Nuclear Physics B169 (1980) 415–429

THE THREE-LOOP β-FUNCTION FOR THE WESS-ZUMINO MODEL

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Received 11 March 1980

We use superfield techniques to calculate the β-function for the Wess-Zumino model at the three-loop level. These techniques permit an easy evaluation of the three-loop diagrams and allow us to exhibit in analytic form all relevant quantities. The scheme dependence of the β-function is exhibited and three subtraction schemes (minimal, modified minimal and momentum space) are discussed in detail. In particular we note that $\beta_{\mathrm{MS}} = \beta_{\overline{\mathrm{MS}}}$.

1. Introduction

It is known that many of the simplifications and cancellations that occur in theories which are supersymmetric can readily be exhibited by formulating the theories in terms of superfields. Recently, simplifications in Feynman rules for superfields have been proposed [1]. The new rules have the property that chiral (scalar multiplet) propagators are δ-functions in θ-space with no exponentials of θ present. Consequently, manifest supersymmetry can be maintained at all stages of calculation and all possible simplifications due to supersymmetry occur automatically. We have taken advantage of this to calculate the three-loop β-function for the massless Wess-Zumino model [2] which consists of scalar, pseudoscalar and Majorana fields with supersymmetric Yukawa and quartic couplings.

There are several reasons for performing and presenting this calculation. First, it exhibits the remarkable power of the superfield techniques: our three-loop calculation involves only four superfield diagrams which are immediately expressible in terms of simple scalar integrals for which the infinite parts are easily calculable. Second, it provides a nice explicit example of how the renormalization program is carried out at the three-loop level. Third, since the whole calculation can be done analytically, the results allow us to examine in detail the expected scheme dependence of the three-loop β-function. Finally, we have performed the calculation for the sheer pleasure of doing something easily with the superfield techniques which otherwise would have been extremely difficult to do.

[1] Research supported in part by the US Department of Energy under contract no. E(11-1)3230.
[2] Research supported in part by the National Science Foundation under grant no. PHY78-07757.

Ours is not the first calculation of a higher-loop β-function. The β-function through four loops has been calculated for a scalar field with ϕ^4 coupling [3, 4] and the results are summarized in ref. [4]. Although our model, in component language, looks much more complicated, in terms of superfields a four-loop calculation is quite manageable; however, we are not planning to undertake it. We also note that the one- and two-loop results for the β-function of the Wess-Zumino model have been previously obtained in the component field formalism [5].

In our calculation we dimensionally regularize the momentum integrals after all the supersymmetry manipulations are carried out in four dimensions. We know that this procedure maintains supersymmetry [1] and at least for the scalar multiplet it is doubtful that potential problems with the dimensional reduction regularization technique [6] will arise. Our results agree of course with the existing one- and two-loop results [5].

Our paper is organized as follows: in sect. 2 we describe the model both in superfield and component form and summarize its renormalization properties. Sect. 3 is devoted to a description of the superfield Feynman diagram techniques and a calculation of the relevant one- and two-loop quantities. In sect. 4 we discuss the three-loop diagrams and use the results to compute the three-loop β-function in the minimal subtraction scheme. Finally, in sect. 5 we discuss the scheme dependence of the β-function and exhibit its value in different subtraction schemes. In the appendix we evaluate the integrals which we need for our calculations including a calculation using the technique of Chebyshev polynomial expansion [7, 8].

2. The model

In superfield language the lagrangian of the Wess-Zumino model is (we use two-component spinors and the conventions of ref. [1])

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta}\, \bar{\phi}_0\phi_0 - \frac{\lambda_0}{3!}\int d^2\theta\, \phi_0^3 - \frac{\lambda_0}{3!}\int d^2\bar{\theta}\, \bar{\phi}_0^3 . \tag{2.1}$$

In terms of component fields

$$\phi_0 = \exp\left[i\theta^\alpha\bar{\theta}^{\dot{\beta}}\sigma^a_{\alpha\dot{\beta}}\partial_a\right]\left[\sqrt{\tfrac{1}{2}}\,(A+iB) + \theta^\alpha\psi_\alpha + \sqrt{\tfrac{1}{2}}\,(F-iG)\theta^2\right], \tag{2.2}$$

and the lagrangian becomes

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\gamma\cdot\partial\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \\ & + \frac{\lambda_0}{2\sqrt{2}}\left[-F(A^2 - B^2) + 2GAB + \bar{\psi}(A + i\gamma_5 B)\psi\right]\end{aligned} \tag{2.3}$$

or, after eliminating the auxiliary fields F, G

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\gamma\cdot\partial\psi$$
$$-\tfrac{1}{16}\lambda_0^2(A^2+B^2)^2 + \frac{\lambda_0}{2\sqrt{2}}\bar{\psi}(A+i\gamma_5 B)\psi . \tag{2.4}$$

