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CALCULATING QUANTUM CORRECTIONS TO THE MASS OF A SOLITON WITHOUT COLLECTIVE COORDINATES *

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Feynman rules are derived for computing quantum corrections to the mass of a soliton in quantum field theory. These rules exhibit a finite propagator, but in contrast to previous methods, no additional effective vertices are introduced beyond those present in the original shifted Lagrangian. The derivation is based on imposing end-point boundary conditions appropriate to a soliton state on the functional integral representing the soliton-to-soliton transition amplitude.

1. Introduction

During the past few years, a great deal of attention has been given to the problem of constructing a perturbation expansion around a non-dissipative, finite-energy solution to the classical field equations of a quantum field theory [1–9] ***. Classical "lump-like" solutions are interpreted as representing particles called solitons. To investigate the quantum corrections to various physical processes one must derive Feynman rules for the one-soliton sector of the theory. One can begin by shifting the quantum field operator by a particular c-number solution to the field equations. This defines a shifted Lagrangian from which one can read off a set of interaction vertices and a propagator. However, the shifted field operator possesses certain zero-frequency excitations correponding to symmetry transformations of the original solution, and these appear to contribute a divergent piece to the propagator. The zero-frequency mode problem seems to render the naive Feynman rules useless.

This problem has been resolved by introducing collective coordinates and making a canonical transformation to a new set of dynamical variables [4,9]. This approach

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*** For reviews see ref. [1]. The approach of Goldstone and Jackiw [2a] is based on the method of Kerman and Klein [2b].

eliminates *all* of the contributions of the zero-frequency modes from the propagator including the troublesome divergences. However, the propagator is made finite at the expense of introducing a complicated set of effective vertices in addition to those of the original shifted Lagrangian. Furthermore, in a path-integral approach to quantization, great care must be taken in performing the canonical transformation to collective coordinates [9].

Recently, it has been shown that if we regulate the divergences of the original naive propagator and proceed with our calculations, then in a computation of the soliton-to-soliton transition amplitude all divergences cancel [5,6]. In this direct approach, no additional effective vertices are introduced besides those of the original shifted Lagrangian. Instead, a *finite* contribution from the zero-frequency modes to the propagator reproduces the extra terms which come from the new effective vertices in the collective-coordinate approach. Although the two methods give the same results, the direct method is much simpler. Jevicki has pointed out certain ambiguities in this procedure, and has proposed a scheme in which a non-canonical collective coordinate is introduced [7]. Jevicki's approach again introduces new effective vertices coming from a Jacobian factor in the functional integral.

The great simplicity of the direct approach makes it a convenient method for computing quantum corrections to the soliton-to-soliton transition amplitude. This is the approach developed in this paper. Rather than using a limiting procedure which results in the type of ambiguities discussed by Jevicki [7], we derive the Feynman rules by carefully analyzing the end-point conditions imposed on the functional integral in order to define the soliton-to-soliton transition amplitude. This leads to a simple set of Feynman rules with no additional effective vertices besides those of the original shifted Lagrangian, and a finite propagator free from any ambiguities of definition.

This direct method was first studied by Creutz in an operator formalism [8]. Below, we present the method in a path-integral form. In ref. [8], various cumbersome operations on δ functions occur in the Feynman rules and momentum conservation is not explicitly displayed. These problems are eliminated in the present discussion.

The key to understanding how the divergences and ambiquities of the naive propagator are eliminated lies in an appreciation of the role of the soliton end-point conditions in defining the functional integral representing the soliton-to-soliton transition amplitude. In a perturbative approach these end-point conditions along with the quadratic part of the Lagrangian serve to *define* the propagator. In particular, when we evaluate the soliton-to-soliton transition amplitude at finite time and correctly handle the soliton end-point conditions all divergences are unambiguously regulated and can be eliminated in a well-defined way. Our Feynman rules will define a perturbation expansion for this soliton-to-soliton amplitude at zero soliton momentum and hence serve to calculate the mass of a soliton. In sect. 2, the soliton end-point conditions are discussed and applied to a derivation of the Feynman rules for computing the soliton-to-soliton transition amplitude. In sect. 3, we present a heuristic argument to explain how the physics of the one-soliton sector acts by means of these end-point conditions to modify the pole prescription for the propagator and render it finite.

