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THE BACKGROUND FIELD METHOD BEYOND ONE LOOP

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The background field approach to multi-loop calculations in gauge field theories is presented. A relation between the gauge-invariant effective action computed using this method and the effective action of the conventional functional approach is derived. Feynman rules are given and renormalization is discussed. It is shown that the renormalization programme can be carried out without any reference to fields appearing inside loops. Finally, as an explicit example, the two-loop contribution to the β function of pure Yang-Mills theory is calculated using the background field method.

1. Introduction

Explicit gauge invariance, which is present at the classical level in gauge field theories, is normally lost when quantum corrections are included. The background field method [1,2] is a technique which allows one to fix a gauge, and thereby compute quantum effects, without losing explicit gauge invariance. It thus makes calculations in gauge theories easier both technically and conceptually. The background field method is used extensively in analyses of gravity and supergravity and has been used by Weinberg [3] to construct light effective field theories from grand unified models. In its original formulation, the method was applicable only to one-loop processes. However, the extension to multi-loop effects has been made by 't Hooft [4] and by DeWitt [5]. (In addition, very recently a discussion of the gauge-invariant effective action by Boulware [6] has appeared.) Here, the background field method which is applicable to multi-loop processes will be presented in detail. The result is a prescription, including Feynman rules and a renormalization scheme, for computing an explicitly gauge-invariant effective action. (Feynman rules for a general gauge theory have also been given by DeWitt [5].) The method is equivalent to that of 't Hooft [4], although the formulation (like that of DeWitt [5]) follows more closely the conventional functional approach.

The basic idea of the background field method is to write the gauge field appearing in the classical action as $A + Q$, where A is the background field and Q is the quantum field which is the variable of integration in the functional integral.

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Then, a gauge is chosen (the background field gauge) which breaks the gauge invariance of the Q field, but retains gauge invariance in terms of the A field. Background field gauge invariance is further assured by coupling external sources only to the Q field [4]. Thus, quantum calculations can be performed, yet explicit gauge invariance in the background field variable is not lost.

The generating functionals and effective action of the conventional functional approach to field theory are reviewed in sect. 2. The analogous quantities used in the background field method are then introduced. They are defined exactly as in the conventional approach except that, as outlined above, the gauge field appearing in the classical action is written as $A + Q$. The generating functionals and effective action thus become functionals of the background field A as well as of their usual arguments. Furthermore, in the background field gauge they are gauge-invariant functionals of A. The gauge-invariant effective action is just the background field effective action considered as a functional of A and evaluated with vanishing quantum field. It is shown in sect. 3 that this gauge-invariant effective action is equal to the conventional effective action evaluated in an unusual, but nevertheless valid gauge. It can thus be used in the normal manner to generate the S-matrix of the theory.

In the background field approach, it is desirable to work only with background field Green functions and not with those of the quantum field, as it is only the background field gauge invariance which is retained. The renormalization programme beyond one loop would then seem to present a problem, since calculation of the quantum field renormalization factor necessarily involves working directly with Q field Green functions. However, as shown in sect. 4, it is not necessary to renormalize the quantum fields. The only renormalizations required are those of the gauge coupling constant, background field and gauge-fixing parameter. Furthermore, the gauge-fixing parameter renormalization can be avoided by going to the Landau-type background field gauge (after the calculation has been performed with an arbitrary bare gauge-fixing parameter; see sect. 4). The coupling constant and background field renormalizations can be determined from the A field two-point function. Thus, it is possible to carry out the renormalization programme without any reference to quantum field Green functions.

Because explicit gauge invariance is retained, the gauge coupling and background field renormalization in the background field approach are related. This allows one to determine the β function from a calculation of the background field two-point function alone; no vertex functions need to be considered. This leads to a considerable simplification in the background field calculation over those performed using conventional methods [7,8]. The one- and two-loop contributions to the β function for pure Yang-Mills theory are calculated using the background field approach in sect. 5. Since the subtleties of the renormalization programme first arise at the two-loop level, it is reassuring to see the formalism verified in an explicit example.

For simplicity, throughout the paper only pure Yang-Mills theory is considered. The inclusion of fermions or scalars is straightforward.