It is known [9] that the model involves only one renormalization constant. Equivalently superfield power counting rules [1] immediately lead to the conclusion that the three-point function is finite. If we define the multiplicative renormalizations in $n = 4 - 2\varepsilon$ dimensions

$$\lambda_0 = \mu^\varepsilon Z_\lambda \lambda , \tag{2.5}$$

$$\phi_0 = Z_\phi^{1/2}\phi , \tag{2.6}$$

then the fact that $\lambda_0\phi_0^3$ is finite means that we can relate Z_λ and Z_ϕ by

$$Z_\lambda = Z_\phi^{-3/2} . \tag{2.7}$$

This in turn implies a relationship between

$$\gamma = \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi , \tag{2.8}$$

and

$$\beta = -\lambda\mu\frac{\partial}{\partial\mu}\ln Z_\lambda , \tag{2.9}$$

namely that [10]

$$\beta = 3\lambda\gamma . \tag{2.10}$$

Therefore, we can calculate the β-function by calculating Z_ϕ, which requires knowledge of the two-point function only, or equivalently, of that part of the effective action which is linear in ϕ and $\bar{\phi}$. In momentum space with euclidean signature this is

$$\Gamma_2(\phi,\bar{\phi}) = \int\frac{d^4p}{(2\pi)^4} d^2\theta\, d^2\bar{\theta}\,\bar{\phi}(-p,\theta,\bar{\theta})\phi(p,\theta,\bar{\theta})\Delta^{-1}(p)$$
$$= \int\frac{d^4p}{(2\pi)^4} d^2\theta\, d^2\bar{\theta}\,\bar{\phi}(-p)\phi(p)$$
$$\times\left\{Z_\phi + \frac{1}{2}\left(\frac{\lambda}{4\pi}\right)^2 F(p^2) + \left(\frac{\lambda}{4\pi}\right)^4\left[\tfrac{1}{2}G(p^2) - Z_1F(p^2)\right]\right.$$
$$\left. + \left(\frac{\lambda}{4\pi}\right)^6\left[H(p^2) - \tfrac{5}{2}Z_1G(p^2) - (Z_2 - \tfrac{3}{2}Z_1^2)F(p^2)\right] + O\left(\left(\frac{\lambda}{4\pi}\right)^8\right)\right\} . \tag{2.11}$$

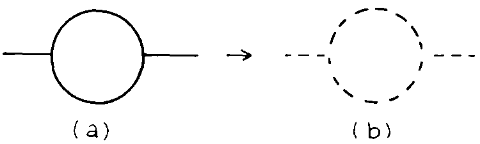

Fig. 1. One-loop contribution to the propagator correction: (a) superfield diagram; (b) equivalent scalar diagram.

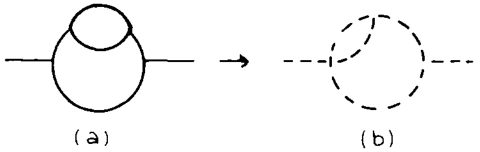

Fig. 2. Two-loop contribution to the propagator correction: (a) superfield diagram; (b) equivalent scalar diagram.

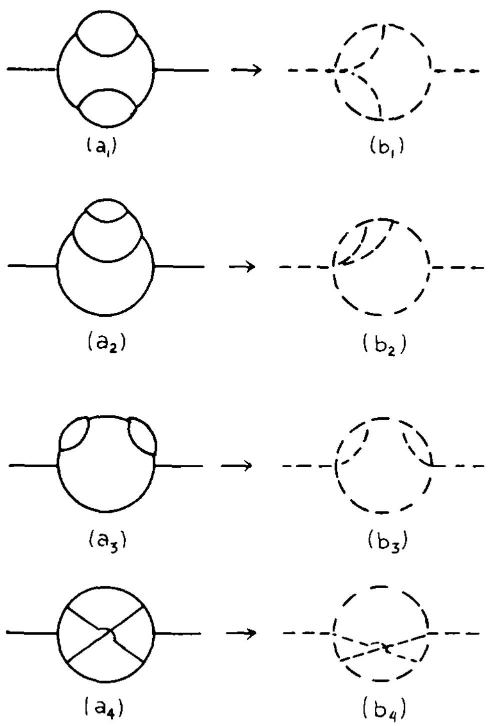

Fig. 3. Three-loop contributions to the propagator correction: (a) superfield diagrams; (b) equivalent scalar diagrams.

We are using renormalized quantities with

$$Z_\phi = 1 + \left(\frac{\lambda}{4\pi}\right)^2 Z_1 + \left(\frac{\lambda}{4\pi}\right)^4 Z_2 + \left(\frac{\lambda}{4\pi}\right)^6 Z_3 + \cdots , \tag{2.12}$$

and we have organized the contributions as follows: $Z_\phi \bar{\phi}(-p)\phi(p)$ is the renormalized kinetic term; $\frac{1}{2}(\lambda/4\pi)^2 F(p^2)$ is the contribution of the one-loop diagram fig. 1a, the factor of $\frac{1}{2}$ being a symmetry number. The two contributions at order $(\lambda/4\pi)^4$ are $\frac{1}{2}G(p^2)$ from the two-loop diagram of fig. 2a and $-Z_1 F(p^2)$ which comes from a Z_ϕ insertion into the one-loop diagram, fig. 1a. Finally, the $(\lambda/4\pi)^6$ terms are $H(p^2)$ coming from figs. 3a and the corresponding one- and two-loop Z_ϕ insertion terms.

To understand the Z_1, Z_2 factors we observe that in the renormalized lagrangian Z_ϕ only appears in the kinetic term $\bar{\phi}_0\phi_0 = Z_\phi \bar{\phi}\phi = \bar{\phi}\phi + (Z_\phi - 1)\bar{\phi}\phi$. Instead of making $(Z_\phi - 1)$ insertions into diagrams it is simpler to use Z_ϕ^{-1} times the usual Feynman propagator. Thus in the diagrams we have a factor of Z_ϕ^{-1} for each propagator. Applying this to figs. 1a and 2a we see that the complete contribution of these diagrams is $\frac{1}{2}(\lambda/4\pi)^2 Z_\phi^{-2} F$ and $\frac{1}{2}(\lambda/4\pi)^4 Z_\phi^{-5} G$. Using the expansion (2.12) for Z_ϕ one obtains the Z_1 and Z_2 factors in eq. (2.11).