To simplify and clarify our discussion, we will consider the case of a single scalar field in two spacetime dimensions and a time-independent soliton solution. As a result there will be a *single* zero-frequency mode corresponding to translations of the soliton in space. The generalization to more complicated cases is straightforward.

2. The soliton-to-soliton transition amplitude

The transition amplitude (in a two-dimensional scalar field theory with Lagrangian $\mathcal{L}$) from a field configuration ϕ_1 at time $-\frac{1}{2}T$ to another configuration ϕ_2 at time $\frac{1}{2}T$ is given in the path-integral formalism by the functional integral

$$K(\phi_1, \phi_2; T) = N \int_{\phi_1}^{\phi_2} \delta[\phi] \exp(i \int_{-T/2}^{T/2} dt \int dx\, \mathcal{L}(\phi)) , \tag{2.1}$$

where N is a normalization factor. The end-point conditions on the functional integral are thus determined by the properties of the states $|\phi_1\rangle$ and $|\phi_2\rangle$ which define a particular transition amplitude. Our problem is to extract the transition amplitude from a soliton state of momentum p_1 to another of momentum p_2 from expressions like (2.1).

Let us consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - V(\phi) , \tag{2.2}$$

and assume the existence of a time-independent soliton solution to the classical field equations

$$-\frac{d^2}{dx^2}\phi_c(x) + V^{(1)}(\phi_c(x)) = 0 , \tag{2.3}$$

where we use the notation

$$V^{(n)}(\phi) = \frac{d^n V(\phi)}{d\phi^n} . \tag{2.4}$$

The classical energy of $\phi_c(x)$ is just the lowest-order approximation to the soliton mass

$$M = \int dx \left[\frac{1}{2}\left(\frac{d\phi_c}{dx}\right)^2 + V(\phi_c)\right] . \tag{2.5}$$

Of course, a particular solution $\phi_c(x)$ is only one member of a family of soliton solutions generated by translations and boosts. The general form is

$$\phi_S = \phi_c\left(\frac{x - a + vt}{\sqrt{1 - v^2}}\right). \tag{2.6}$$

In the following discussion we will need only those solutions with $v = 0$. The role of solutions with $v \neq 0$ is, however, indicated below.

We may now write

$$\phi(x, t) = \phi_c(x - a) + \eta(x - a, t). \tag{2.7}$$

Then, from (2.2) we find that the original Lagrangian in terms of the shifted field $\eta(x, t)$ is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \eta)^2 - \tfrac{1}{2} V^{(2)}(\phi_c)\, \eta^2 - U(\phi_c, \eta), \tag{2.8}$$

where

$$U(\phi_c, \eta) = \sum_{n=3}^{\infty} \frac{1}{n!} V^{(n)}(\phi_c)\, \eta^n. \tag{2.9}$$

Let us introduce eigenfunctions and eigenvalues appropriate to a normal-mode decomposition of (2.8):

$$\left[-\frac{d^2}{dx^2} + V^{(2)}(\phi_c)\right] \psi_n = \omega_n^2 \psi_n. \tag{2.10}$$

The zero-frequency mode corresponding to translations of the original soliton solution, obtained by shifting the parameter a, is the $n = 0$ mode with

$$\psi_0(x) = \frac{d}{dx} \phi_c(x), \tag{2.11}$$

$$\omega_0 = 0. \tag{2.12}$$

All of the other modes are taken to satisfy

$$\omega_n > 0 \qquad \text{for} \qquad n \neq 0. \tag{2.13}$$

From eqs. (2.3) and (2.5) with (2.11) we find

$$\int dx\, \psi_0^2(x) = M. \tag{2.14}$$

Let us now isolate the zero-frequency mode from $\phi(x, t)$ and write

$$\phi(x, t) = \phi_c(x - a) + q(t)\, \psi_0(x - a) + \rho(x - a, t), \tag{2.15}$$

