic satisfying

$$D^2 Q_i^{(1)} + k^2 Q_i^{(1)} = 0\,, \tag{44a}$$

and

$$D^i Q_i^{(1)} = 0\,. \tag{44b}$$

Defining

$$Q_{ij}^{(1)} = -\frac{1}{k}\frac{1}{2}\,[D_i Q_j^{(1)} + D_j Q_i^{(1)}]\,, \tag{45}$$

we can write the fluctuation $h_{\mu\nu}$ in the form

$$h_{oo} = 0\,, \qquad h_{oi} = -B^{(1)}(\tau)Q_i^{(1)}(x)\,, \qquad h_{ij} = 2H_T^{(1)}(\tau)Q_{ij}^{(1)}(x)\,. \tag{46}$$

We next rewrite equation (7) in terms of these variables to obtain

$$\frac{\delta T_o}{T_o} = \int_0^{\tau_o - \tau_e} dy\left[\frac{1}{k}\,\gamma^2 \dot{H}_T^{(1)} D_i Q_j^{(1)} e^i e^j - \gamma \dot{B}^{(1)} Q_i^{(1)} e^i\right]. \tag{47}$$

Integrating this by parts and using the identity (22), this becomes

$$\frac{\delta T_o}{T_o} = \frac{1}{k}\,\dot{H}_T^{(1)} Q_i^{(1)} e^i \gamma \Big|_{y=0}^{y=\tau_o - \tau_e} + \int_0^{\tau_o - \tau_e} dy\left[\frac{1}{k}\,\ddot{H}_T^{(1)} - \dot{B}^{(1)}\right]\gamma Q_i^{(1)} e^i\,. \tag{48}$$

For vector harmonics, Bardeen (1980) defines the gauge-invariant variables

$$\Psi = B^{(1)} - \frac{1}{k}\,\dot{H}_T^{(1)} \tag{49a}$$

and

$$v_c = v^{(1)} - B^{(1)}\,, \tag{49b}$$

where $v^{(1)}$ is defined by the energy-momentum tensor perturbation

$$\delta T_i^0 = [\rho(\tau) + p(\tau)][v^{(1)}(\tau) - B^{(1)}(\tau)]Q_i^{(1)}(x)\,. \tag{50}$$

However, in the gauge we have chosen $v^{(1)} = 0$, so equation (48) can be rewritten in terms of the gauge-invariant variables as

$$\frac{\delta T_o}{T_o} = -\gamma(v_c + \Psi)Q_i^{(1)} e^i \Big|_{y=0}^{y=\tau_o - \tau_e} - \int_0^{\tau_o - \tau_e} dy[\dot{\Psi}\gamma Q_i^{(1)} e^i]\,. \tag{51}$$

b) *Evolution of Perturbations*

For a pressureless, perfect fluid the gauge-invariant variable Ψ and v_c satisfy the evolution equations (Bardeen 1980)

$$\frac{1}{2}\frac{k^2-2K}{S^2}\Psi = \rho v_c \,, \tag{52a}$$

and

$$\dot{v}_c = -\frac{\dot{S}}{S} v_c \,. \tag{52b}$$

The second equation can be solved immediately to give the shrinking mode

$$v_c \propto \frac{1}{S} \,. \tag{53}$$

There is no growing mode for the vector perturbations.

c) *Vector Harmonics*

Because the unit vector e^i appearing in equation (51) which points from the observer out to the source is in the radial direction, we only need the radial component of the vector harmonic $Q_i^{(1)}$. The radial component $Q_r^{(1)}$ is given by

$$Q_r^{(1)} = N_y^{(1)}\Phi_\beta^l(r)Y_{lm}(\theta, \phi) \,, \tag{54}$$

where

$$N_y = \frac{1}{\beta}[l(l+1)]^{1/2}\begin{cases} \dfrac{1}{\sinh y}\,, & K=-1\,, \\ \dfrac{1}{y}\,, & K=0\,, \\ \dfrac{1}{\sin y}\,, & K=+1\,. \end{cases} \tag{55}$$

The function $\Phi_\beta{}^l$ and the variable y are as defined in § IIc, with β given by

$$\beta^2 = k^2 + 2K \,. \tag{56}$$

IV. TENSOR PERTURBATIONS

a) *Sachs-Wolfe Formula*

Tensor perturbations are characterized by tensor harmonics $Q_{ij}^{(2)}(x)$ satisfying

$$D^2 Q_{ij}^{(2)} + k^2 Q_{ij}^{(2)} = 0\,, \qquad Q_{ij}^{(2)} = Q_{ji}^{(2)}\,, \qquad D^i Q_{ij}^{(2)} = Q_i^{(2)i} = 0\,. \tag{57}$$

For tensor perturbations the variable of relevance is $H_T^{(2)}$, defined by writing the perturbations as

$$h_{oo} = h_{oi} = 0\,, \qquad h_{ij} = 2H_T^{(2)}(\tau)Q_{ij}^{(2)}(x)\,, \qquad \delta T_\alpha{}^\beta = 0\,. \tag{58}$$

The quantity $H_T^{(2)}$ is already gauge-invariant, so we immediately obtain a gauge-invariant form for $\delta T_o/T_o$ from equations (1) and (7),

$$\frac{\delta T_o}{T_o} = -\int_0^{\tau_o-\tau_e} dy\gamma^2 \dot{H}_T^{(2)} Q_{ij}^{(2)} e^i e^j \,. \tag{59}$$

b) *Evolution of $H_T^{(2)}$*

The quantity $H_T^{(2)}$ satisfies the equation

$$\ddot{H}_T^{(2)} + 2\frac{\dot{S}}{S}\dot{H}_T^{(2)} + (k^2+2K)H_T^{(2)} = 0\,. \tag{60}$$

For a matter-dominated universe we have a shrinking mode solution

$$H_T^{(2)} \propto \Phi_{2\beta}^{-1}[C(\tau)]\,,$$

which can be ignored, and a growing mode solution

$$H_T^{(2)} \propto \frac{S^{-1/2}}{\beta}\Phi_{2\beta}^1[C(\tau)]\,, \tag{61}$$

where $\Phi^1_{2\beta}$ are the scalar radial harmonic functions given in § II*c* with index 2β. The relation between β and k is

$$k^2 = \beta^2 + 3K , \tag{62}$$

for tensor perturbations, and $C(\tau)$ is given by

$$C(\tau) = \begin{cases} \cosh \tau/2 & \text{for} \quad K = -1 , \\ \tau/2 & \text{for} \quad K = 0 , \\ \cos \tau/2 & \text{for} \quad K = +1 . \end{cases} \tag{63}$$