Our program now is to calculate $F(p^2)$ and $G(p^2)$ and the divergent part of $H(p^2)$. Then Z_ϕ is determined up to three loops by requiring that it cancels all the divergence of Γ. Finally we will calculate the β-function from eqs. (2.8) and (2.10).

3. Feynman rules and one- and two-loop contributions

The Feynman rules for the model were presented in ref. [1] and are as follows. The only propagator is from ϕ to $\bar{\phi}$ and is

$$\langle \mathrm{T}(\bar{\phi}(-p,\theta_1)\phi(p,\theta_2))\rangle = \frac{1}{p^2}\delta^4(\theta_1 - \theta_2) = \frac{1}{p^2}(\theta_1 - \theta_2)^2(\bar{\theta}_1 - \bar{\theta}_2)^2 . \tag{3.1}$$

At an internal vertex corresponding to the interaction term $-(\lambda_0/3!)\phi_0^3$ we have a factor $\mu^\epsilon\lambda$, and for any two of the lines entering the vertex, factors of $-\frac{1}{4}\bar{D}^2$ acting on the propagator δ-functions for those lines. At a vertex containing external ϕ lines one omits one such factor for each external line. Correspondingly, at a $\bar{\phi}^3$ vertex one has $-\frac{1}{4}D^2$ factors. One then integrates over loop momenta with a factor $(2\pi)^{-n}\,\mathrm{d}^n q$ and over vertex θ's with $\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}$. Here the D's are covariant derivatives

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma\cdot\partial)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} , \\ \bar{D}_{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha(\sigma\cdot\partial)_{\alpha\dot{\alpha}} , \\ \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= 2i(\sigma\cdot\partial)_{\alpha\dot{\alpha}} , \end{aligned} \tag{3.2}$$

but the only relevant properties are the following:

$$\bar{D}_{\dot{\alpha}}\phi = D_{\alpha}\bar{\phi} = 0, \qquad \text{(chirality of } \phi\text{)}, \tag{3.3}$$

$$D^2\bar{D}^2D^2 = 16\Box D^2, \qquad \bar{D}^2D^2\bar{D}^2 = 16\Box\bar{D}^2, \tag{3.4}$$

$$D^2D_{\alpha} = \bar{D}^2\bar{D}_{\dot{\alpha}} = 0, \tag{3.5}$$

$$D_{\theta}\delta^4(\theta-\theta') = -\delta^4(\theta-\theta')\overleftarrow{D}_{\theta'}, \qquad \bar{D}_{\theta}\delta^4(\theta-\theta') = -\delta^4(\theta-\theta')\overleftarrow{\bar{D}}_{\theta'}; \tag{3.6}$$

and

$$\delta^4(\theta-\theta')D^2\bar{D}^2\delta^4(\theta-\theta') = 16\delta^4(\theta-\theta'), \tag{3.7}$$

while any similar expression with fewer D's gives zero. Furthermore, the D's obey usual integration by parts rules (except for possible minus signs for anti-commutativity).

We are now in a position to calculate the one- and two-loop two-point diagrams. Application of the rules to the one-loop diagram fig. 1a gives

$$\Gamma_2^{(\text{one-loop})} = \mu^{2\varepsilon}\lambda^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^nq}{(2\pi)^n}d^4\theta\,d^4\theta'\bar{\phi}(-p,\theta)$$
$$\times\left(-\tfrac{1}{4}D^2\right)\frac{\delta^4(\theta-\theta')}{q^2}\left(-\tfrac{1}{4}\overleftarrow{\bar{D}}^2\right)\frac{\delta^4(\theta-\theta')}{(p+q)^2}\phi(p,\theta'). \tag{3.8}$$

Using eqs. (3.6), (3.7) we immediately find,

$$\Gamma_2^{(\text{one-loop})} = \lambda^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{d}^4\theta\bar{\phi}(-p,\theta)\phi(p,\theta)\mu^{2\varepsilon}\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{1}{q^2(p+q)^2}, \tag{3.9}$$

so that

$$F = (4\pi)^2\mu^{2\varepsilon}\int\frac{\mathrm{d}^nq}{(2\pi)^n}\frac{1}{q^2(p+q)^2} = \left(\frac{p^2}{4\pi\mu^2}\right)^{-\varepsilon}\frac{\Gamma(\varepsilon)[\Gamma(1-\varepsilon)]^2}{\Gamma(2-2\varepsilon)}$$
$$= \frac{1}{\varepsilon}\left[1+\varepsilon\left(2-\gamma_{\mathrm{E}}-\ln\frac{p^2}{4\pi\mu^2}\right)\right.$$
$$\left.+\varepsilon^2\left(4-2\gamma_{\mathrm{E}}+\gamma_{\mathrm{E}}^2-\delta+\tfrac{1}{2}\ln^2\frac{p^2}{4\pi\mu^2}+(\gamma_{\mathrm{E}}-2)\ln\frac{p^2}{4\pi\mu^2}\right)+\mathrm{O}(\varepsilon^3)\right]. \tag{3.10}$$

The factor $(4\pi)^2$ is present because we have removed $(4\pi)^{-2}$ in our definition of F in eq. (2.11). The factor $\mu^{2\varepsilon}$ comes from the relation between bare and renormalized couplings of eq. (2.5).

$$\Gamma(1+\varepsilon) = 1 - \varepsilon\gamma_E + \varepsilon^2\delta + O(\varepsilon^3)\,, \tag{3.11}$$

where γ_E is the Euler constant and $\delta = \frac{1}{12}\pi^2 + \frac{1}{2}\gamma_E^2$. Note that after taking care of the supersymmetry algebra, the evaluation of fig. 1a reduces to that of an ordinary massless scalar field loop. We have indicated this as fig. 1b.

