Class. Quantum Grav. 4 (1987) L201-L204. Printed in the UK

LETTER TO THE EDITOR

Vacuum decay in curved backgrounds

L F Abbott†‡, Diego Harari§ and Q-Han Park||

† Physics Department, Boston University, Boston, MA 02215, USA
§ Physics Department, University of Florida, Gainesville, FL 32611, USA
|| Physics Department, Brandeis University, Waltham, MA 02254, USA

Received 24 June 1987

Abstract. We evaluate the effect of gravity on the probability of vacuum decay through bubble nucleation for a scalar field in an arbitrary curved background up to first order in an expansion in powers of $R\bar{\rho}^2$, where R is the scalar curvature of the background metric at the location of the bubble and $\bar{\rho}$ is the size of the bubble as it would appear in a flat space.

The mechanism by which a scalar field in a false vacuum can decay into its true ground state is well known [1-3] and many cosmological implications (e.g. inflation, domain walls, cosmic strings, etc) have been considered. Qualitatively, the decay proceeds through the materialisation of bubbles of true vacuum within the false vacuum phase, which is a quantum tunnelling effect. Once they are formed the bubbles expand, asymptotically approaching the speed of light.

The probability per unit time and per unit volume for the materialisation of a bubble, Γ/V, is given in the semiclassical limit by an expression of the form

$$\Gamma/V = A\,\mathrm{e}^{-B}[1+\mathrm{O}(\hbar)]. \tag{1}$$

The theory for the coefficient B (ignoring the effects of gravity) was developed in [1] and the theory for A in [2]. The effects of gravity on bubble nucleation were first considered by Coleman and de Luccia [4]. In their work the scalar field is the only source for gravity, and the potential energy of either the false or true vacuum is taken to be zero. Hence their analysis corresponds to the appearance of bubbles with an interior Minkowski metric in a de Sitter space, or to the materialisation of anti-de Sitter bubbles in a flat space. Both the Euclidean metric and the scalar-field configurations are assumed O(4) invariant.

Here, we consider bubble nucleation in an arbitrary background spacetime produced by an external source and unaffected by the scalar field itself. If the background geometry is not maximally symmetric the evaluation of the vacuum decay probability introduces new problems, since bubbles centred at different points are no longer equivalent. Thus in order to get solutions of the Euclidean equations of motion representing bubbles centred around a given point one must introduce constraints in the action.

It is our purpose in this letter to evaluate the leading-order correction to the vacuum decay probability with respect to its flat space value when a scalar field evolves in a fixed but otherwise arbitrary curved background. We show that it is possible to set

‡ On leave from Brandeis University.

up a perturbative calculation and that to leading order there is no need to consider a constrained solution. The perturbative expansion makes sense in those regions of spacetime where the absolute value of the components of the curvature tensor are much smaller than the inverse of the size of the bubble (as it would appear in a flat space) squared, $|R_{\mu\nu\alpha\beta}| \ll \bar{\rho}^{-2}$.

The system under consideration is a scalar field ϕ described by the action

$$S[\phi; g_{\mu\nu}] = \int d^4x \sqrt{-g}[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)]. \tag{2}$$

The potential $V(\phi)$ is assumed to have a global minimum $\phi = \phi_-$ (a true vacuum in the quantum theory) and a local minimum $\phi = \phi_+$ (a false vacuum) with a small energy splitting $\varepsilon = V(\phi_+) - V(\phi_-)$. The two vacua are assumed to be separated by a barrier large enough to allow the use of the thin-wall approximation. The background metric $g_{\mu\nu}$ is arbitrary, but fixed (the back reaction of ϕ on the metric is assumed to be negligible). The exponential suppression factor B of the vacuum decay rate equation (1) is given by

$$B = S_E[\bar{\phi}] - S_E[\phi_+]. \tag{3}$$

The subscript E denotes quantities evaluated with the metric analytically continued so as to have an Euclidean signature. $\bar{\phi}$ is the bounce, the solution of the Euclidean equation of motion with minimum action among all non-trivial solutions that tend to ϕ_+ (the false vacuum) at large Euclidean distances.

In order to carry out the calculation we use Riemann normal coordinates [5-7]. They are special among all local inertial frames in that geodesics crossing the origin are not only parametrised as straight lines, but with a special parameter: the proper distance along the geodesic. Given two points x_0 and x_1 close enough that they can be joined by only one geodesic, the normal coordinates of x_1 with origin at x_0, that we shall denote y^μ, are the components of the vector tangent to that geodesic at the origin whose norm equals the distance between x_0 and x_1 as measured along the geodesic. One of the most useful properties of Riemann normal coordinates is that the Taylor expansion of a tensorial quantity has, in such frame, explicitly covariant coefficients. In particular, the Taylor expansion in normal coordinates of the metric tensor $g_{\mu\nu}$ around the origin x_0 is

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \tfrac{1}{3}R_{\alpha\mu\beta\nu}(x_0)y^\alpha y^\beta + O(R_{\mu\alpha\nu\beta,\gamma}(x_0)y^\alpha y^\beta y^\gamma) \tag{4}$$

where $\eta_{\mu\nu}$ is the flat space metric tensor and $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of $g_{\mu\nu}$.