As in § II*b*, we characterize the size of the fluctuations by their amplitude at the horizon crossing time by dividing out the value of $H_T^{(2)}$ at τ such that $\dot{S}/S = k$ and introducing a constant H_H in analogy with ϵ_H so that

$$H_T^{(2)} = H_H \frac{S^{-1/2}}{\beta} \Phi^1_{2\beta}[C(\tau)] \frac{\hat{a}_{lm}(\beta)}{h(\beta)} , \tag{64}$$

where

$$h(\beta) = \Phi^1_{2\beta}[C(\tau)] \Big|_{\tau \text{ such that } \dot{S}/S = k} . \tag{65}$$

c) Tensor Harmonics

As in the case of vector harmonics, we will only need the radial component of the tensor harmonics since e^i in equation (59) points in the radial direction. This component can be written as

$$Q_{rr}^{(2)} = N_y^{(2)} \Phi_\beta^l(r) Y_{lm}(\theta, \phi) , \tag{66}$$

where

$$N_y^{(2)} = \left[\frac{(l+2)(l+1)l(l-1)}{2\beta^2(\beta^2 - K)}\right]^{1/2} \begin{cases} \dfrac{1}{\sinh^2 y} , & \text{for} \quad K = -1 , \\ \dfrac{1}{y^2} , & \text{for} \quad K = 0 , \\ \dfrac{1}{\sin^2 y} , & \text{for} \quad K = +1 , \end{cases} \tag{67}$$

and ${\Phi_\beta}^l$ is as given in § II*c*, but now β is defined by $\beta^2 = k^2 + 3K$.

V. DIPOLE AND QUADRUPOLE MOMENTS

We now apply the formalism developed in the previous sections to the dipole and quadrupole moments of the microwave background. Higher moments are considered in the next section.

Unlike the other moments of the microwave background temperature which we consider, the dipole moment is sensitive to short-wavelength fluctuations. This is most easily seen by noting that the dipole moment depends on local motions of the observer. Since short-wavelength perturbations have gone nonlinear, the analysis of the previous sections is not applicable to the entire dipole calculation. However, the contribution of long-wavelength fluctuations which are still evolving linearly to the dipole moment can be computed, and the result will place a limit on the magnitude of these fluctuations. The basic assumption which we must make is that the contribution to the dipole moment from the long-wavelength linear fluctuations and that from the short-wavelength nonlinear fluctuations are uncorrelated.[4] In this case the probability distribution for values of the dipole moment a_1 is given by a convolution

$$P(a_1) = \int dy G(|\boldsymbol{x} - \boldsymbol{y}|) H(\boldsymbol{y}) , \tag{68}$$

where $\boldsymbol{x}$ is the vector $(a_{10}, a_{11}, a_{1-1})$, with $|\boldsymbol{x}| = a_1$ and $\boldsymbol{y}$ another vector $({a_{10}}', {a_{11}}', {a_{1-1}}')$. In equation (68), G is the Gaussian probability distribution for long-wavelength fluctuations computed as described in §§ I–IV. The quantity H is an unknown distribution for the nonlinear modes. The result which enables us to derive a bound from the dipole distribution is that the distribution P is always broader than the distribution G, that is,

$$\int_{|\boldsymbol{x}| < a_1} d\boldsymbol{x} P(\boldsymbol{x}) < \int_{|\boldsymbol{x}| < a_1} d\boldsymbol{x} G(|\boldsymbol{x}|) , \tag{69}$$

for all values of a_1. This means that the full measured dipole moment determined from P should be larger than the value of the dipole moment determined from G by considering only linear fluctuations. This is the basis for our bound. To prove equation (69)

[4] The following analysis was worked out in collaboration with Mark Wise.

we note that

$$\int_{|x|<a_1} dx P(x) = \int_{|x|<a_1} dx \int dy G(|x-y|)H(y)$$
$$= \int dy H(y) \int_{|x|<a_1} dx G(|x-y|) < \int dy H(y) \int_{|x|<a_1} dx G(|x|) = \int_{|x|<a_1} dx G(|x|) . \tag{70}$$

The inequality in equation (70) follows from the fact that G is peaked at the origin and the last equality follows from the normalization of H.

From the above proof it is clear what our program should be. We determine the value of the dipole moment a_1 predicted using long-wavelength linear fluctuations and require that this be smaller than the measured dipole moment. This restricts the amplitude of the spectrum. For example, for a scale-invariant spectrum, the value of ϵ_H (as defined in eq. [33]) is, at the 90% confidence level, restricted to lie below the curve shown in Figure 1 for various values of the density ratio Ω. It is clear that the observed dipole moment places a severe restriction on ϵ_H, especially for the inflationary case $\Omega = 1$.

The results of Figure 1 clearly depend on how we split what we call linear fluctuations from those that we call nonlinear. We have chosen the cutoff wavelength to be $60h^{-1}$ Mpc (where $h = H_0/100$ km s^{-1} Mpc^{-1} and H_0 is the present value of the Hubble constant) since at this length scale fluctuations are still linear as the amount of excess mass within a volume of that radius is very small ($[\delta M/M]^2 < 0.01$). Thus, our bounds arise from requiring that the contribution to the dipole moment from wavelengths longer than $60h^{-1}$ Mpc be less than the observed value. In this analysis we have included the effect of the bend in the spectrum of cold dark matter fluctuations coming from wavelengths which cross the horizon before the time of matter domination. This is especially relevant for values of Ω less than 1. We note that the wavelength at which this bend occurs (i.e., the horizon size when the densities of matter and radiation are equal) is sensitive to the value of h. For smaller values of h the bend occurs at larger wavelengths, thus reducing the contribution to the dipole on scales at which it is most sensitive. In Figure 1 we have plotted the numbers for $h = 1$. Using $h = \frac{1}{2}$ would produce a dipole reduced by a factor which goes roughly (to $\sim$5% accuracy) as $\Omega^{0.3}$ for $\Omega > 0.05$. Thus the value of ϵ_H for $h = \frac{1}{2}$ will be allowed to be larger than that shown in Figure 1 by a factor of $\Omega^{-0.3}$.