There is only one two-loop diagram, fig. 2a. This is due to the absence of a $\phi\phi$ or $\bar{\phi}\bar{\phi}$ propagator. The full expression for the contribution to the effective action is

$$\Gamma_2^{(\text{two-loop})} = \frac{\mu^{4\varepsilon}\lambda^4}{(4)^6}\int \frac{d^4p}{(2\pi)^4}\frac{d^nq}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}$$
$$\times \int d^4\theta\, d^4\theta'\, d^4\theta_1\, d^4\theta_2\, \bar{\phi}(-p,\theta) D^2 \frac{\delta^4(\theta-\theta_1)}{q^2}\overleftarrow{\bar{D}}^2$$
$$\times \bar{D}^2\frac{\delta^4(\theta_1-\theta_2)}{k^2}\overleftarrow{D}^2 D^2\frac{\delta^4(\theta_2-\theta')}{q^2}\overleftarrow{\bar{D}}^2\phi(p,\theta')\frac{\delta^4(\theta_1-\theta_2)}{(q+k)^2}\frac{\delta^4(\theta-\theta')}{(p+q)^2}\,. \tag{3.12}$$

Eqs. (3.6), (3.7) can be used to replace the $\bar{D}^2\delta(\theta_1-\theta_2)\overleftarrow{D}^2$ factor by 16, after which the θ_2 integration can be done using the other $\delta^4(\theta_1-\theta_2)$ factor. One next uses integration by parts to move all the D's onto $\delta^4(\theta-\theta_1)$. Eq. (3.5) can now be used to replace $D^2\bar{D}^2D^2\bar{D}^2$ by $-16q^2D^2\bar{D}^2$ and finally one uses eq. (3.7) again to get rid of the remaining $D^2\bar{D}^2$. We find then that $G(p^2)$, as defined in eq. (2.11) is given by

$$G(p^2) = (4\pi)^4\mu^{4\varepsilon}\int\frac{d^nq}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}\frac{1}{k^2q^2(k+q)^2(p+q)^2} \tag{3.13}$$

which is just the integral associated with the diagram for a scalar field of fig. 2b. The integral in eq (3.13) can be readily performed and we find

$$G = -\left(\frac{p^2}{4\pi\mu^2}\right)^{-2\varepsilon}\frac{\Gamma(\varepsilon)\Gamma(2\varepsilon)[\Gamma(1-\varepsilon)]^3}{(1-2\varepsilon)\Gamma(1+\varepsilon)\Gamma(2-3\varepsilon)}$$
$$= -\frac{1}{2\varepsilon^2}\left[1+\varepsilon\left(5-2\gamma_E-2\ln\frac{p^2}{4\pi\mu^2}\right)\right.$$
$$\left.+\varepsilon^2\left(19-10\gamma_E+3\gamma_E^2-2\delta+2\ln^2\frac{p^2}{4\pi\mu^2}+(4\gamma_E-10)\ln\frac{p^2}{4\pi\mu^2}\right)+O(\varepsilon^3)\right]. \tag{3.14}$$

We chose Z_1, Z_2 so as to render $\Gamma_2(\phi, \bar{\phi})$ in eq. (2.11) finite through order λ^4. In the minimal subtraction scheme (MS) this is achieved by choosing Z's so as to cancel just the pole parts coming from F and G so that

$$Z_1 = -\frac{1}{2\varepsilon}\,, \tag{3.15}$$

$$Z_2 = -\frac{1}{4\varepsilon^2} + \frac{1}{4\varepsilon}\,. \tag{3.16}$$

In the minimal subtraction scheme, Z_ϕ is given by a sum of poles in ε

$$Z_\phi = 1 + \sum_{i=1}^{\infty} \frac{Z_\phi^{(i)}}{(\varepsilon)^i}\,. \tag{3.17}$$

In general, an i-loop diagram contributes to $Z^{(1)}$ through $Z^{(i)}$. Now combining eq. (2.8) with the renormalization group equation for the coupling constant [11],

$$\mu\frac{\partial\lambda}{\partial\mu} = -\varepsilon\lambda + \beta\,, \tag{3.18}$$

and using the chain rule of differentiation one finds that

$$2Z\gamma = (-\varepsilon\lambda + \beta)\frac{\partial Z}{\partial\lambda}\,, \tag{3.19}$$

or, using (3.17) and extracting each pole term,

$$\left(2\gamma - \beta\frac{\partial}{\partial\lambda}\right)Z_\phi^{(i)} = -\lambda\frac{\partial}{\partial\lambda}Z_\phi^{(i+1)}\,. \tag{3.20}$$

Combining this with eq. (2.10) we get

$$\beta\left(\frac{2}{3} - \lambda\frac{\partial}{\partial\lambda}\right)Z_\phi^{(i)} = -\lambda^2\frac{\partial}{\partial\lambda}Z_\phi^{(i+1)}\,. \tag{3.21}$$

This is a useful result which allows us to check our two- and three-loop calculations. For example, setting $i = 1$ in eq. (3.21) one can check that the coefficient of $1/\varepsilon^2$ in Z_2 [eq. (3.16)] must be $-\frac{1}{4}$. Eq. (3.21) also provides an easy way to calculate the β-function. Setting $i = 0$ and recalling that $Z_\phi^{(0)} = 1$, we find

$$\beta = -\tfrac{3}{2}\lambda^2\frac{\partial}{\partial\lambda}Z_\phi^{(1)}\,. \tag{3.22}$$

From eqs. (3.15) and (3.16) we find that up to the two-loop level

$$Z_\phi^{(1)} = -\frac{1}{2}\left(\frac{\lambda}{4\pi}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{4\pi}\right)^4, \tag{3.23}$$

so that

$$\beta = \lambda\left[\frac{3}{2}\left(\frac{\lambda}{4\pi}\right)^2 - \frac{3}{2}\left(\frac{\lambda}{4\pi}\right)^4\right], \tag{3.24}$$

which agrees with previous one- and two-loop results computed using component fields [5].