2. The method

In the conventional functional approach to field theory, one defines the generating functional (throughout, the letter Q is used to denote the quantum gauge field which is the variable of integration in the functional integral)

$$Z[J]=\int \delta Q \det\left[\frac{\delta G^a}{\delta\omega^b}\right]\exp i\int d^4x\left[\mathcal{L}(Q)-\frac{1}{2\alpha}(G^a)^2+J^a_\mu Q^a_\mu\right], \tag{2.1}$$

where

$$\mathcal{L}(Q)=-\tfrac{1}{4}\left(F^a_{\mu\nu}\right)^2, \tag{2.2}$$

with

$$F^a_{\mu\nu}=\partial_\mu Q^a_\nu-\partial_\nu Q^a_\mu+gf^{abc}Q^b_\mu Q^c_\nu. \tag{2.3}$$

G^a is the gauge-fixing term (for example, $G^a=\partial_\mu Q^a_\mu$ is a typical choice) and $\delta G^a/\delta\omega^b$ is the derivative of the gauge-fixing term under an infinitesimal gauge transformation

$$\delta Q^a_\mu=-f^{abc}\omega^b Q^c_\mu+\frac{1}{g}\partial_\mu\omega^a. \tag{2.4}$$

The functional derivatives of $Z[J]$ with respect to J are the disconnected Green functions of the theory. The connected Green functions are generated by

$$W[J]=-i\ln Z[J]. \tag{2.5}$$

Finally, one defines the effective action by making the Legendre transformation

$$\Gamma[\bar{Q}]=W[J]-\int d^4x\, J^a_\mu\bar{Q}^a_\mu, \tag{2.6}$$

where

$$\bar{Q}^a_\mu=\frac{\delta W}{\delta J^a_\mu}. \tag{2.7}$$

The derivatives of the effective action with respect to $\bar{Q}$ are the one-particle-irreducible Green functions of the theory.

We now define quantities analogous to Z, W, and Γ in the background field method. We denote these by $\tilde{Z}$, $\tilde{W}$, and $\tilde{\Gamma}$. They are defined exactly like the conventional generating functionals except that the field in the classical lagrangian is written not as Q but as $A+Q$, where A is the background field. Following 't Hooft [4] we do not couple the background field to the source J. Thus, we define

$$\tilde{Z}[J,A]=\int \delta Q \det\left[\frac{\delta G^a}{\delta \omega^b}\right] \exp i\int d^4x\left[\mathcal{L}(A+Q)-\frac{1}{2\alpha}(G^a)^2+J^a_\mu Q^a_\mu\right], \quad (2.8)$$

where $\delta G^a/\delta\omega^b$ is the derivative of the gauge-fixing term under the infinitesimal gauge transformation $\delta Q^a_\mu = -f^{abc}\omega^b(A^c_\mu + Q^c_\mu)+(1/g)\partial_\mu\omega^a$. Then, just as in the conventional approach, we define

$$\tilde{W}[J,A]=-i\ln\tilde{Z}[J,A] \quad (2.9)$$

and the background field effective action

$$\tilde{\Gamma}[\tilde{Q},A]=\tilde{W}[J,A]-\int d^4x J^a_\mu \tilde{Q}^a_\mu, \quad (2.10)$$

where

$$\tilde{Q}^a_\mu=\frac{\delta\tilde{W}}{\delta J^a_\mu}. \quad (2.11)$$

Since there are several field variables being used here, it is worthwhile to summarize them:

Q^a_μ = the quantum field, the variable of integration in the functional integral;

A^a_μ = the background field;

$\bar{Q}^a_\mu = \delta W/\delta J^a_\mu$ = the argument of the conventional effective action, $\Gamma[\bar{Q}]$;

$\tilde{Q}^a_\mu = \delta\tilde{W}/\delta J^a_\mu$ = the quantum field argument of the background field effective, action, $\tilde{\Gamma}[\tilde{Q},A]$.

One now chooses the background field gauge condition

$$G^a=\partial_\mu Q^a_\mu + g f^{abc}A^b_\mu Q^c_\mu \quad (2.12)$$

in eq. (2.8). By making the change of variables $Q^a_\mu \to Q^a_\mu - f^{abc}\omega^b Q^c_\mu$ it is easy to show that $\tilde{Z}[J,A]$ and hence $\tilde{W}[J,A]$ are invariant under the infinitesimal transformations

$$\delta A^a_\mu = -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a,$$
$$\delta J^a_\mu = -f^{abc}\omega^b J^c_\mu \tag{2.13}$$

when this gauge-fixing term is used. It then follows that $\tilde{\Gamma}[\tilde{Q},A]$ is invariant under

$$\delta A^a_\mu = -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a, \tag{2.14}$$

$$\delta \tilde{Q}^a_\mu = -f^{abc}\omega^b \tilde{Q}^c_\mu \tag{2.15}$$

in the background field gauge. In particular, $\tilde{\Gamma}[0,A]$ must be an explicitly gauge-invariant functional of A since (2.14) is just an ordinary gauge transformation of the background field. The quantity $\tilde{\Gamma}[0,A]$ is the gauge-invariant effective action which one computes in the background field method. In sect. 3 it will be shown that $\tilde{\Gamma}[0,A]$ is equal to the usual effective action $\Gamma[\bar{Q}]$, with $\bar{Q}=A$, calculated in an unconventional gauge which depends on A. Thus, $\tilde{\Gamma}[0,A]$ can be used to generate the S-matrix of a gauge theory in exactly the same way as the usual effective action is employed. Furthermore, it is explicitly gauge-invariant. The advantages of this will become apparent when the two-loop β function is calculated in sect. 5.