A more physical way of representing the results of our dipole bound is to limit the value of the quadrupole moment coming from various spectra which are constrained to satisfy the dipole bound.[5] This gives the 90% confidence level upper limits for the quadrupole moment shown in Figure 2 for various Ω and the spectra:

$$\epsilon_H \propto k^{\alpha} , \tag{71}$$

with $\alpha = +1/2$, 0, or $-1/2$. (This corresponds to Peebles's 1980 spectra $|\delta k^2| \approx k^{2\alpha+1}$; see § II*b*). Also shown is the present experimental upper limit, and the expected sensitivity of the *COBE* satellite measurement (Mather 1982). The value of the quadrupole and, as we shall point out in the following section, the next several higher multipole moments ($l < 20$) are insensitive to the value of h, but since we have effectively normalized the quadrupole to the dipole, the values predicted for the quadrupole in Figure 2 are sensitive to the quantity h. Since the dipole is reduced for $h < 1$, the predicted value of our quadrupole will be increased for $h < 1$. To get the analog of Figure 2 for $h = \frac{1}{2}$, multiply the given curves in Figure 2 by the following factors: $\Omega^{-0.4}$ for $\epsilon_H \propto k^{-1/2}$, $\Omega^{-0.6}$ for $\epsilon_H \propto k^0$, and $\Omega^{-0.8}$ for $\epsilon_H \propto k^{+1/2}$. This will yield values accurate to $\sim$10% in the square of the quadrupole.

It should be stressed that the limits of Figure 2 apply to scalar perturbations. If the perturbations are predominantly tensor then, since tensors do not contribute to the dipole moment, there is no quadrupole bound besides the observational one. This is also true in weakly anisotropic cosmologies (Tolman and Matzner 1984; Fabbri and Melchiorri 1981), since tensor perturbations are analogous to weakly anisotropic cosmologies. In fact, if a large quadrupole moment is found, then the way inflation can be saved is to assume that inflation-generated tensor perturbations dominate.

VI. HIGHER MULTIPOLE MOMENTS

Unlike the dipole moment, the moments $l = 2$ through $\sim$20 are relatively insensitive to short-wavelength perturbations which have gone nonlinear. This yields two benefits: they can be accurately determined by a purely linear treatment, and they are

[5] This is in contrast to the standard quadrupole normalization which uses the galaxy-galaxy correlation function to fix the amplitude of ϵ_H.

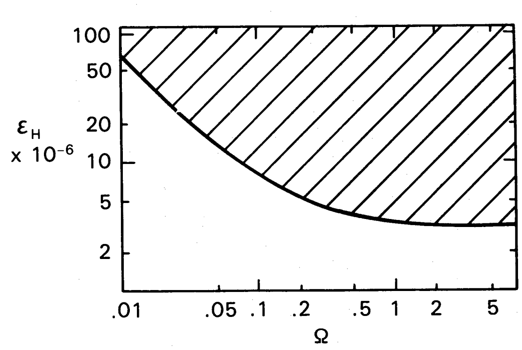

FIG. 1.—Dipole bound on ϵ_H. With 90% confidence, the allowed value of ϵ_H lies below the curve. This bound assumes a Harrison-Zel'dovich adiabatic spectrum.

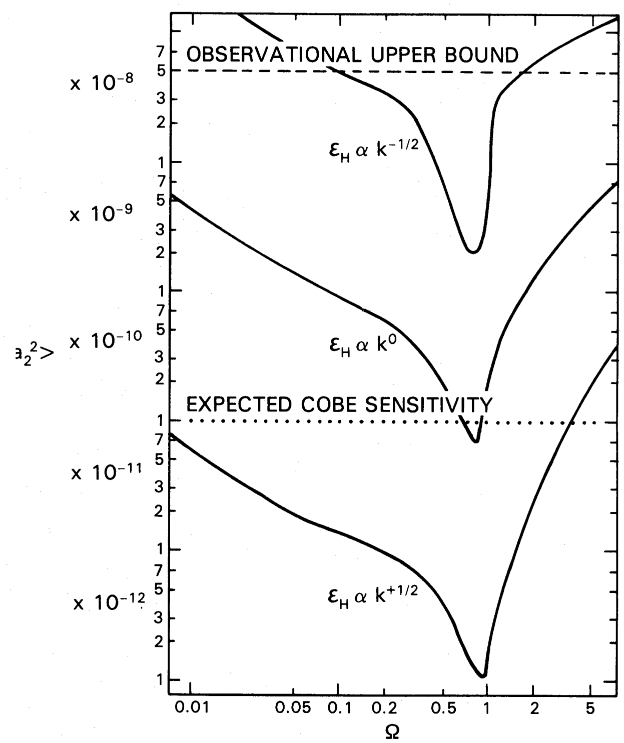

FIG. 2.—Quadrupole to dipole ratio. Curves represent 90% confidence upper bounds on the quadrupole moment for spectra which are consistent with the observed dipole moment. Three curves are for $\alpha = +\frac{1}{2}$, Harrison-Zel'dovich, and $\alpha = -\frac{1}{2}$ spectra. Quadrupole moment must lie *below* the lines drawn.

insensitive to the spectral effects of the radiation dominated era and hence to the value of the Hubble constant. One can see from equation (28) that these moments depend only on the scalar curvature parameter K, and the values of the conformal times at observation and emission of the radiation, τ_o and τ_e. These conformal times in turn depend only on the current value of Ω and the effective temperature of decoupling, taken here to be 4000 K. Thus, the results of §§ I–IV along with knowledge of the perturbation spectrum give an unambiguous prediction for the higher moments. The purpose of this section is to see if future measurements of these moments could distinguish between different spectra and different values of Ω with particular emphasis on testing the predictions of inflationary cosmology. We also compare the results for scalar perturbations with those for tensor fluctuations.