4. Three-loop contributions

Four diagrams are present at the three-loop level, as depicted in figs. 3a. We denote the contributions to H by H_1, H_2, H_3, H_4 and, by using the Feynman rules and simple manipulations analogous to those performed to obtain G in sect. 3 we find that they reduce to momentum integrals represented by the scalar field diagrams in figs. 3b. For H_1, H_2 and H_3 we obtain

$$\begin{aligned}
H_1 &= \tfrac{1}{8}(4\pi)^6\mu^{6\varepsilon}\int\frac{d^nk\,d^nl\,d^nq}{(2\pi)^{3n}}\,\frac{1}{q^2k^2l^2(q+l)^2(p+q)^2(p+q+k)^2} \\
&= \frac{1}{8}\left(\frac{p^2}{4\pi\mu^2}\right)^{-3\varepsilon}\frac{[\Gamma(\varepsilon)]^2[\Gamma(1-\varepsilon)]^4\Gamma(3\varepsilon)[\Gamma(1-2\varepsilon)]^2}{[\Gamma(2-2\varepsilon)]^2[\Gamma(1+\varepsilon)]^2\Gamma(2-4\varepsilon)} \\
&= \frac{1}{24\varepsilon^3}\left[1+\varepsilon\left(8-3\gamma_E-3\ln\frac{p^2}{4\pi\mu^2}\right)\right. \\
&\quad\left.+\varepsilon^2\left(44-24\gamma_E+6\gamma_E^2-3\delta+\tfrac{9}{2}\ln^2\frac{p^2}{4\pi\mu^2}+(9\gamma_E-24)\ln\frac{p^2}{4\pi\mu^2}\right)+O(\varepsilon^3)\right],
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
H_2 &= \tfrac{1}{2}(4\pi)^6\mu^{6\varepsilon}\int\frac{d^nk\,d^nl\,d^nq}{(2\pi)^{3n}}\,\frac{1}{q^2k^2l^2(p+q)^2(q+l)^2(l+k)^2} \\
&= \frac{1}{2}\left(\frac{p^2}{4\pi\mu^2}\right)^{-3\varepsilon}\frac{[\Gamma(1-\varepsilon)]^4\Gamma(3\varepsilon)}{2\varepsilon^2(1-3\varepsilon)(1-2\varepsilon)\Gamma(2-4\varepsilon)} \\
&= \frac{1}{12\varepsilon^3}\left[1+\varepsilon\left(9-3\gamma_E-3\ln\frac{p^2}{4\pi\mu^2}\right)\right. \\
&\quad\left.+\varepsilon^2\left(55-27\gamma_E+6\gamma_E^2-3\delta+\tfrac{9}{2}\ln^2\frac{p^2}{4\pi\mu^2}+(9\gamma_E-27)\ln\frac{p^2}{4\pi\mu^2}\right)+O(\varepsilon^3)\right],
\end{aligned} \tag{4.2}$$

$$H_3 = \tfrac{1}{4}(4\pi)^6 \mu^{6\varepsilon} \int \frac{d^n k\, d^n l\, d^n q}{(2\pi)^{3n}} \frac{1}{q^2 k^2 l^2 (q+l)^2 (q+k)^2 (p+q)^2}$$

$$= \frac{1}{4}\left(\frac{p^2}{4\pi\mu^2}\right)^{-3\varepsilon} \frac{[\Gamma(\varepsilon)]^2 [\Gamma(1-\varepsilon)]^5 \Gamma(3\varepsilon)\Gamma(1-3\varepsilon)}{[\Gamma(2-2\varepsilon)]^2 \Gamma(1+2\varepsilon)\Gamma(2-4\varepsilon)}$$

$$= \frac{1}{12\varepsilon^3}\left[1 + \varepsilon\left(8 - 3\gamma_E - 3\ln\frac{p^2}{4\pi\mu^2}\right)\right.$$

$$\left. + \varepsilon^2\left(44 - 24\gamma_E + 6\gamma_E^2 - 3\delta + \tfrac{9}{2}\ln^2\frac{p^2}{4\pi\mu^2} + (9\gamma_E - 24)\ln\frac{p^2}{4\pi\mu^2}\right) + O(\varepsilon^3)\right]. \tag{4.3}$$

H_4 is less straightforward and we give some details of its evaluation. To begin with, the Feynman rules give an integrand

$$I = \frac{1}{2}\frac{1}{(4)^{10}} \bar{\phi}(-p,\theta) D^2 \frac{\delta^4(\theta-\theta_1)}{q^2} \overleftarrow{\bar{D}}^2 \bar{D}^2 \frac{\delta^4(\theta_1-\theta_2)}{l^2} \overleftarrow{\bar{D}}^2 D^2 \frac{\delta^4(\theta_2-\theta')}{k^2} \overleftarrow{\bar{D}}^2 \phi(p,\theta')$$

$$\times \frac{\delta^4(\theta_1-\theta_4)}{(k-l)^2} \frac{\delta^4(\theta-\theta_3)}{(p+q)^2} \overleftarrow{\bar{D}}^2 \bar{D}^2 \frac{\delta^4(\theta_3-\theta_4)}{(p+l)^2} \overleftarrow{\bar{D}}^2 D^2 \frac{\delta^4(\theta_4-\theta')}{(p+k)^2} \frac{\delta^4(\theta_2-\theta_3)}{(q-l)^2}, \tag{4.4}$$