3. The relation between $\tilde{\Gamma}[0,A]$ and $\Gamma[\bar{Q}]$

We now derive relationships between Z, W, and Γ and the analogous quantities $\tilde{Z}$, $\tilde{W}$, and $\tilde{\Gamma}$ of the background field method. This is done by making the change of variables $Q \to Q - A$ in eq. (2.8). One then finds that when $\tilde{Z}[J,A]$ is calculated in the background field gauge of eq. (2.12),

$$\tilde{Z}[J,A] = Z[J]\exp\left(-i\int d^4x\, J^a_\mu A^a_\mu\right), \tag{3.1}$$

where $Z[J]$ is the conventional generation functional of eq. (2.1) evaluated with the gauge-fixing term

$$G^a = \partial_\mu Q^a_\mu - \partial_\mu A^a_\mu + g f^{abc} A^b_\mu Q^c_\mu. \tag{3.2}$$

One can verify that the ghost determinant of $\tilde{Z}$ in the background field gauge goes over into the correct ghost determinant for Z in the gauge of eq. (3.2). Note that because of the presence of the background field A in the gauge-fixing term (3.2),

$Z[J]$ will actually be a functional of A as well as of J. It follows from eq. (3.1) that W and $\tilde{W}$ are related by

$$\tilde{W}[J,A] = W[J] - \int d^4x J^a_\mu A^a_\mu . \tag{3.3}$$

Like $Z[J]$, $W[J]$ depends on A through the gauge-fixing term. Taking a derivative of (3.3) with respect to J and recalling that $\bar{Q} = \delta W/\delta J$ and $\tilde{Q} = \delta\tilde{W}/\delta J$ we find that

$$\tilde{Q}^a_\mu = \bar{Q}^a_\mu - A^a_\mu . \tag{3.4}$$

Finally, performing a Legendre transformation on the relation (3.3) we have a relation between the background field effective action and the conventional effective action

$$\tilde{\Gamma}[\tilde{Q},A] = \Gamma[\bar{Q}]|_{\bar{Q}=\tilde{Q}+A} . \tag{3.5}$$

The gauge-invariant effective action is just $\tilde{\Gamma}[0,A]$ so from (3.5) we have the identity we need

$$\tilde{\Gamma}[0,A] = \Gamma[\bar{Q}]|_{\bar{Q}=A} . \tag{3.6}$$

In this identity, $\tilde{\Gamma}$ is calculated in the background field gauge of eq. (2.12) and Γ in the gauge of eq. (3.2). Thus, in eq. (3.6), Γ depends on A both through this gauge-fixing term and because $\bar{Q} = A$.

The connection with the formalism of 't Hooft [4] can be made using the above results. First, note that the gauge-invariant effective action, $\tilde{\Gamma}[0,A]$ is given according to (2.10) by

$$\tilde{\Gamma}[0,A] = \tilde{W}[J,A] . \tag{3.7}$$

However, the condition $\tilde{Q} = 0$ must still be imposed. By eq. (3.4), $\tilde{Q} = 0$ is equivalent to $\bar{Q} = A$ which, in turn, implies that A and J are related through the dependence of $\bar{Q}$ on J. Thus, when we take derivatives of W with respect to J, we must include the J dependence which enters through the presence of A in the gauge-fixing term (3.2). With this in mind, the condition $\bar{Q} = A$ is just

$$\frac{\delta W}{\delta J^a_\mu} + \int d^4y \left[\frac{\delta W}{\delta A^b_\nu(y)} \frac{\delta A^b_\nu(y)}{\delta J^a_\mu} \right] = A^a_\mu . \tag{3.8}$$

Finally, using (3.3) to relate W to $\tilde{W}$, eq. (3.8) can be written in the equivalent form

$$\frac{\delta \tilde{W}}{\delta A^a_\mu} + \int d^4y \left[\frac{\delta \tilde{W}}{\delta J^b_\nu(y)} \frac{\delta J^b_\nu(y)}{\delta A^a_\mu} \right] = -J^a_\mu . \tag{3.9}$$

Eqs. (3.7) and (3.9) are the basis of 't Hooft's formulation [4] of the background field method. According to the language used here, (3.7) is just the usual Legendre transformation and (3.9) is the condition $\tilde{Q} = 0$. The formulation given here is thus equivalent to 't Hooft's [4] although it follows more closely the conventional functional approach. One advantage of this is that it allows for an easy derivation of eq. (3.6) relating the background field and usual methods.