We begin by considering spectra of the form

$$\epsilon_H = \epsilon_0 k^\alpha , \tag{72}$$

for various values of α (see § II*b*). The scale-invariant spectrum of inflationary cosmology corresponds to $\alpha = 0$. In the case $\Omega = 1$ the integrals in equation (28) can be analytically determined in the limit $\tau_e \ll \tau_o$. This gives the following expression for the moments for $\Omega = 1, l \geq 2$,

$$\langle a_l^2 \rangle = C \frac{(2l+1)(l+\alpha-1)(l+\alpha-2)\cdots(1+\alpha)}{(l-\alpha+1)(l-\alpha)(l-\alpha-1)\cdots(1-\alpha)} , \tag{73}$$

where C is a constant which is independent of l. Equation (73) is accurate to $\sim$20% for l = 2–9. Results for $\alpha = 0, \alpha = \frac{1}{2}$, and $\alpha = -\frac{1}{2}$ with one standard deviation error bars are plotted in Figure 3. It is clear that, even with the statistical uncertainties in the predictions, these three cases could be distinguished by a measurement of these moments. In Figure 3 we have normalized the different spectra so that the quadrupole moments agree for all three spectra. We will continue to do this in the other figures, since a measurement of the quadrupole moment will serve to normalize the predictions and then measurement of higher moments will distinguish the l-dependences which characterize the different cases.

For $\Omega \neq 1$ we cannot derive an analytic expression for the l-dependence of the moments, but all the needed integrals can be done numerically. The results for $\alpha = 0, \frac{1}{2}$, and $-\frac{1}{2}$ in the case $\Omega = 0.2$ and $\Omega = 2.0$ are shown in Figures 4 and 5. For a scale-invariant spectrum, $\alpha = 0$, results for $\Omega = 0.2$, 1.0, and 2.0 are shown in Figure 6. It is easy to distinguish the closed case, $\Omega = 2.0$, from the critical universe, but $\Omega = 1.0$ and $\Omega = 0.2$ cannot be distinguished from observation of these moments.

In an inflationary cosmology both scalar and tensor perturbations are produced, and both have a scale-invariant spectrum. As seen in Figure 7, it is impossible to determine which is causing the microwave anisotropy from observing moments l = 2–9. However, as mentioned before, tensor perturbations do not contribute to the dipole moment, and so they provide a way for inflation to escape the dipole to quadrupole ratio test.

We wish to thank Mark Wise for many valuable contributions to this work.

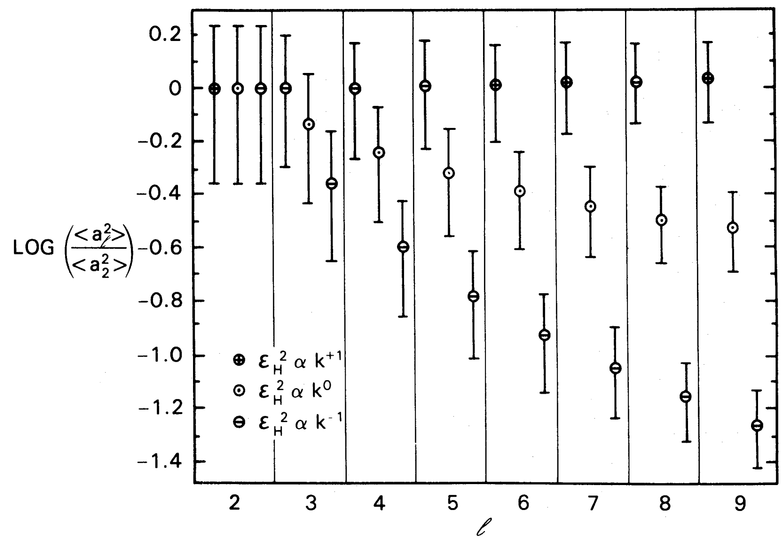

FIG. 3.—Comparison of the moment predictions for the three different spectra $k^{+1/2}$, k^0, and $k^{-1/2}$ in a critical density universe. As can be seen, the $k^{+1/2}$ puts more power into the higher moments, and the $k^{-1/2}$ puts less power in the higher moments than inflation (k^0).

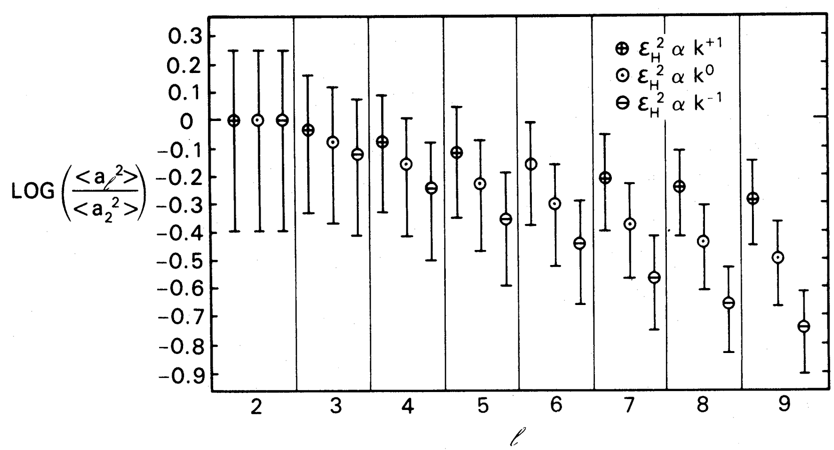

FIG. 4.—Comparison of the moment predictions from the different spectra $k^{+1/2}$, k^0, and $k^{-1/2}$ in a low ($\Omega = 0.2$) density universe