or equivalently, from eq. (2.7),

$$\eta(x, t) = q(t)\, \psi_0(x) + \rho(x, t). \tag{2.16}$$

By definition, the field $\rho(x, t)$ will be constrained to satisfy the orthogonality condition

$$\int \mathrm{d}x\, \rho(x, t)\, \psi_0(x) = 0 \,. \tag{2.17}$$

As we recall from eq. (2.1), a particular transition amplitude can be defined by specifying two field configurations ϕ_1 and ϕ_2. By eq. (2.15) this is equivalent to specifying a, $q(\pm\frac{1}{2}T)$ and a field $\rho(x, \pm\frac{1}{2}T)$ subject to condition (2.17). The field $\rho(x, t)$ is associated with the normal-mode oscillations of eq. (2.10) with non-zero frequencies, and thus represents the physical meson excitations in the presence of the soliton. On the other hand, the zero-frequency mode is *not* associated with a physical meson state, but rather with translations of the soliton itself. Since we are interested in computing the transition amplitude between states with one soliton and *no mesons* at times $\pm\frac{1}{2}T$, the end-point conditions for the field $\rho(x, t)$ are those of the ordinary no-meson vacuum. It is well-known that the no-meson end-point conditions on the functional integration over $\rho(x, t)$ serve only to determine the $i\epsilon$ pole prescription in the ρ propagator [10]. Thus, instead of constraining the end-points of the functional integral over $\rho(x, t)$ we can add a term $\frac{1}{2}i\epsilon\rho^2$ to the Lagrangian and integrate freely over the ρ field, subject only to the condition (2.17). This is the usual convention used to derive Feynman rules in the vacuum sector. However, to complete our specification of the end-point conditions in the one-soliton sector we must also specify a and $q(\pm\frac{1}{2}T)$. We set

$$q(-\tfrac{1}{2}T) = -\tfrac{1}{2}\bar{q} \,, \qquad q(\tfrac{1}{2}T) = \tfrac{1}{2}\bar{q} \,. \tag{2.18}$$

These boundary conditions will assure that the variable a of eq. (2.15) will serve as the c.m. coordinate for the soliton, while for small values q represents fluctuations around the c.m. Of course, for large values q does not represent a translation coordinate, but the correctness of the boundary conditions of (2.18) will nevertheless be evident by the correct momentum conservation relation which will appear in the soliton-to-soliton transition amplitude. These end-point conditions define a transition amplitude which, when we write the original Lagrangian (2.2) in terms of q and ρ, is given by

$$K(a, \bar{q}, T) = N \exp(-iMT) \int_{-\bar{q}/2}^{\bar{q}/2} \delta[q] \int \delta[\rho]\, \delta\left(\int \mathrm{d}x\, \rho(x, t)\, \psi_0(x)\right)$$

$$\times \exp\left(i \int_{-T/2}^{T/2} \mathrm{d}t\, \{\tfrac{1}{2}M\dot{q}^2 + \int \mathrm{d}x\,[\tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}V^{(2)}(\phi_c)\,\rho^2 + \tfrac{1}{2}i\epsilon\rho^2 - U(\phi_c, \eta)]\}\right) \tag{2.19}$$

where a is the position variable in eq. (2.15).

The perturbation expansion for the amplitude (2.19) is defined by introducing a source term

$$\mathcal{L}_{\text{source}} = J(x-a, t)\,\eta(x-a, t)\,. \tag{2.20}$$

Note that the source has been defined as $J(x-a, t)$ and not $J(x, t)$. The perturbation expansion for the amplitude (2.19) is then

$$K(\bar{q}, T) = \exp(-iMT)\exp\left[-i\int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}x\, U\left(\phi_c, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] \tilde{K}_0(\bar{q}; J)\Big|_{J=0}\,, \tag{2.21}$$