Note that because A appears in the gauge condition (3.2) and acts as a source there, the one-particle-irreducible Green functions calculated from the gauge-invariant effective action will be very different from those calculated by conventional methods in normal gauges. Nevertheless, the relation (3.6) assures us that all gauge-independent physical quantities will come out the same in either approach.

4. Feynman rules and renormalization

The gauge-invariant effective action, $\tilde{\Gamma}[0,A]$, is computed by summing all one-particle irreducible diagrams with A fields on external legs and Q fields inside loops. No Q field propagators appear on external lines (since $\tilde{Q} = 0$) and likewise no A field propagators occur inside loops (since the functional integral is only over Q). To derive the Feynman rules one must write the determinant factor appearing in the functional integral in terms of an anticommuting scalar ghost field. From the background field gauge-fixing term of (2.12), using the gauge transformation

$$\delta Q^a_\mu = -f^{abc}\omega^b\left(A^c_\mu + Q^c_\mu\right) + \frac{1}{g}\partial_\mu \omega^a , \tag{4.1}$$

one derives the ghost lagrangian

$$\mathrm{L}_{\text{ghost}} = -\theta^\dagger_a\Big[\Box^2\delta^{ab} - g\overleftarrow{\partial}_\mu f^{acb}\left(A^c_\mu + Q^c_\mu\right) + g f^{acb} A^c_\mu \overrightarrow{\partial}_\mu$$
$$+ g^2 f^{acx} f^{xdb} A^c_\mu\left(A^d_\mu + Q^d_\mu\right)\Big]\theta_b . \tag{4.2}$$

The complete Feynman rules are those given in fig. 1. Wavy lines represent quantum gauge propagators whereas wavy lines terminating in an A denote the external background field. Ghost propagators are represented by dashed lines. Because the effective action involves only one-particle-irreducible diagrams, vertices with only one outgoing quantum line will never contribute. Consequently, they have not been included in the Feynman rules.

It was pointed out in the introduction that the renormalization of quantum fields was undesirable because it required the calculation of quantum field Green functions. However, since the ghost and quantum gauge fields appear only inside loops, it is not necessary to renormalize them and, in fact, they are best left as bare quantities. To see this, imagine that we *did* renormalize these fields θ and Q by writing

$$\theta_0 = Z_\theta^{1/2}\theta, \qquad Q_0 = Z_Q^{1/2}Q. \tag{4.3}$$

Then, one has a factor of $Z_Q^{1/2}$ at each end of a gauge propagator coming from renormalizing the field at each vertex, and a factor Z_Q^{-1} from renormalizing the propagator. The two factors $Z_Q^{1/2}$ and the Z_Q^{-1} associated with each propagator then cancel exactly. Likewise, the two $Z_\theta^{1/2}$ factors at the ends of each ghost line cancel with the Z_θ^{-1} renormalization of the ghost propagator. Thus, the renormalization of eq. (4.3) is completely irrelevant and it is better to leave the ghost and quantum gauge fields unrenormalized. However, it is still necessary to renormalize the gauge-fixing parameter for the quantum gauge field due to the fact that the longitudinal part of the gauge field propagator is not renormalized. Thus, coupling constant, background field and gauge-fixing parameter renormalizations given by

$$g_0 = Z_g g, \qquad A_0 = Z_A^{1/2}A, \qquad \alpha_0 = Z_\alpha \alpha \tag{4.4}$$

are required.

In sect. 5, the renormalizations of eq. (4.4) are explicitly carried out in the Feynman-type gauge $\alpha = 1$. However, in principle, it is possible to completely avoid the gauge-fixing parameter renormalization by calculating with an arbitrary bare gauge-fixing parameter α_0. Then, one can either extract physical quantities which are independent of α_0 or one can go to the Landau-type gauge $\alpha_0 = \alpha = 0$. In either case, the renormalization of the gauge-fixing parameter need not be performed. Because of the presence of vertices proportional to $1/\alpha$, one cannot go to the limit $\alpha = 0$ until after the calculation has been performed and all $1/\alpha$ factors have cancelled. Thus, during the calculation, an arbitrary α_0 should be retained.