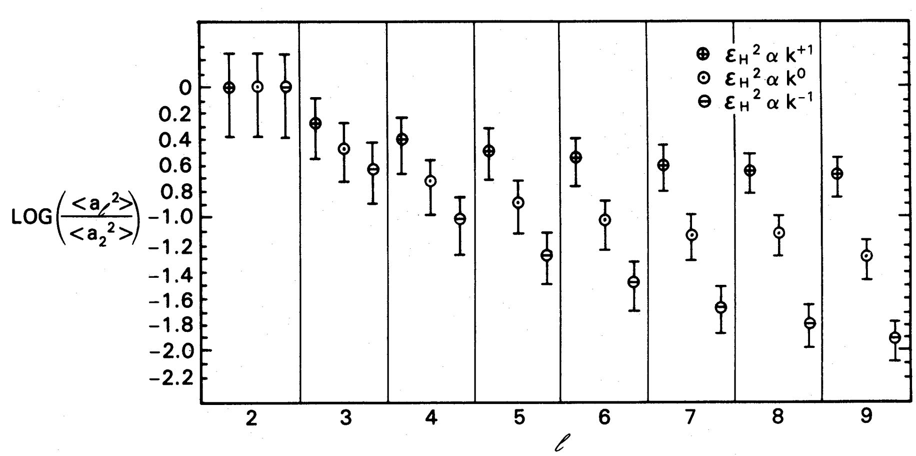

FIG. 5.—Comparison of the moment predictions of the different spectra $k^{+1/2}$, k^0, and $k^{-1/2}$ for a high-density ($\Omega = 2$) universe

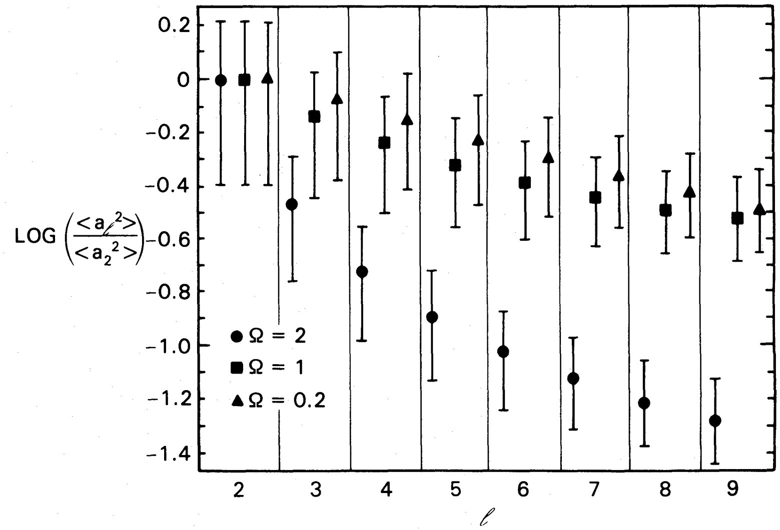

FIG. 6.—Comparison of moment predictions from a Harrison-Zel'dovich spectrum for low-, critical, and high-density universes (Ω = 0.2, 1, 2, respectively)

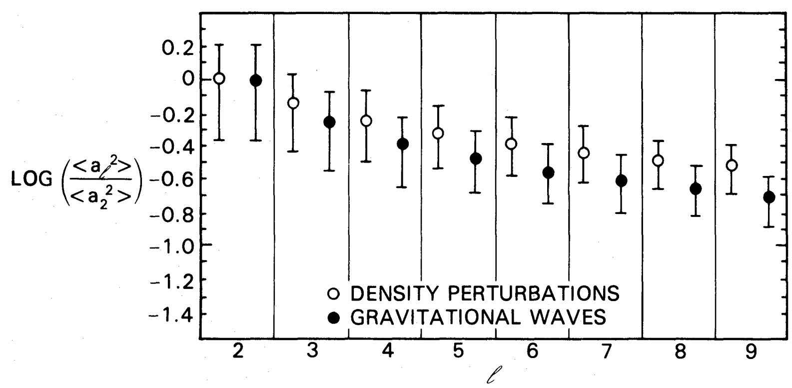

FIG. 7.—Comparison of moment predictions from gravitational waves and density perturbations in an inflationary universe. The two distributions of moments are displayed with identical values of the quadrupole because the overall normalization for either set of moments is unknown. In an inflationary universe the moments could be distributed as an arbitrary linear combination of the two sets.

APPENDIX

We present the derivation and explicit forms of the spatial harmonic functions which are solutions of a generalized Helmholtz equation. We will separate the discussion into three parts, one for each type of harmonic: scalar, vector, and tensor. We will present the scalar harmonics $Q(\boldsymbol{x})$ derivation in some detail, because they form the basis for the vector and tensor harmonics as well. The treatment of the scalar functions follows that of Harrison (1967).

AI. SCALAR HARMONIC FUNCTIONS

To find the eigenfunctions $Q(\boldsymbol{x})$ of the covariant Laplacian in a Robertson-Walker (with no cosmological constant) space, we need to solve the Helmholtz equation

$$(D^2 + k^2)Q(\boldsymbol{x}) = 0 \,, \tag{A1}$$

where $D^2 = D^i D_i$ and D_i is a spatial covariant derivative of the space defined by the spatial part of our metric

$$ds^2 = S^2(\tau)\left(-d\tau^2 + \frac{1}{\gamma^2}\, d\boldsymbol{x} \cdot d\boldsymbol{x}\right), \tag{A2}$$

with $\gamma = 1 + Kr^2/4$. With this metric we can easily see that only the radial dependence of the $Q(\boldsymbol{x})$ will change with the value of the curvature constant K. In flat space ($K = 0$), equation (A1) becomes

$$(\nabla^2 + k^2)Q(\boldsymbol{x}) = 0 \,, \tag{A3}$$

which is easily solved by $Q(\boldsymbol{x}) = \exp(i\boldsymbol{k} \cdot \boldsymbol{x})$. We can express this form of Q in terms of spherical coordinates by remembering the relation

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi \sum_{l,m} i^l Y^*_{lm}(\theta_k, \phi_k) j_l(kr) Y_{lm}(\theta, \phi) \,, \tag{A4}$$

with the Y_{lm} defined by

$$Y_{lm}(\theta, \phi) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} . \tag{A5}$$

Thus the purely spatial dependence of each mode of oscillation in spherical coordinates is represented in the form

$$Q(x) = j_l(kr)Y_{lm}(\theta, \phi) . \tag{A6}$$

Since in our coordinates only the radial dependence of the metric changes with nonzero curvature constant K, the general form for our solutions will be

$$Q(x) = \Phi_\beta{}^l(r)Y_{lm}(\theta, \phi) , \tag{A7}$$

where $\Phi_\beta{}^l = j_l(kr)$ for $K = 0$ and $\beta^2 = k^2 + K$. Using our metric (eq. [A2]), the radial harmonic equation is given by

$$\frac{\gamma^3}{r^2}\frac{d}{dr}\frac{r^2}{\gamma}\frac{d}{dr}\Phi_\beta{}^l + \left[k^2 - \gamma^2\frac{l(l+1)}{r^2}\right]\Phi_\beta{}^l = 0 . \tag{A8}$$