The Euclidean action of the bounce can be expanded around the flat Euclidean metric $\delta_{\mu\nu}$ as

$$S_E[\bar{\phi}; g_{\mu\nu}] = S_E[\bar{\phi}_0; \delta_{\mu\nu}] + \left.\frac{\delta S_E[\phi, \delta_{\mu\nu}]}{\delta\phi}\right|_{\phi=\bar{\phi}_0} (\bar{\phi} - \bar{\phi}_0)$$
$$+ \tfrac{1}{2}\int d^4y\, T^{\mu\nu}[\bar{\phi}_0]\delta g_{\mu\nu} + O((\bar{\phi} - \bar{\phi}_0)^2; (\delta g_{\mu\nu})^2). \tag{5}$$

Here $\bar{\phi}$ is the exact bounce solution including the effects of gravity while $\bar{\phi}_0(\rho)$ is the bounce solution, when the background geometry is flat, written as a function of $\rho = \delta_{\mu\nu}y^\mu y^\nu$, the Euclidean geodesic distance (i.e. $\bar{\phi}_0$ is the minimum action, non-trivial solution of the equation $\ddot{\phi}_0 + (3/\rho)\dot{\phi}_0 = dV/d\phi_0$ with boundary condition $\bar{\phi}_0 \to \phi_+$ as $\rho \to \infty$). $T_{\mu\nu} \equiv (2/\sqrt{-g})(\delta S/\delta g^{\mu\nu})$ is the energy-momentum tensor of the field ϕ. Note that the term $\delta S[\phi, \delta_{\mu\nu}]/\delta\phi|_{\phi=\bar{\phi}_0}=0$. The next term in equation (5), however, is a non-trivial correction to $S_E[\bar{\phi}_0; \delta_{\mu\nu}]$ that arises using just the zeroth-order approximation for the bounce solution. It is linear in the curvature of the background geometry since $\delta g_{\mu\nu}=-\frac{1}{3}R_{\mu\alpha\nu\beta}(x_0)y^\alpha y^\beta$. The remainder of equation (5), $O((\bar{\phi}-\bar{\phi}_0)^2, (\delta g_{\mu\nu})^2)$, is a higher-order correction, since $(\bar{\phi}-\bar{\phi}_0)$ is also of the order of $R\bar{\rho}^2$. Indeed, in an arbitrary background, $\bar{\phi}$ satisfying the equation $\Box\phi=\mathrm{d}V/\mathrm{d}\phi$, will be a function not only of the geodesic distance ρ but also of three other coordinates x^i. Expanding around x_0 this equation is

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{3}{\rho}\frac{\partial\phi}{\partial\rho}(1-\tfrac{1}{9}R_{\mu\nu}(x_0)y^\mu y^\nu+\ldots)+\nabla^{(3)}\phi=\mathrm{d}V/\mathrm{d}\phi$$

where $\nabla^{(3)}$ is the three-dimensional Laplacian with respect to the coordinates x^i. We have used the fact that $\Box\rho=(3/\rho)(1-\frac{1}{9}R_{\mu\nu}(x_0)y^\mu y^\nu+\ldots)$. Since $\bar{\phi}_0(\rho)$ satisfies the equation $\ddot{\phi}_0+(3/\rho)\dot{\phi}_0=\mathrm{d}V/\mathrm{d}\phi_0$, it is clear that $\bar{\phi}$ differs from $\bar{\phi}_0(\rho)$ by terms of the order of $R_{\mu\nu}(x_0)y^\mu y^\nu$.

Taking into account that $\bar{\phi}_0$ is a function of ρ only, after integration over the angular coordinates we get from equation (5)

$$S_E[\bar{\phi}; g_{\mu\nu}]=S_E[\bar{\phi}_0; \delta_{\mu\nu}]-\tfrac{1}{12}\pi^2R_{\mu\nu}(x_0)\int \mathrm{d}\rho\,\rho^5T^{\mu\nu}[\bar{\phi}_0(\rho)]+\ldots. \qquad (6)$$