Because explicit gauge invariance is retained in the background field method, the renormalization factors Z_A and Z_g are related. The infinities appearing in the gauge-invariant effective action $\tilde{\Gamma}[0,A]$ must take the gauge-invariant form of a divergent constant times $(F^a_{\mu\nu})^2$. Now, according to (4.4), $F^a_{\mu\nu}$ is renormalized by

$$\left(F^a_{\mu\nu}\right)_0 = Z_A^{1/2}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gZ_g Z_A^{1/2} f^{abc}A^b_\mu A^c_\nu\right]. \tag{4.5}$$

This will only take on the gauge-covariant form of a constant time $F^a_{\mu\nu}$ if

$$Z_g = Z_A^{-1/2}. \tag{4.6}$$

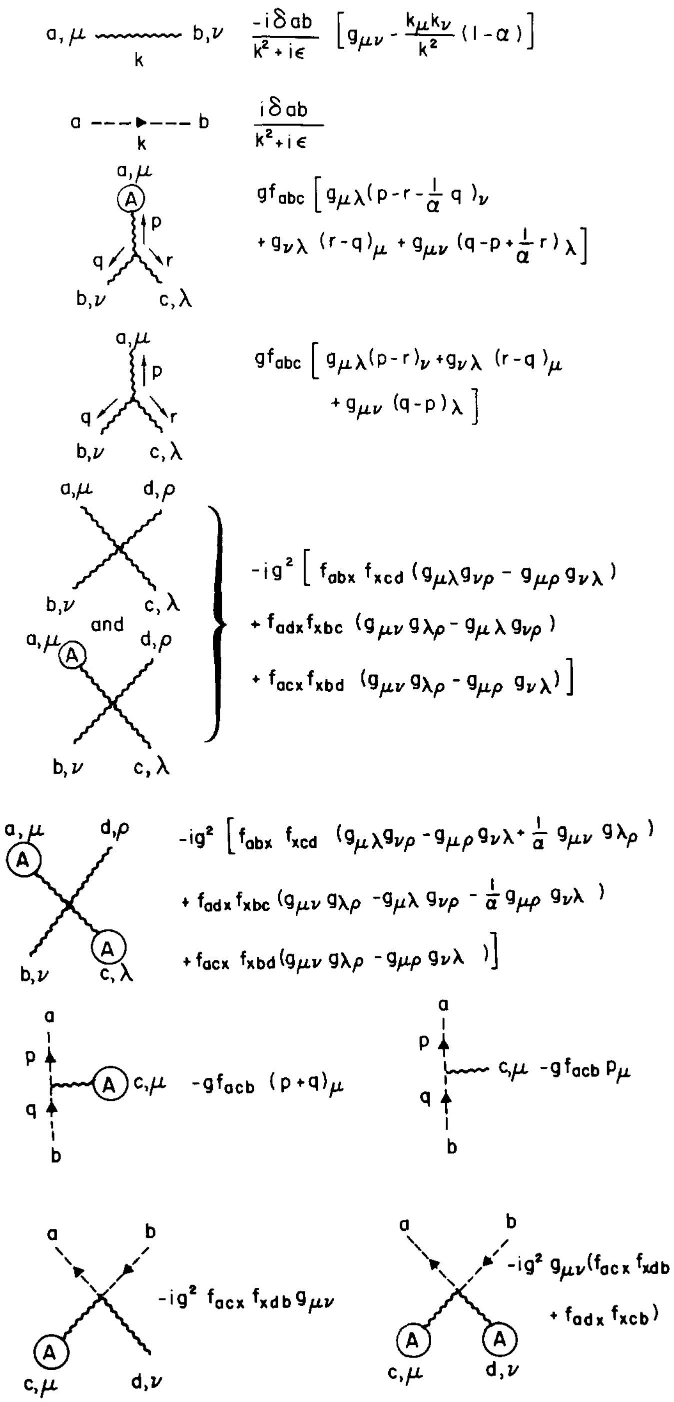

Fig. 1. Feynman rules for background field calculations in Yang-Mills theory. Wavy lines are quantum gauge propagators, wavy lines ending with an A are external background fields and dashed lines are ghost propagators.

This is the relation between the charge and background field renormalizations in the background field gauge.

5. Calculation of the two-loop β function

As an explicit example of the background field formalism in use, we now compute the β function for pure Yang-Mills theory up to the two-loop level. The β function is related to the dependence of the coupling constant renormalization, Z_g, on the renormalization mass parameter, μ, by

$$\beta = -g\mu \frac{\partial}{\partial \mu} \ln Z_g . \tag{5.1}$$

Likewise, the anomalous dimension γ_A is defined by

$$\gamma_A = \tfrac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_A . \tag{5.2}$$

Because $Z_g = Z_A^{-1/2}$ in the background field method, γ_A and β are related by

$$\beta = g\gamma_A . \tag{5.3}$$

The β function can thus be determined by calculating Z_A which only requires a knowledge of the background field two-point function. In contrast to conventional methods, no vertex functions need to be considered. The great simplification provided by the method thus becomes apparent. In previous calculations [7,8], the gauge propagator, ghost propagator and gauge-ghost-ghost vertex all had to be computed. Here, only the gauge propagator is needed.