For $K = 0$ we see that this reduces to the spherical Bessel equation for the argument kr, i.e.,

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi_\beta^l + \left[k^2 - \frac{l(l+1)}{r^2}\right]\Phi_\beta^l = 0 . \tag{A9}$$

We can treat the cases $K = \pm 1$ simultaneously by changing from the variable r to the variable $\xi = yK^{1/2}$, where y is the affine variable parameterizing the null geodesics as described in § I. The change of variables is defined by

$$K^{1/2}\frac{r}{\gamma} = \sin\xi . \tag{A10}$$

If we take the limit $K \to 0$ in the above relation, we recover the flat space limit $y = r$. Using our variable ξ, equation (A8) is replaced by

$$\frac{1}{\sin^2\xi}\frac{d}{d\xi}\left(\sin^2\xi\frac{d}{d\xi}\Phi_\beta^l\right) + \left[Kk^2 - \frac{l(l+1)}{\sin^2\xi}\right]\Phi_\beta^l = 0 . \tag{A11}$$

Now let $\Phi_\beta^l = \Pi(\xi)\sin^{-1/2}\xi$, and equation (A11) becomes

$$\frac{1}{\sin\xi}\frac{d}{d\xi}\left(\sin\xi\frac{d}{d\xi}\Pi\right) + \left[\lambda(\lambda+1) - \frac{(l+\frac{1}{2})^2}{\sin^2\xi}\right]\Pi = 0 , \tag{A12}$$

where $\lambda(\lambda + 1) = Kk^2 + \frac{3}{4}$, and now equation (A12) has the solutions which are associated Legendre functions $P_\nu{}^\mu(\cos\xi)$ and $Q_\nu{}^\mu(\cos\xi)$, with $\mu = \pm(l + \frac{1}{2})$ and $\nu = \lambda$. The variable has two solutions:

$$\lambda_{1,2} = -\tfrac{1}{2} \pm (1 + Kk^2)^{1/2} . \tag{A13}$$

We now impose the boundary conditions on our solutions, as did Harrison (1967). Since $P_\nu{}^\mu = P_{-1-\nu}{}^\mu$ or $P_{\lambda_1}{}^\mu = P_{\lambda_2}{}^\mu$, we will use $\nu = \lambda_1$. Also since μ is a half-integer, we can use $P_\nu{}^\mu$ and $P_\nu{}^{-\mu}$ as independent solutions instead of $P_\nu{}^\mu$ and $Q_\nu{}^\mu$. We will demand that $\Phi_\beta{}^l$ be regular at the origin. This boundary condition eliminates $P_\nu{}^{l+1/2}$ as a solution and leaves us with $P_\nu{}^{-l-1/2}$ as the only solution we need to consider.

When discussing these eigenfunctions and their eigenvalues it is better to label them with the index β instead of the comoving wavenumber k. For $K = +1$, $\beta^2 = k^2 + 1$, and the radial function is

$$\Phi_\beta^l = \frac{\text{norm. const.}}{(\sin y)^{1/2}} P_{-1/2+\beta}^{-1/2-l}(\cos y) . \tag{A14}$$

In order that $\Phi_\beta{}^l$ be single valued, β must be an integer, i.e. $\Phi_\beta{}^l$ must satisfy the periodic boundary condition

$$\Phi_\beta{}^l(-\cos y) = \cos[(\beta - 1 - l)\pi]\Phi_\beta{}^l(\cos y) . \tag{A15}$$

It can be shown that the values $\beta = 1$ and $\beta = 2$ correspond to modes which are pure gauge terms (Lifshitz and Khalatnikov 1963; Bardeen 1980). Thus our spectrum of eigenvalues for $K = +1$ are

$$k^2 = \beta^2 - 1 , \quad \text{with} \quad \beta = 3, 4, 5, \ldots \quad \text{and} \quad \beta > l . \tag{A16}$$

For $K = -1$, $\beta^2 = k^2 - 1$, so the eigenfunctions are

$$\Phi_\beta{}^l = \frac{\text{norm. const.}}{(\sinh y)^{1/2}} P_{-1/2+i\beta}^{-1/2-l}(\cosh y) . \tag{A17}$$

There are no periodic boundary conditions to satisfy because space is open, so β can take on any positive real value. Thus the spectrum for $K = -1$ is

$$k^2 = \beta^2 + 1 ; \qquad \beta \geq 0 . \tag{A18}$$

We have yet to discuss our normalization condition so we can fix our normalization constants. To make the curved space radial function normalizations similar to $j_l(kr)$ we use the condition

$$\int_{\text{all space}} \frac{d^3x}{\gamma^3} [Q^*(x)]_{\beta lm}[Q(x)]_{\beta' l' m'} = \frac{\pi}{2} \delta(\beta, \beta')\delta_{ll'} \delta_{mm'} , \tag{A19}$$

where $\beta^2 = k^2 + K$ and $\delta(\beta, \beta')$ is the delta "function" with respect to the measure $\mu(\beta)$:

$$\int d\mu(\beta) f(\beta)\delta(\beta, \beta') = f(\beta') . \tag{A20}$$

The measure $\mu(\beta)$ is defined according to the curvature as in Table 1.

We will use the definition $\beta^2 = k^2 + K$ for the rest of this section on scalar harmonics. We have specified our normalization so our functions take the properly normalized form

$$\Phi_\beta^l = \begin{cases} \left(\dfrac{\pi N_\beta^l}{2\beta^2 \sinh y}\right)^{1/2} P^{-1/2-l}_{-1/2+i\beta}(\cosh y) , & K = -1 , \\ j_l(\beta y) = j_l(kr) , & K = 0 , \\ \left(\dfrac{\pi M_\beta^l}{2\beta^2 \sin y}\right)^{1/2} P^{-1/2-l}_{-1/2+\beta}(\cos y) , & K = +1 . \end{cases} \tag{A21}$$

where

$$N_\beta^l = \prod_{n=0}^{l} (\beta^2 + n^2) . \qquad M_\beta^l = \prod_{n=0}^{l} (\beta^2 - n^2) , \tag{A22}$$

after Harrison (1967) and y is defined as in § I.