where

$$\tilde{K}_0(\bar{q}; J) = N \int_{-\bar{q}/2}^{\bar{q}/2} \delta[q] \int \delta[\rho]\, \delta\left(\int \mathrm{d}x\, \rho(x, t)\, \psi_0(x)\right) \exp(i \int_{-T/2}^{T/2} \mathrm{d}t$$
$$\times \{\tfrac{1}{2}M\dot{q}^2 + J_0 q + \int \mathrm{d}x [\tfrac{1}{2}(\partial_\mu \rho)^2 - \tfrac{1}{2}V^{(2)}(\phi_c)\,\rho^2 + \tfrac{1}{2}i\epsilon\rho^2 + J\rho]\})\,, \tag{2.22}$$

with

$$J_0(t) = \int \mathrm{d}x\, J(x, t)\, \psi_0(x) \tag{2.23}$$

The integrals in eq. (2.22) can easily be performed. The result is [11]

$$\tilde{K}_0(\bar{q}; J) = \sqrt{\frac{M}{2\pi i T}} \exp\left(i\left\{\frac{M\bar{q}^2}{2T} + \frac{\bar{q}}{T}\int_{-T/2}^{T/2} \mathrm{d}t\, J_0(t)\, t\right.\right.$$
$$\left.\left. + \frac{1}{2M}\int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{d}s\, J_0(t)\, J_0(s)\left[\frac{1}{2}|t-s| + \frac{ts}{T} - \frac{1}{4}T\right]\right\}\right)$$
$$\times \exp(i\{-\tfrac{1}{2}\int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{d}s \int \mathrm{d}x\, \mathrm{d}y\, [J(x, t)\, J(y, s)\, D(x, y, t-s)]\})\,, \tag{2.24}$$

where

$$D(x, y, t-s) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{i\omega(t-s)} \sum_{n=1}^{\infty} \frac{\psi_n(x)\, \psi_n(y)}{\omega^2 - \omega_n^2 + i\epsilon}\,. \tag{2.25}$$

At first sight, the perturbation expansion defined by (2.21) and (2.24) would appear to have all the problems outlined in sect. 1. Although the vertices are clearly those of the shifted Lagrangian given by $U(\phi_c, \eta)$, the propagator obtained from eq. (2.24) would naively appear to be divergent. From eq. (2.24) we obtain the propagator

$$\tilde{\Delta}(x, y, t, s) = \frac{\psi_0(x)}{\sqrt{M}}\frac{\psi_0(y)}{\sqrt{M}}\left[\tfrac{1}{4}T - \tfrac{1}{2}|t-s| - \frac{ts}{T}\right] + D(x, y, t-s)\,. \tag{2.26}$$

Note that it is the contributions of the $n = 0$ mode which appears to cause problems with the propagator (2.26). First as $T \to \infty$ we have the divergent piece $\frac{1}{4}T$. The last term ts/T contributes the kind of ambiguities discussed by Jevicki when it multiplies a term $\frac{1}{4}T$ in a diagram involving a product of two propagators. Finally, we note that $\tilde{K}_0(\bar{q}; J)$ has a factor $(M/2\pi iT)^{1/2}$ multiplying it which actually vanishes as $T \to \infty$. This is clearly the zero coming from the $n = 0$, zero eigenvalue in the functional determinant normally obtained from performing quadratic functional integrals.

All of these problems are eliminated when one correctly defines the transition amplitude between soliton momentum eigenstates. This is done by defining the transition amplitude between a soliton of momentum p_1 and another of momentum p_2 by the Fourier transform

$$K(p, P, T) = \int \mathrm{d}a\, \mathrm{d}\bar{q}\; \mathrm{e}^{ipa}\mathrm{e}^{iP\bar{q}} K(\bar{q}, T)\,, \tag{2.27}$$

with

$$p = p_1 - p_2\,, \tag{2.28}$$

$$P = \tfrac{1}{2}(p_1 + p_2)\,. \tag{2.29}$$

Using the perturbation expansion of eqs. (2.21) and (2.24) and performing the integrals in eq. (2.27), we find

$$K(p, P, T) = 2\pi\delta(p) \exp(-iMT) \exp\left[-i \int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}x\, U\left(\phi_c, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] \tilde{K}_0(P; J)\Big|_{J=0} \tag{2.30}$$