We will use the dimensional regularization procedure [9] in $4-2\varepsilon$ dimensions and the minimal subtraction scheme [10] in which Z_A is written as a series of poles in ε

$$Z_A = 1 + \sum_{i=1}^{\infty} \frac{Z_A^{(i)}}{(\varepsilon)^i} . \tag{5.4}$$

By using the chain rule of differentiation on eq. (5.2) and the result

$$\mu \frac{\partial}{\partial \mu} g = -\varepsilon g + \beta , \tag{5.5}$$

one can derive a relationship between the various $Z_A^{(i)}$, namely [11]

$$\left(2\gamma_A - \beta \frac{\partial}{\partial g}\right) Z_A^{(i)} = -g \frac{\partial}{\partial g} Z_A^{(i+1)} \tag{5.6}$$

or using (5.3), in the background field method,

$$\beta\left(2-g\frac{\partial}{\partial g}\right)Z_A^{(i)}=-g^2\frac{\partial}{\partial g}Z_A^{(i+1)}. \tag{5.7}$$

Recalling that $Z_A^{(0)}=1$, this gives

$$\beta=-\tfrac{1}{2}g^2\frac{\partial}{\partial g}Z_A^{(1)}. \tag{5.8}$$

Also note that for the piece of $Z_A^{(1)}$ proportional to g^2, $(2-g\partial/\partial g)Z_A^{(1)}=0$. This means that the term in $Z_A^{(2)}$ proportional to g^4 will be zero. Thus, in our calculation of Z_A, there will be no $1/\varepsilon^2$ pole at the two-loop level. From this fact and eq. (5.8) we find that up to two loops, if we write the β function as

$$\beta=-g\left[\beta_0\left(\frac{g}{4\pi}\right)^2+\beta_1\left(\frac{g}{4\pi}\right)^4\right], \tag{5.9}$$

then Z_A must be

$$Z_A=1+\frac{\beta_0}{\varepsilon}\left(\frac{g}{4\pi}\right)^2+\frac{\beta_1}{2\varepsilon}\left(\frac{g}{4\pi}\right)^4. \tag{5.10}$$

The diagrams needed to compute Z_A at the one-loop level are given in fig. 2. The divergent contribution of fig. 2a is

$$\frac{ig^2C_A\delta^{ab}}{(4\pi)^2}\left(\frac{1}{3\varepsilon}\right)\left[g_{\mu\nu}k^2-k_\mu k_\nu\right], \tag{5.11}$$

and that of fig. 2b

$$\frac{ig^2C_A\delta^{ab}}{(4\pi)^2}\left(\frac{10}{3\varepsilon}\right)\left[g_{\mu\nu}k^2-k_\mu k_\nu\right]. \tag{5.12}$$

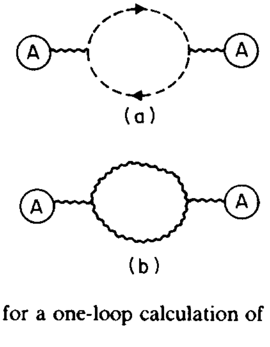

Fig. 2. Graphs for a one-loop calculation of the β function.

Adding these together, one determines Z_A and hence the well-known one-loop result $\beta_0 = \frac{11}{3} C_A$ [7].

The two-loop graphs for computing Z_A are given in fig. 3. The divergent contributions are all of the form

$$\frac{ig^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left[A g_{\mu\nu} k^2 - B k_\mu k_\nu\right]. \tag{5.13}$$

The individual contributions to A and B as well as the totals are given in table 1. In table 1, $\rho = \gamma_E - \ln 4\pi + \ln(k^2/\mu^2)$. Because the calculation was performed in the Feynman-type gauge, $\alpha = 1$, the gauge-fixed renormalization insertion diagrams of figs. 3l and 3m are included. The renormalization factor Z_α is determined from the