While equation (A21) provides a compact form of the $\Phi_\beta{}^l$ in terms of well-known functions, it is of little value for numerical computation. We will present some useful properties of the $\Phi_\beta{}^l$, which are derivable from the tabulated properties of the Bessel and Legendre functions (Arfken 1966; Magnus, Oberhettinger, and Soni 1966).

First of all we can rewrite the derivative of the radial functions which appears in the scalar formula for $\delta T_o/T_o$

$$\gamma \frac{d}{dr} \Phi_\beta^l = \frac{d}{dy} \Phi_\beta^l = \begin{cases} l(\coth y)\Phi_\beta^l - [\beta^2 + (l+1)^2]^{1/2}\Phi_\beta^{l+1} , & K = -1 , \\ \dfrac{l}{y} \Phi_\beta^l - \beta\Phi_\beta^{l+1} , & K = 0 , \\ l(\cot y)\Phi_\beta^l - [\beta^2 - (l+1)^2]^{1/2}\Phi_\beta^{l+1} , & K = +1 . \end{cases} \tag{A23}$$

This leaves us with the task of calculating the $\Phi_\beta{}^l$ values.

We can do this recursively using the relations

$$\Phi_\beta^l = \begin{cases} (\beta^2 + l^2)^{-1/2}\{(2l-1)(\coth y)\Phi_\beta^{l-1} - [\beta^2 + (l-1)^2]^{1/2}\Phi_\beta^{l-2}\} , & K = -1 , \\ \dfrac{1}{\beta}\left[(2l-1)\dfrac{1}{y}\Phi_\beta^{l-1} - \beta\Phi_\beta^{l-2}\right] , & K = 0 , \\ (\beta^2 - l^2)^{-1/2}\{(2l-1)(\cot y)\Phi_\beta^{l-1} - [\beta^2 - (l-1)^2]^{1/2}\Phi_\beta^{l-2}\} , & K = +1 , \end{cases} \tag{A24}$$

TABLE 1

WAVENUMBER SPACE MEASURE

K	$\int d\mu(\beta)$	$\delta(\beta, \beta')$	Function Name
-1..........	$\int_0^\infty \beta^2 d\beta$	$\frac{1}{\beta^2}\delta(\beta - \beta')$	"Radial" Dirac delta
0..........	$\int_0^\infty \beta^2 d\beta$	$\frac{1}{\beta^2}\delta(\beta - \beta')$	"Radial" Dirac delta
$+1$..........	$\sum_{\beta=3}^\infty \beta^2$	$\frac{1}{\beta^2}\delta_{\beta\beta'}$	Kronecker delta

and exact formulae for two of the $\Phi_\beta{}^l$ values, e.g. the $l = 0$ and $l = 1$ modes:

$$\Phi_\beta^0 = \frac{1}{\beta}\sin(\beta y) \times \begin{cases} \dfrac{1}{\sinh y}, & K = -1, \\ \dfrac{1}{y}, & K = 0, \\ \dfrac{1}{\sin y}, & K = +1, \end{cases} \tag{A25}$$

and

$$\Phi_\beta^1 = \Phi_\beta^0 \times \begin{cases} (\beta^2+1)^{-1/2}[\coth y - \beta \cot(\beta y)], & K = -1, \\ \dfrac{1}{\beta}\left[\dfrac{1}{y} - \beta\cot(\beta y)\right], & K = 0, \\ (\beta^2-1)^{-1/2}[\cot y - \beta\cot(\beta y)], & K = +1. \end{cases} \tag{A26}$$

A closed form for any other $\Phi_\beta{}^l$ can be found from the generating functions

$$\Phi_\beta^l = \begin{cases} (-1)^{l+1}\dfrac{\sinh^l y}{\beta(N_\beta^l)^{1/2}}\left(\dfrac{d}{d\cosh y}\right)^{l+1}\cos(\beta y), & K = -1, \\ (-1)^{l+1}\dfrac{y^l}{\beta^{l+2}}\left(\dfrac{1}{y}\dfrac{d}{dy}\right)^{l+1}\cos(\beta y), & K = 0, \\ \dfrac{\sin^l y}{\beta(M_\beta^l)^{1/2}}\left(\dfrac{d}{d\cos y}\right)^{l+1}\cos(\beta y), & K = +1. \end{cases} \tag{A27}$$

For small values of βy it is better to use the Taylor series for $\Phi_\beta{}^l$ since the recursion relations would then involve subtractions of numbers very close in magnitude, resulting in the loss of significant figures. The Taylor series are

$$\Phi_\beta^l = \begin{cases} (N_\beta^l)^{1/2}\dfrac{\sinh^l y}{\beta(2l+1)!!}\left[1 + \displaystyle\sum_{n=1}^{\infty}\dfrac{C_n(-1)}{D_n}\dfrac{(-1)^n}{n!}\sinh^{2n}\left(\dfrac{y}{2}\right)\right], & K = -1, \\ \beta^l\dfrac{y^l}{(2l+1)!!}\left[1 + \displaystyle\sum_{n=1}^{\infty}\dfrac{C_n(0)}{D_n}\dfrac{(-1)^n}{n!}\left(\dfrac{y}{2}\right)^{2n}\right], & K = 0, \\ (M_\beta^l)^{1/2}\dfrac{\sin^l y}{\beta(2l+1)!!}\left[1 + \displaystyle\sum_{n=1}^{\infty}\dfrac{C_n(+1)}{D_n}\dfrac{(-1)^n}{n!}\sin^{2n}\left(\dfrac{y}{2}\right)\right], & K = +1. \end{cases} \tag{A28}$$

where

$$C_n(K) = \prod_{j=1}^{n}[\beta^2 - K(l+j)^2], \quad \text{and} \quad D_n = \prod_{j=0}^{n}\left(l + j + \frac{1}{2}\right).$$

In this form it is now clear that at small distances and short wavelengths (small y and large β), we recover the flat space limit:

$$\lim_{\beta\to\infty,\, y\to 0} \Phi_\beta^l = j_l(\beta y) = j_l(kr). \tag{A29}$$

as we must since space is locally flat.