When the thin-wall approximation is applicable this expression can be further evaluated by dividing the range of integration into three regions: inside the bubble, through the wall and outside the bubble. In the thin-wall approximation the field is everywhere constant except in the wall of the bubble, where $|\dot{\bar{\phi}}_0(\rho)|^2=2|V(\phi)-V(\phi_+)|$. The energy-momentum tensor of the bounce is

$$T_{\mu\nu}[\bar{\phi}_0(\rho)]=\begin{cases} V(\phi_-)\delta_{\mu\nu} & \text{if } \rho<\bar{\rho} \\ |\dot{\bar{\phi}}_0(\rho)|^2\delta_{\mu\nu} & \text{if } \rho\approx\bar{\rho} \\ V(\phi_+)\delta_{\mu\nu} & \text{if } \rho>\bar{\rho} \end{cases}$$

and so for B, a given by equation (1), we find

$$B=\tfrac{1}{2}\pi^2\bar{\rho}^3[(4S_1-\bar{\rho}\varepsilon)-\tfrac{1}{6}R(x_0)\bar{\rho}^2(S_1-\tfrac{1}{6}\bar{\rho}\varepsilon)+\ldots]. \qquad (7)$$

The value of $\bar{\rho}$, the radius of the bubble, is evaluated by extremising the flat space action; $\bar{\rho}=3S_1/\varepsilon$, where $S_1=\int_{\phi_-}^{\phi_+}\mathrm{d}\phi[2(V(\phi)-V(\phi_+))]^{1/2}$ is the surface energy density in the bubble wall and ε is the energy difference between ϕ_+ and ϕ_-. Extremising just the flat-space action or the whole expression (7) makes no difference, up to the order considered. Corrections to $\bar{\rho}$ of order $R\bar{\rho}_0^2$ will only affect the action by terms proportional to $(R\bar{\rho}_0^2)^2$.

The final result for B including first-order corrections is

$$B=B_0[1-\tfrac{1}{12}R(x_0)\bar{\rho}^2+\ldots] \qquad (8)$$

where $B_0=\frac{1}{2}\pi^2\bar{\rho}^3S_1$. In a region where the scalar curvature is positive the probability of bubble nucleation is enhanced. The fact that the Euclidean action depends on the spacetime point x_0 that we choose as our origin of normal coordinates (and thus as the centre of the bubble) simply expresses the fact that the vacuum decay probability per unit time and per unit volume is different when that volume is centred on different regions of spacetime, since no translational invariance was assumed. It still makes sense to evaluate the vacuum decay probability per unit volume in the simple way we did because any bubble centred at an arbitrary point inside the normal neighbourhood where the expansion of the metric in normal coordinates makes sense will contribute the same amount to the action, up to the order considered. This analysis applies, of course, only as long as $R\bar{\rho}^2 \ll 1$. The dependence of the action on x_0 clearly shows that we are not fully extremising the action. However, as said before, the action of an actual solution, which must eventually include a constraint to force the bubble to be centred around x_0, will differ only by higher-order terms. The correction to the flat space value B_0 as given by equation (8) is then basically a geometric effect, since no modification on the dynamics of ϕ is involved.

Our result equation (8) for B agrees, in the particular case of a de Sitter background or anti-de Sitter background, with the appropriate limit $\varepsilon \ll V(\phi_+)$, $R\bar{\rho}_0^2 \ll 1$, to the results of [8], where the calculations of Coleman and de Luccia [4] were reproduced including also the possibility of a cosmological constant term in the Einstein equations.

Finally, a word about the coefficient A. All our calculations would be worthless if the change in A from its flat-space value could overwhelm that of B. However, we expect the correction to be the form $A = A_0(1 + cR\bar{\rho}^2 + \ldots)$, with c a number of order unity. Since $\Gamma/V = A_0(1 + cR\bar{\rho}^2 + \ldots)e^{-B_0}(1 - B_0R\bar{\rho}^2/12 + \ldots)$ and the semiclassical approach we have been considering presupposes $B_0 \gg 1$, we can expect the change on B to be more relevant than the one on A.

We wish to thank S Coleman for helpful discussions. LFA was supported in part by an Alfred P Sloan Foundation Fellowship. LFA and DH were supported in part by the US Department of Energy under contract no AC02-ER0320. Q-HP was supported in part by the US Department of Energy under contract no FG05-86-ER40272.

References

[1] Coleman S 1977 *Phys. Rev.* D **15** 2929; 1979 *The Whys of Subnuclear Physics* ed A Zichichi (New York: Plenum)

[2] Callan C and Coleman S 1977 *Phys. Rev.* D **16** 1762

[3] Coleman S, Glaser V and Martin A 1978 *Commun. Math. Phys.* **58** 211

[4] Coleman S and de Luccia F 1980 *Phys. Rev.* D **21** 3305

[5] Schouten J A 1954 *Ricci Calculus* (Berlin: Springer)

[6] Misner C, Thorne K and Wheeler J 1973 *Gravitation* (San Francisco: Freeman)

[7] Bunch T and Parker L 1979 *Phys. Rev.* D **20** 2499

[8] Parke S 1983 *Phys. Lett.* **121B** 313