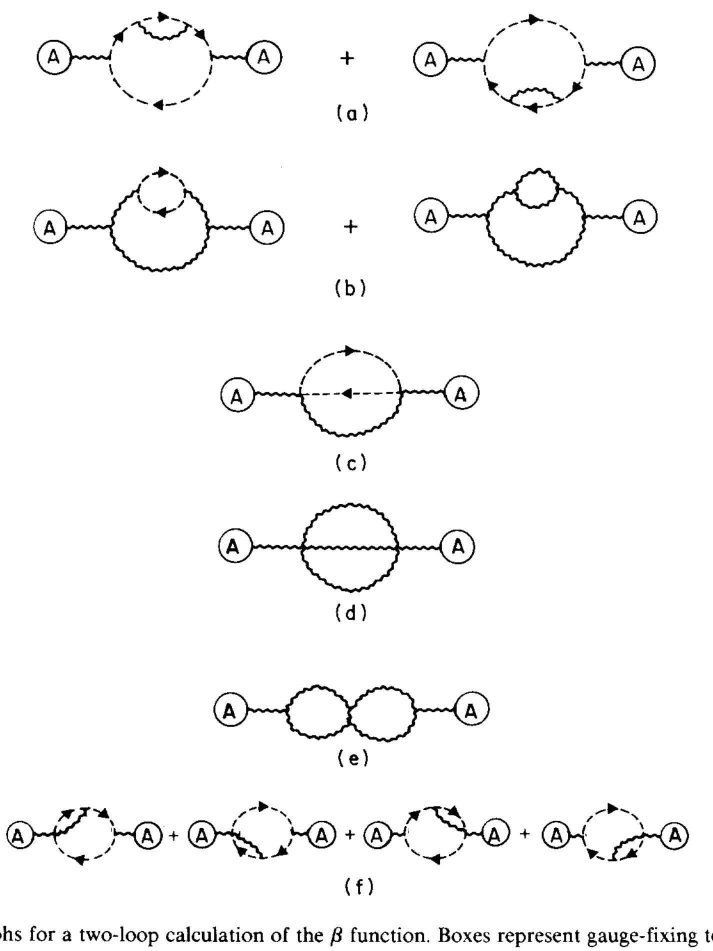

Fig. 3. Graphs for a two-loop calculation of the β function. Boxes represent gauge-fixing term insertions resulting from renormalization of the gauge-fixing parameter.

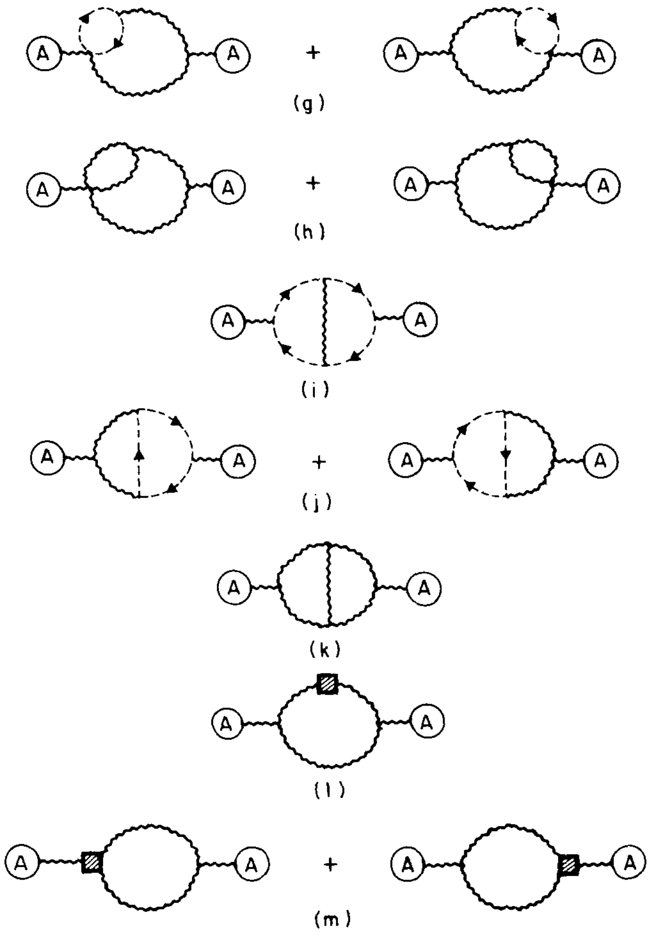

Fig. 3. (continued)

quantum gauge field propagator corrections appearing in fig. 3b. (Recall that if the calculation had been performed with an arbitrary α_0 rather than in the Feynman-type gauge $\alpha = 1$, this step could have been avoided.) It is

$$Z_\alpha = 1 + \left(\frac{5}{3\varepsilon}\right)\frac{g^2 C_A}{(4\pi)^2} \tag{5.14}$$

The insertions result from a counter-term of the form

$$\frac{1}{2}\left(\frac{5g^2C_A}{3(4\pi)^2\varepsilon}\right)\left(\partial_\mu Q_\mu^a + gf^{abc}A_\mu^b Q_\mu^c\right)^2 \tag{5.15}$$

TABLE 1
Contributions to eq. (5.13) of the text from the graphs of fig. 3: $\rho = \gamma_E - \ln 4\pi + \ln(k^2/\mu^2)$