to be integrated over θ's and loop momenta. We use the last two δ-functions to integrate over θ_3 and θ_4 and rewrite I as

$$I = \frac{1}{2}\frac{1}{(4)^{10}} \bar{\phi}(-p,\theta) D^2 \bar{D}^2 \frac{\delta^4(\theta-\theta_1)}{q^2} \bar{D}^2 D^2 \frac{\delta^4(\theta_1-\theta_2)}{l^2} \bar{D}^2 D^2 \frac{\delta^4(\theta'-\theta_2)}{k^2} \phi(p,\theta')$$

$$\times \bar{D}^2 \frac{\delta^4(\theta-\theta_2)}{(p+q)^2} \bar{D}^2 D^2 \frac{\delta^4(\theta_2-\theta_1)}{(p+l)^2} D^2 \frac{\delta^4(\theta'-\theta_1)}{(p+k)^2} \frac{1}{(q-l)^2(k-l)^2}. \tag{4.5}$$

Now we integrate by parts the $\bar{D}^2$ acting on $\delta^4(\theta-\theta_2)$. In principle the derivative can act either on $\bar{\phi}(-p,\theta)$ or $D^2\bar{D}^2\delta(\theta-\theta_1)$. However, the diagram is only logarithmically divergent so that infinite integrals will only occur in the latter case. Thus, we retain only the term $\bar{D}^2 D^2 \bar{D}^2 \delta^4(\theta-\theta_1) = -16q^2 \bar{D}^2 \delta^4(\theta-\theta_1)$ and in a similar fashion $D^2 \bar{D}^2 D^2 \delta^4(\theta'-\theta_2) = -16k^2 D^2 \delta^4(\theta'-\theta_2)$. After that the remaining manipulations are obvious and we find a contribution given by fig. $3b_4$ which can be written *after relabeling momenta*

$$H_4 = \tfrac{1}{2}(4\pi)^6\mu^{6\epsilon}\int \frac{\mathrm{d}^n k\,\mathrm{d}^n l\,\mathrm{d}^n q}{(2\pi)^{3n}}\,\frac{1}{q^2k^2(p-l)^2(l-q)^2(l-k)^2(q-k)^2}\,. \tag{4.6}$$

We discuss the evaluation of the integral in the appendix. We find

$$H_4 = \frac{1}{\varepsilon}\left[3\zeta(3) + \mathrm{O}(\varepsilon)\right], \tag{4.7}$$

where ζ is the Riemann zeta-function. Adding together the results of eqs. (4.1), (4.2), (4.3) and (4.7) we obtain H, and from eq. (2.11) we deduce

$$Z_3 = -\frac{1}{24\varepsilon^3}\left[5 - 9\varepsilon + (5 + 3\zeta(3))\varepsilon^2\right]. \tag{4.8}$$

The coefficients of $1/\varepsilon^2$ and $1/\varepsilon^3$ in eq. (4.8) can be checked by using the relation (3.21) with $i = 1$ and 2. This relates these coefficients to the known values of Z_1 and Z_2.

Finally, knowledge of Z_3 allows us to compute the three-loop β-function. Together with the result (3.17) we find, in the minimal subtraction scheme

$$\beta_{\mathrm{MS}} = \lambda\left[\frac{3}{2}\left(\frac{\lambda}{4\pi}\right)^2 - \frac{3}{2}\left(\frac{\lambda}{4\pi}\right)^4 + \left(\tfrac{15}{8} + \tfrac{9}{8}\zeta(3)\right)\left(\frac{\lambda}{4\pi}\right)^6 + \mathrm{O}\left(\left(\frac{\lambda}{4\pi}\right)^8\right)\right]. \tag{4.9}$$

5. Scheme dependence of the three-loop β-function

At the three-loop level the value of the β-function depends on the subtraction scheme used to define the coupling constant λ. If one uses two schemes which define coupling constants λ and λ' related by

$$\lambda' = \lambda\left[1 + a\left(\frac{\lambda}{4\pi}\right)^2 + b\left(\frac{\lambda}{4\pi}\right)^4 + \cdots\right], \tag{5.1}$$

and if

$$\beta(\lambda) = \lambda\left[\beta_0\left(\frac{\lambda}{4\pi}\right)^2 + \beta_1\left(\frac{\lambda}{4\pi}\right)^4 + \cdots\right], \tag{5.2}$$

with a similar expansion for $\beta'(\lambda')$, then

$$\begin{aligned} \beta_0' &= \beta_0\,, \\ \beta_1' &= \beta_1\,, \\ \beta_2' &= \beta_2 - 2a\beta_1 + 2b\beta_0 - 3a^2\beta_0\,. \end{aligned} \tag{5.3}$$

For example let us consider a scheme where we perform a momentum-space subtraction at $p^2 = \mu^2$. In other words we define Z_ϕ such that the effective action assumes its classical value at $p^2 = \mu^2$. In this scheme, Z_ϕ can be obtained up to two loops from the results of sect. 3 and is given by $Z_\phi = 1 + (\lambda/4\pi)^2 Z_1 + (\lambda/4\pi)^4 Z_2$ with

$$Z_1 = -\frac{1}{2\varepsilon} + \tfrac{1}{2}(\gamma_E - \ln 4\pi) - 1\,, \tag{5.4}$$

$$Z_2 = -\frac{1}{4\varepsilon^2} - \frac{3}{4\varepsilon} + \frac{1}{2\varepsilon}(\gamma_E - \ln 4\pi) - \tfrac{1}{4}(\gamma_E - \ln 4\pi)^2 + \tfrac{1}{2}(\gamma_E - \ln 4\pi) + \tfrac{3}{4}\,. \tag{5.5}$$