with

$$\tilde{K}_0(P; J) = \exp\left(-i\frac{P^2}{2M}T\right) \exp\left[-i\frac{P}{M} \int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}x\, t\, J(x, t)\, \psi_0(x)\right]$$
$$\times \exp(-\tfrac{1}{2}i \int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{d}s \int \mathrm{d}x\, \mathrm{d}y\, [J(x, t)\, J(y, s)\, \tilde{\Delta}(x, y, t-s)])\,, \tag{2.31}$$

where

$$\tilde{\Delta}(x, y, t-s) = -\tfrac{1}{2}|t-s| \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} + D(x, y, t-s) + \tfrac{1}{4}T \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}}\,. \tag{2.32}$$

In deriving this result we have used the definition of $J_0(t)$, eq. (2.23). Note the appearance of the factor $P^2/2M$ and the higher-order kinematic corrections expressed by the tadpole term in (2.31). The potentially ambiguous term ts/T has been removed from the propagator and the troublesome $(M/2\pi iT)^{1/2}$ factor has been cancelled. Furthermore, momentum conservation is explicitly displayed.

Note that eq. (2.31) contains a tadpole term proportional to P/M. This term tells us that if we are calculating amplitudes with $P \neq 0$, we should expand around one of the moving soliton solutions of eq. (2.6) with $v \neq 0$. For the present, it is more convenient to restrict ourselves to those frames for which $P = 0$. This eliminates the tadpole and gives for the soliton-to-soliton transition amplitude,

$$K(p, 0, T) = 2\pi\delta(p) \exp(-iMT) \exp\left[-i \int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}x\, U\left(\phi_c, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] \tilde{K}_0(0;J)\Big|_{J=0}, \tag{2.33}$$

with

$$\tilde{K}_0(0;J) = \exp(-\tfrac{1}{2}i \int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{d}s \int \mathrm{d}x\, \mathrm{d}y\, [J(x,t)\, J(y,s)\, \tilde{\Delta}(x,y,t-s)]) \,. \tag{2.34}$$

Finally, we must deal with the problem of the term in the propagator of eq. (2.32) which is proportional to $\frac{1}{4}T$ and thus which diverges for infinite time $T \to \infty$. It has been proven elsewhere [5,6] that the perturbation expansion defined by eqs. (2.33) and (2.34) remains unchanged when the propagator is transformed by

$$\Delta(x,y,t-s) \to \Delta(x,y,t-s) + c\,\frac{\psi_0(x)}{\sqrt{M}}\,\frac{\psi_0(y)}{\sqrt{M}} \,. \tag{2.35}$$

Therefore, choosing

$$c = -\tfrac{1}{4}T \tag{2.36}$$

we find the finite propagator

$$\Delta(x,y,t-s) = -\tfrac{1}{2}|t-s|\,\frac{\psi_0(x)}{\sqrt{M}}\,\frac{\psi_0(y)}{\sqrt{M}} + D(x,y,t-s) \,. \tag{2.37}$$

Thus, a perturbation expansion which gives results identical to those of eqs. (2.33) and (2.34) but which features a finite propagator is defined by

$$K(p, 0, T) = 2\pi\delta(p) \exp(-iMT) \exp\left[-i \int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}x\, U\left(\phi_c, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] K_0(0;J)\Big|_{J=0}, \tag{2.38}$$

with

$$K_0(0;J) = \exp(-\tfrac{1}{2}i \int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{d}s \int \mathrm{d}x\, \mathrm{d}y\, [J(x,t)\, J(y,s)\, \Delta(x,y,t-s)]) \,. \tag{2.39}$$

This gives us a simple set of Feynman rules with vertices given in terms of the original shifted Lagrangian by $U(\phi_c, \eta)$ and a propagator $\Delta(x, y, t - s)$ given in eq. (2.37). No new vertices have been added, and no divergences or ambiguous terms appear.