Graph	A	B
a	$\frac{1}{6\varepsilon^2}(1+\frac{13}{2}\varepsilon-2\rho\varepsilon)$	$\frac{1}{6\varepsilon^2}(1+6\varepsilon-2\rho\varepsilon)$
b	$\frac{25}{6\varepsilon^2}(1+\frac{43}{10}\varepsilon-2\rho\varepsilon)$	$\frac{25}{6\varepsilon^2}(1+\frac{22}{5}\varepsilon-2\rho\varepsilon)$
c	$\frac{1}{8\varepsilon}$	0
d	$-\frac{9}{8\varepsilon}$	0
e	$-\frac{6}{\varepsilon^2}(1+4\varepsilon-2\rho\varepsilon)$	$-\frac{6}{\varepsilon^2}(1+4\varepsilon-2\rho\varepsilon)$
f	$-\frac{1}{6\varepsilon^2}(1+\frac{13}{2}\varepsilon-2\rho\varepsilon)$	$-\frac{1}{6\varepsilon^2}(1+\frac{9}{2}\varepsilon-2\rho\varepsilon)$
g	$-\frac{5}{24\varepsilon^2}(1+\frac{41}{10}\varepsilon-2\rho\varepsilon)$	$-\frac{5}{24\varepsilon^2}(1+\frac{9}{2}\varepsilon-2\rho\varepsilon)$
h	$-\frac{9}{8\varepsilon^2}(1+\frac{19}{6}\varepsilon-2\rho\varepsilon)$	$-\frac{9}{8\varepsilon^2}(1+\frac{31}{6}\varepsilon-2\rho\varepsilon)$
i	$\frac{1}{24\varepsilon^2}(1+\frac{19}{2}\varepsilon-2\rho\varepsilon)$	$\frac{1}{24\varepsilon^2}(1+\frac{15}{2}\varepsilon-2\rho\varepsilon)$
j	$-\frac{1}{4\varepsilon^2}(1+\frac{9}{2}\varepsilon-2\rho\varepsilon)$	$-\frac{1}{4\varepsilon^2}(1+\frac{9}{2}\varepsilon-2\rho\varepsilon)$
k	$\frac{27}{8\varepsilon^2}(1+\frac{233}{54}\varepsilon-2\rho\varepsilon)$	$\frac{27}{8\varepsilon^2}(1+\frac{245}{54}\varepsilon-2\rho\varepsilon)$
l	$\frac{25}{9\varepsilon^2}(1+\frac{46}{15}\varepsilon-\rho\varepsilon)$	$\frac{25}{9\varepsilon^2}(1+\frac{46}{15}\varepsilon-\rho\varepsilon)$
m	$-\frac{25}{9\varepsilon^2}(1+\frac{28}{15}\varepsilon-\rho\varepsilon)$	$-\frac{25}{9\varepsilon^2}(1+\frac{28}{15}\varepsilon-\rho\varepsilon)$
Total	$\frac{17}{3\varepsilon}$	$\frac{17}{3\varepsilon}$

arising from the renormalization of α_0 by (5.14). Because of the relation $Z_g = Z_A^{-1/2}$, there are no counter-term insertion diagrams from the coupling and background field renormalizations of fig. 2. From the totals given in table 1 and eq. (5.10) one determines that $\beta_1 = \frac{34}{3} C_A^2$ in agreement with previous results [8]. The algebraic manipulation programme Schoonschip [12] was used in the evaluation of fig. 3k.

A curious feature of this calculation is that the leading pole ($1/\varepsilon$ at one loop and $1/\varepsilon^2$ at two loops) takes the transverse form $[g_{\mu\nu}k^2 - k_\mu k_\nu]$ diagram by diagram as can be verified from eqs. (5.11) and (5.12) and table 1. In addition, the sum of the non-insertion diagrams and the insertion diagrams in fig. 3 separately are of the form $[g_{\mu\nu}k^2 - k_\mu k_\nu]$. This remarkable uniformity is probably a result of both background field techniques and the choice of the Feynman-type gauge.

6. Conclusions

The background field formalism for generating a gauge-invariant effective action* has been presented and employed in an explicit example. Although the Feynman-type gauge may be easier for calculations, it is important that, in principle, the renormalization programme can be carried out without reference to any renormalization of fields inside loops. Sect. 5 gave an example of the simplifications provided by the method for gauge theory calculations.

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* A gauge-covariant two-loop effective action has been computed in the background field gauge in ref. [13]. In this work, the background field is treated exactly and is required to be covariantly constant. In this paper, as in refs. [1–6], the treatment of the background field is quite different: the background field is allowed to be arbitrary and is considered perturbatively as a source for the gauge field.