Requiring that

$$\lambda_0 = \mu^\varepsilon Z_\lambda \lambda = \mu^\varepsilon Z'_{\lambda'} \lambda'\,, \tag{2.5'}$$

and using eq. (2.7) we derive the relationship between the λ's in the minimal subtraction (MS) and momentum-space (MOM) subtraction schemes,

$$\lambda_{\mathrm{MOM}} = \lambda_{\mathrm{MS}}\left[1 + \left(\frac{\lambda_{\mathrm{MS}}}{4\pi}\right)^2\left(\tfrac{3}{4}(\gamma_E - \ln 4\pi) - \tfrac{3}{2}\right) + \left(\frac{\lambda_{\mathrm{MS}}}{4\pi}\right)^4\left(\tfrac{27}{32}(\gamma_E - \ln 4\pi)^2 - \tfrac{33}{8}(\gamma_E - \ln 4\pi) + 6\right) + \cdots\right]. \tag{5.6}$$

Finally, from eq. (5.3) we find

$$\beta_2^{\mathrm{MOM}} = \beta_2^{\mathrm{MS}} + \tfrac{27}{8}\,. \tag{5.7}$$

We consider now a second subtraction scheme, the $\overline{\mathrm{MS}}$ scheme [12]. In this scheme the ubiquitous factors of γ_E and $\ln 4\pi$ which are artifacts of the dimensional regularization method are subtracted out along with the pole terms. In this scheme

$$Z_1 = -\frac{1}{2\varepsilon} + \tfrac{1}{2}(\gamma_E - \ln 4\pi)\,, \tag{5.8}$$

$$Z_2 = -\frac{1}{4\varepsilon^2} + \frac{1}{4\varepsilon} + \frac{1}{2\varepsilon}(\gamma_E - \ln 4\pi) - \tfrac{1}{4}(\gamma_E - \ln 4\pi)^2 - \tfrac{1}{2}(\gamma_E - \ln 4\pi)\,. \tag{5.9}$$

Note that there are no factors of δ [recall eq. (3.11)] in these expressions or in those of eqs. (5.4) and (5.5). These factors cancel when the various contributions of order

$(\lambda/4\pi)^4$ are summed. From eqs. (5.8) and (5.9) we obtain

$$\lambda_{\overline{\rm MS}}=\lambda_{\rm MS}\left[1+\left(\frac{\lambda_{\rm MS}}{4\pi}\right)^2\tfrac{3}{4}(\gamma_{\rm E}-\ln 4\pi)\right.$$
$$\left.+\left(\frac{\lambda_{\rm MS}}{4\pi}\right)^4\left(\tfrac{27}{32}(\gamma_{\rm E}-\ln 4\pi)^2-\tfrac{3}{4}(\gamma_{\rm E}-\ln 4\pi)\right)+\cdots\right]. \tag{5.10}$$

Finally, using eq. (5.3), we find

$$\beta_2^{\overline{\rm MS}}=\beta_2^{\rm MS}. \tag{5.11}$$

Appendix

We present here some details of the momentum integrations we carried out. Except for the integral in H_4, eq. (4.6), all other integrals only involve repeated evaluation of expressions of the form

$$J=\int\frac{{\rm d}^nq}{(2\pi)^n}(q^2)^{-\alpha}[(k+q)^2]^{-\beta}, \tag{A.1}$$

where factors such as $(q^2)^{-\alpha}$ come about from performing a similar integral for a subgraph. Such integrals are easily performed and give [14]

$$J=\frac{\Gamma(\alpha+\beta-\frac{1}{2}n)\Gamma(\frac{1}{2}n-\alpha)\Gamma(\frac{1}{2}n-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}\frac{1}{(4\pi)^{n/2}}\left(\frac{1}{k^2}\right)^{\alpha+\beta-n/2}. \tag{A.2}$$

For the integral in H_4 we use Rosner's technique of expanding the propagators in Chebyshev polynomials [7]. This technique has been generalized to the n-dimensional momentum integration case by Celmaster and Gonsalves [8]. However, since we need only the singular (simple pole) part of H_4, it is sufficient to work in four dimensions and replace the final, divergent momentum integral by an appropriate pole in ε.

To begin we observe that H_4, as given by (4.6) depends only on p^2, so that we can average over the (four-dimensional) directions of p. We write [7]

$$\frac{1}{(p-l)^2}=\frac{1}{pl}\sum_n\langle p|l\rangle^{n+1}C_n\left(\frac{p\cdot l}{pl}\right), \tag{A.3}$$

with

$$\langle p|l\rangle=\frac{p}{l}\quad \text{if } p<l$$
$$=\frac{l}{p}\quad \text{if } p>l, \tag{A.4}$$

and use the orthogonality relations for the Chebyshev polynomials

$$\int \frac{d\Omega_p}{2\pi^2} C_n\left(\frac{p\cdot l}{pl}\right) C_m\left(\frac{p\cdot l}{pl}\right) = \delta_{nm}, \tag{A.5}$$