Calculations using the Feynman rules of eqs. (2.38), (2.39) and (2.37) have previously been performed and give results in agreement with those of the collective coordinate method, as has been explicitly demonstrated up to the two-loop level [5,7,12]. As outlined in sect. 1, the extra term

$$-\tfrac{1}{2}|t - s| \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}}$$

in the propagator (2.37) provides the additional factors obtained by using the propagator $D(x, y, t - s)$ and the extra vertices of the collective coordinate method.

3. The propagator pole prescription

As we have seen in sect. 2, a careful analysis of the end-point conditions imposed on the functional integral to define the soliton-to-soliton transition amplitude yields a simple set of Feynman rules with a finite and well-defined propagator. We now present a heuristic argument which demonstrates how these Feynman rules including the finite propagator can easily be obtained by considering the physical properties of the one-soliton sector. This should not be viewed as an independent derivation, but rather as a discussion of the physics contained in the more rigorous derivation of sect. 2.

After expanding $\phi(x, t)$ around a soliton solution $\phi_c(x)$ we find the Lagrangian for the shifted field $\eta(x, t)$ given by eq. (2.8):

$$\mathcal{L} = -\tfrac{1}{2}\eta[\Box^2 + V^{(2)}(\phi_c)]\, \eta - U(\phi_c, \eta)\,. \tag{3.1}$$

This Lagrangian determines a set of vertices given by $U(\phi_c, \eta)$ as in eq. (2.9), and a propagator given by

$$-\Delta^{-1} = \Box^2 + V^{(2)}(\phi_c)\,. \tag{3.2}$$

Our problem is to invert this result. Defining the normal modes as in eq. (2.10),

$$\left[-\frac{d^2}{dx^2} + V^{(2)}(\phi_c)\right] \psi_n = \omega_n^2 \psi_n\,, \tag{3.3}$$

we can write the propagator *formally* as

$$\Delta(x, y, t - s) = \int \frac{d\omega}{2\pi}\, e^{i\omega(t-s)} \left[\frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} \left(\frac{1}{\omega^2 - \omega_0^2}\right) + \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(y)}{\omega^2 - \omega_n^2}\right]. \tag{3.4}$$

Actually, it will be convenient to introduce a small mass parameter μ satisfying

$$0 < \mu \ll \omega_n \qquad \text{for all } n \neq 0\,, \tag{3.5}$$

and define our propagator as

$$\Delta(x, y, t-s) = \int \frac{d\omega}{2\pi} e^{i\omega(t-s)} \left[\frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} \left(\frac{1}{\omega^2 - \mu^2} \right) + \sum_{n=1}^{\infty} \frac{\psi_n(x)\, \psi_n(y)}{\omega^2 - \omega_n^2} \right] \tag{3.6}$$

(recall that $\omega_0 = 0$). Of course, at the end of our calculation we will take $\mu \to 0$. This will allow us to explicitly display any divergences present.

Now eq. (3.6) does not actually define a propagator until we specify how the integration around the poles at $\omega = \pm\omega_n$ for $n \neq 0$ and $\omega = \pm\mu$ for $n = 0$ is to be handled. It is well-known that the specification of end-point conditions on the functional integral which defines the Feynman rules in the vacuum sector is completely equivalent to the specification of the pole prescription in eq. (3.6) [10]. We have already noted in sect. 2 that the no-meson end-point condition in fact determines the $i\epsilon$ pole prescription of the usual Feynman propagator [10]. This is in accordance with the fact that the $n \neq 0$ modes represent physical particles with causal propagation. However, the mode $n = 0$ does *not* correspond to a particle state in the one-soliton sector but instead represents a spacial translation of the soliton. This has been explicitly demonstrated in the work of Goldstone and Jackiw [2]. As a result, it is *incorrect* to assume that the poles at $\omega = \pm\mu$ associated with the $n = 0$ mode should also be integrated with an $i\epsilon$ prescription. Note that if we *do* use an $i\epsilon$ prescription for the $n = 0$ translational mode we obtain the propagator

$$\tilde{\Delta}(x, y, t-s) = \frac{-i}{2\mu} e^{-i\mu|t-s|} \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} + D(x, y, t-s)\,, \tag{3.7}$$

with $D(x, y, t-s)$ given as in eq. (2.25) by

$$D(x, y, t-s) = \int \frac{d\omega}{2\pi} e^{i\omega(t-s)} \sum_{n=1}^{\infty} \left[\frac{\psi_n(x)\psi_n(y)}{\omega^2 - \omega_n^2 + i\epsilon} \right]. \tag{3.8}$$

Note that this propagator diverges as $\mu \to 0$.