AII. VECTOR HARMONIC FUNCTIONS

We now want to find the solutions of the vector Helmholtz equation

$$(D^2 + k^2)Q_i^{(1)}(x) = 0, \tag{A30}$$

where $Q_i^{(1)}$ is divergenceless:

$$D^i Q_i^{(1)} = 0. \tag{A31}$$

We present the derivation of only the radial component $Q_1^{(1)}$ since this is all we need in our formula for $\delta T_o/T_o$.

When we use equation (A31) in equation (A30), we find we can get an equation which involves solely the radial component $Q_1^{(1)}$. We can expand this component in terms of spherical harmonics for each mode of oscillation

$$Q_i^{(1)} = \psi_\beta^l(\xi) Y_{lm}(\theta, \phi), \tag{A32}$$

using the coordinate ξ as defined in equation (A10). The equation for the radial function part of $Q_1^{(1)}$

$$\frac{d}{d\xi}\frac{d}{d\xi}\psi_\beta^l + 4\cot\xi\frac{d}{d\xi}\psi_\beta^l + \left[\frac{k^2}{K} - 2 - \frac{l(l+1)-2}{\sin^2\xi}\right]\psi_\beta^l = 0 . \tag{A33}$$

Equation (A33) has the solution

$$\psi_\beta^l = N^{(1)}\Phi_\beta^l \begin{cases} 1/\sin\xi , & K = \pm 1 , \\ 1/y , & K = 0 , \end{cases} \tag{A34}$$

where the $\Phi_\beta{}^l$ are the solutions of the scalar equation defined in equation (A24) and β is defined for vector harmonics as

$$\beta^2 = k^2 + 2K \tag{A35}$$

and $N^{(1)}$ is a normalization constant. We get only one solution to this second-order differential equation because we enforce the same boundary conditions as in the scalar case. Using equations (A32) and (A34) in equations (A30) and (A31), we can find the other components of $Q_i^{(1)}$. These other components are needed to determine the normalization $N^{(1)}$. The nonradial components have been given previously (Tomita 1982) and in this notation (Schaefer 1985), and there is no real need to repeat their representation here. Instead we will just give the result which is obtained by replacing $Q_{\beta lm}^* Q_{\beta' l' m'}$ in the normalization condition (A19) with $Q_i^{*(1)}{}_{\beta lm} Q^{i\,(1)}{}_{\beta' l' m'}$,

$$N^{(1)} = \frac{1}{\beta}[l(l+1)]^{1/2} .$$

So $Q_1^{(1)}$ is given by

$$Q_1^{(1)}(x) = \frac{1}{\beta}[l(l+1)]^{1/2}\Phi_\beta^l Y_{lm} \begin{cases} \dfrac{1}{\sinh y} , & K = -1 , \\ \dfrac{1}{y} , & K = 0 , \\ \dfrac{1}{\sin y} , & K = +1 . \end{cases} \tag{A36}$$

AIII. TENSOR HARMONIC FUNCTIONS

We now want to find the solutions of the generalized Helmholtz equation

$$(D^2 + k^2)Q_{ij}^{(2)} = 0 , \tag{A37}$$

subject to the constraints

$$D^i Q_{ij}^{(2)} = 0 , \qquad Q_{ij}^{(2)} = Q_{ji}^{(2)} , \qquad Q_i^{(2)i} = 0 . \tag{A38}$$

As in the vector case, we present only the derivation of the useful component $Q_{11}{}^{(2)}$.

Using conditions (A38) in equation (A37), we find that the equation for the radial-radial component decouples from the other components and can be expressed for each oscillatory mode

$$Q_{11}^{(2)} = \chi_\beta^l(\xi) Y_{lm}(\theta, \phi) . \tag{A39}$$

The resulting equation for the radial function $\chi_\beta{}^l$ (using the coordinates of eq. [A10])

$$\frac{d}{d\xi}\frac{d}{d\xi}\chi_\beta^l + 6\cot\xi\frac{d}{d\xi}\chi_\beta^l + \left(\frac{k^2}{K} - 6 - \frac{l^2+l-6}{\sin^2\xi}\right)\chi_\beta^l = 0 . \tag{A40}$$

These are solved by

$$\chi_\beta^l = N^{(2)}\Phi_\beta^l \begin{cases} \dfrac{1}{\sin^2\xi} , & K = \pm 1 , \\ \dfrac{1}{y^2} , & K = 0 , \end{cases} \tag{A41}$$

where the Φ_β^l are the radial part of the scalar harmonic functions defined in eq. (A24), $N^{(2)}$ is a normalization factor, and β is now defined for tensor perturbations by

$$\beta^2 = k^2 + 3K . \tag{A42}$$

As in the other cases imposing the boundary conditions leaves only one solution as given in equation (A41). Once this radial-radial component is known, we can use equations (A37) and (A38) to find relationships between the radial-radial component and the other components. These other components are much more tedious to derive than the vector harmonic components and have been given previously (Tomita 1982) and in this notation (Schaefer 1985). They are needed here only to determine the overall normalization factor $N^{(2)}$, which can be found by replacing $Q^*_{\beta l m} Q_{\beta' l' m'}$ with $Q^{*(2)}_{ij}{}_{\beta l m} Q^{ij(2)}{}_{\beta' l' m'}$ in the normalization condition (A19). We just give the result,

$$N^{(2)} = \left[\frac{(l+2)(l+1)l(l-1)}{2\beta^2(\beta^2 - K)}\right]^{1/2} . \tag{A43}$$

So $Q^{(2)}_{11}(\boldsymbol{x})$ is given by

$$Q^{(2)}_{11} = \left[\frac{(l+2)(l+1)l(l-1)}{2\beta^2(\beta^2 - K)}\right]^{1/2} \Phi^l_\beta Y_{lm} \begin{cases} \dfrac{1}{\sinh^2 y} , & K = -1 , \\ \dfrac{1}{y^2} , & K = 0 , \\ \dfrac{1}{\sin^2 y} , & K = +1 . \end{cases} \tag{A44}$$

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