with $m = 0$, $C_0 = 1$ to deduce

$$\begin{aligned}
H_4(p^2) &= \int \frac{d\Omega_p}{2\pi^2} H_4(p) \\
&= \tfrac{1}{2}\int \frac{d^4k\, d^4l\, d^4q}{(2\pi)^{12}} \frac{\langle p|l\rangle}{plq^2k^2(l-q)^2(l-k)^2(q-k)^2} \\
&= \frac{1}{2}\frac{1}{(8\pi^2)^3}\int_0^\infty \frac{dk}{k}\int_0^\infty \frac{dl}{l}\int_0^\infty \frac{dq}{q}\int \frac{d\Omega_k}{2\pi^2}\frac{d\Omega_l}{2\pi^2}\frac{d\Omega_q}{2\pi^2}\frac{\langle p|l\rangle k^2q^2l^3}{p(l-q)^2} \\
&\quad\times \frac{1}{k^2ql}\sum_{n,m}\langle l|k\rangle^{n+1}\langle q|k\rangle^{m+1} C_n\left(\frac{l\cdot k}{lk}\right) C_m\left(\frac{q\cdot k}{qk}\right) \\
&= \frac{1}{2}\frac{1}{(8\pi^2)^3}\int_0^\infty \frac{dk}{k}\int_0^\infty \frac{dl}{l}\int_0^\infty \frac{dq}{q}\int \frac{d\Omega_q}{2\pi^2}\frac{d\Omega_l}{2\pi^2}\frac{\langle p|l\rangle ql^2}{p(l-q)^2} \\
&\quad\times \sum_n \frac{\langle l|k\rangle^{n+1}\langle q|k\rangle^{n+1}}{n+1} C_n\left(\frac{l\cdot q}{lq}\right) \\
&= \frac{1}{2}\frac{1}{(8\pi^2)^3}\int_0^\infty \frac{dk}{k}\int_0^\infty \frac{dl}{l}\int_0^\infty \frac{dq}{q}\int \frac{d\Omega_q}{2\pi^2}\frac{d\Omega_l}{2\pi^2}\langle p|l\rangle \frac{l}{p} \\
&\quad\times \sum_{n,m} \frac{\langle l|k\rangle^{n+1}\langle q|k\rangle^{n+1}\langle q|l\rangle^{m+1}}{n+1} C_n\left(\frac{l\cdot q}{lq}\right) C_m\left(\frac{l\cdot q}{lq}\right) \\
&= \frac{1}{2}\frac{1}{(8\pi^2)^3}\int_0^\infty \frac{dk}{k}\int_0^\infty \frac{dl}{l}\int_0^\infty \frac{dq}{q}\langle p|l\rangle \frac{l}{p} \\
&\quad\times \sum_n \frac{\left[\langle l|k\rangle\langle q|k\rangle\langle q|l\rangle\right]^{n+1}}{n+1}.
\end{aligned} \tag{A.6}$$

We have used the relation

$$\int \frac{d\Omega_k}{2\pi^2} C_n\left(\frac{l\cdot k}{lk}\right) C_m\left(\frac{q\cdot k}{qk}\right) = \frac{\delta_{nm}}{n+1} C_n\left(\frac{l\cdot q}{lq}\right). \tag{A.7}$$

We have to break now the triple integral into the various regions defined by $l \gtrless p$, $l \gtrless q$, $l \gtrless k$, $k \gtrless q$. However, the only divergence comes from the region where q, k, and l are larger than p and by symmetry of the integrand we can write the relevant contribution as

$$H_4 = \frac{3}{(8\pi^2)^3} \int_p^\infty \mathrm{d}l \int_p^l \mathrm{d}q \int_p^q \mathrm{d}k \sum_n \frac{1}{n+1} \frac{k^{2n+1}}{l^{2n+3} q} + \text{finite}$$

$$= \frac{3}{4(8\pi^2)^3} \int_p^\infty \frac{\mathrm{d}l}{l} \sum_n \frac{1}{(n+1)^3} + \text{finite} . \qquad \text{(A.8)}$$

Since in dimensional regularization

$$\int_p^\infty \frac{\mathrm{d}l}{l} \to \int_p^\infty \frac{\mathrm{d}l}{l^{1+2\varepsilon}} = \frac{1}{2\varepsilon} + \text{finite} , \qquad \text{(A.9)}$$

we have for the divergent part of H_4

$$H_4(\text{div}) = \frac{3}{(4\pi)^6} \zeta(3) \frac{1}{\varepsilon} , \qquad \text{(A.10)}$$

where

$$\zeta(x) = \sum_0^\infty \frac{1}{(n+1)^x} \qquad \text{(A.11)}$$

is the Riemann zeta-function.

References

[1] M.T. Grisaru, M. Rocek and W. Siegel, Nucl. Phys. B159 (1979) 429
[2] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39
[3] A.A. Vladimirov, Theor. Math. Phys. 36 (1978) 732;
F.M. Dittes, Yu.A. Kubyshin and O.V. Tarasov, Theor. Math. Phys. 37 (1978) 879
[4] D.I. Kazakov, O.V. Tarasov and D.V. Shirkov, Theor. Math. Phys. 38 (1979) 9
[5] P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D20 (1979) 1832
[6] W. Siegel, Inconsistency of supersymmetric dimensional regularization, I. A. S. preprint (February, 1980)
[7] J. Rosner, Ann. of Phys. 44 (1976) 11
[8] W. Celmaster and R.J. Gonsalves, Fourth order QCD contribution to the e^+e^- annihilation cross section, UCSD preprint-10P10-207
[9] J. Iliopoulos and B. Zumino, Nucl. Phys. B76 (1974) 310
[10] S. Ferrara, J. Iliopoulos and B. Zumino, Nucl. Phys. B77 (1974) 413
[11] D. Gross, *in* Methods in field theory, ed. C.R. Balian and J. Zinn-Justin (North Holland, 1976)
[12] W. Bardeen, A. Buras, D. Duke and T. Muta, Phys. Rev. D18 (1978) 3998
[13] J. Collins and A. Macfarlane, Phys. Rev. D10 (1974) 1201.
[14] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189