However, we have indicated that the $i\epsilon$ prescription should not be naively followed because the $n = 0$ mode does not correspond to a causal particle state. Now according to unitarity the imaginary part of the propagator is proportional to the amplitude obtained by inserting a complete set of states. Since in this complete set of states there is no particle state corresponding to the $n = 0$ mode the contribution of this translational mode to the propagator should be real. This indicates the use of a principal value pole prescription for the poles at $\omega = \pm\mu$ corresponding to the $n = 0$ mode. Such a prescription will assure that no singularities associated with the $n = 0$ mode will appear in the Green functions in accordance with the fact that there is no corresponding particle to enter into the S-matrix.

Using a principal value for the $\omega = \pm\mu$ poles and an $i\epsilon$ prescription for the $\omega = \pm\omega_n$, $n \neq 0$ poles as dictated by the physics of the one-soliton sector, we find from eq. (3.6) that

$$\Delta(x, y, t - s) = \frac{-1}{2\mu} (\sin \mu|t - s|) \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} + D(x, y, t - s) . \tag{3.9}$$

In the limit $\mu \to 0$ this propagator is finite and given by

$$\Delta(x, y, t - s) = -\tfrac{1}{2}|t - s| \frac{\psi_0(x)}{\sqrt{M}} \frac{\psi_0(y)}{\sqrt{M}} + D(x, y, t - s) , \tag{3.10}$$

in exact agreement with eq. (2.37). Thus, we see that a careful consideration of the physics of the one-soliton state, which explicitly takes into account the special role of the translational mode, eliminates all divergences from the propagator and gives us a simple and well-defined set of Feynman rules for computations in the one-soliton sector of the theory.

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References

[1] J.L. Gervais and A. Neveu (eds.), Phys. Reports 23 (1976) 237;
S. Coleman, 1975 Erice Lectures, ed. A. Zichichi (Plenum Press, NY, 1977);
R. Jackiw, Rev. Mod. Phys. 49 (1977) 681;
R. Rajaramin, Phys. Reports 21 (1975) 227.
[2] (a) J. Goldstone and R. Jackiw, Phys. Rev. D11 (1975) 1486.
(b) A. Kerman and A. Klein, Phys. Rev. 132 (1963) 1326.
[3] R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10 (1974) 4114, 4130, 4138.
[4] L.D. Faddeev, P.O. Kulish and V.E. Korepin, JETP Pisma 21 (1975) 302;
N.H. Christ and T.D. Lee, Phys. Rev. D12 (1975) 1606;
C. Callan and D. Gross, Nucl. Phys. B93 (1975) 29;
J.L. Gervais and B. Sakita, Phys. Rev. D11 (1975) 2943;
J.L. Gervais, A. Jevicki and B. Sakita, Phys. Rev. D12 (1975) 1038;
E. Tomboulis, Phys. Rev. D12 (1975) 1678.
[5] L.D. Faddeev and V.E. Korepin, Phys. Lett. 63B (1976) 435.
[6] V.A. Matveev, Nucl. Phys. B121 (1977) 403.
[7] A. Jevicki, Nucl. Phys. B117 (1976) 365.
[8] M. Creutz, Phys. Rev. D12 (1975) 3126.
[9] J.L. Gervais and A. Jevicki, Nucl. Phys. B110 (1976) 93.
[10] L.D. Faddeev, 1975 Les Houches Lectures, ed. R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976);
S. Weinberg, Harvard University lectures, unpublished.
[11] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965).
[12] L. Jacobs, Phys. Rev. D13 (1975) 2278;
H.J. De Vega, Nucl. Phys. B115 (1976) 411.