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STABILITY OF GRAVITY WITH A COSMOLOGICAL CONSTANT

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The stability properties of Einstein theory with a cosmological constant Λ are investigated. For $\Lambda > 0$, stability is established for small fluctuations, about the de Sitter background, occurring inside the event horizon and semiclassical stability is analyzed. For $\Lambda < 0$, stability is demonstrated for all asymptotically anti-de Sitter metrics. The analysis is based on the general construction of conserved flux-integral expressions associated with the symmetries of a chosen background. The effects of an event horizon, which lead to Hawking radiation, are expressed for general field hamiltonians. Stability for $\Lambda < 0$ is proved, using supergravity techniques, in terms of the graded anti-de Sitter algebra with spinorial charges also expressed as flux integrals.

1. Introduction

The vacuum energy density of the universe is known to be of magnitude less than $(0.003\ \mathrm{eV})^4$. To achieve such a small energy density in the standard models of strong, weak and electromagnetic interactions requires extraordinary and unnatural fine tuning of parameters. This fine tuning problem leads us to wonder why the cosmological constant (defined as $8\pi G$ times the vacuum energy density) is so small. In particular, one might ask whether theories with a non-vanishing cosmological constant can be excluded because of some fundamental instability. With this motivation, we examine here their stability properties.

In flat space, stability can be established for a system by proving it has positive energy, the vacuum being defined as the lowest energy state. For gravity, with $\Lambda = 0$, the situation is more complicated because energy is defined with respect to a flat background space. Nevertheless, for asymptotically flat metrics, it is now well established [1–4] that all the desired stability criteria are met, including positivity of the total energy. When a cosmological constant is present, flat space is no longer a relevant background, since it is not a solution of the Einstein equations. It is replaced as the vacuum by either de Sitter [O(4,1)] or anti-de Sitter [O(3,2)] space depending on whether Λ is positive or negative. Since these spaces have non-vanishing constant curvature, there is no asymptotic Poincaré invariance and the conventional energy is not conserved. Instead, the appropriate symmetries are those of the de Sitter or anti-de Sitter group generated by the five-dimensional "rotation"

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operators J_{ab} $(a,b=0,1,\dots,4)$. The question is whether J_{04}, which becomes P_0 as $\Lambda \to 0$, provides a reasonable substitute for the energy. In sect. 2 we will show that it does because J_{04} (unlike J_{ij}, for example) is associated with a Killing vector which is timelike. We will see, however, that for de Sitter space the presence of an event horizon limits the region in which a timelike Killing vector can be defined and thus where stability can be tested in the usual way. We will show that the "Killing energy", J_{04}, has the same desirable properties, for a gravitational system, that the normal energy has in an asymptotically flat space. It is de Sitter (rather than Poincaré) covariant. It can be written as a flux integral over a two-dimensional spatial surface, and it is conserved. These features are demonstrated in sects. 2 and 3, which are also intended to provide a self-contained review of the relevant formalism. The results obtained are, in fact, applicable to all background metrics which solve the field equations and have Killing symmetries.

Once an energy-like quantity (in our case the Killing energy) has been defined, there are several stability tests which can be performed. First, one can check that small oscillations have positive Killing energy. We do this in sect. 4. For de Sitter space, we find that this is the case for excitations inside the event horizon, but not for those beyond it. This is a reflection, in hamiltonian form, of a feature which leads to Hawking radiation [5]. For anti-de Sitter space, small oscillations are everywhere positive. One can also check for semiclassical stability by looking for euclidean "bounce" solutions [6–8] which would signal an instability to quantum-mechanical tunnelling. In sect. 5 we find no evidence of such an instability in de Sitter space. Finally, stability can be established by proving that the Killing energy is positive for all asymptotically vanishing fluctuations, large or small. This is done for the case of anti-de Sitter space in sect. 6, using supergravity techniques which have been used to establish positivity [2] of the energy for $\Lambda = 0$. Throughout this work we will, for simplicity, treat gravity without sources. However, additional couplings to normal matter do not change our arguments about the existence of conserved flux-integrals or about stability.

Thus, while it would have been appealing to be able to exclude the cosmological constant on stability grounds, our results indicate that the situation for $\Lambda \neq 0$ is quite similar to $\Lambda = 0$. To be sure, there are global peculiarities of spaces with a cosmological constant [9], e.g., the event horizon in de Sitter space and the absence of a global Cauchy surface in anti-de Sitter space. However, from the point of view of stability, cosmological theories cannot be excluded.

2. Definition of conserved quantities

In this section we will construct conserved quantities corresponding to symmetries of a background metric satisfying the Einstein equations with a cosmological term,

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0 \tag{2.1}$$

(conventions are $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \sim +\partial_\alpha \Gamma^{\alpha}_{\mu\nu}$, signature $-+++$). Our main interest is with a background de Sitter or anti-de Sitter metric; however the formalism is completely general. These conserved quantities are constructed from the gravitational energy-momentum tensor and the Killing vectors of the background metric and have the essential property that they can be expressed as two-dimensional flux integrals*.

We divide the metric tensor into two parts,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.2}$$

where $\bar{g}_{\mu\nu}$ is any solution of (2.1), for example a de Sitter or anti-de Sitter metric, and $h_{\mu\nu}$ represents deviations which vanish at infinity. The energy-momentum tensor for the gravitational field is defined by expanding the left-hand side of eq. (2.1) into a piece independent of $h_{\mu\nu}$ [which vanishes because $\bar{g}_{\mu\nu}$ itself satisfies the field equation (2.1)], a piece linear in $h_{\mu\nu}$, and terms of quadratic and higher order in $h_{\mu\nu}$. To facilitate this decomposition, we will use the convention throughout this paper that all operations such as index moving or differentiation are with respect to $\bar{g}_{\mu\nu}$. Once this division of eq. (2.1) is made, we define all the terms of second and higher order in $h_{\mu\nu}$ to be the gravitational energy-momentum tensor and write eq. (2.1) as

$$R_{\mathrm{L}}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu} R_{\mathrm{L}} - \Lambda h^{\mu\nu} = (-\bar{g})^{-1/2} T^{\mu\nu}. \tag{2.3}$$

The subscript L refers to terms linear in $h_{\mu\nu}$, and $T^{\mu\nu}$ is the energy-momentum tensor *density* of the gravitational field. Because the l.h.s. of eq. (2.3) obeys the background Bianchi identity, $\bar{D}_\mu(R_{\mathrm{L}}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu} R_{\mathrm{L}} - \Lambda h^{\mu\nu}) = 0$, we find by virtue of the field equations the exact result

$$\bar{D}_\mu T^{\mu\nu} = 0, \tag{2.4}$$

where $\bar{D}_\mu$ is the covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$. However, the derivative in eq. (2.4) is a background covariant, not an ordinary derivative; furthermore, only integrals over divergences of contravariant vector densities have invariant content, so eqs. (2.3) and (2.4) cannot be used directly to construct conserved quantities. This problem is easily resolved. Let us introduce a Killing vector $\bar{\xi}_\mu$ associated with one of the symmetries of the background metric $\bar{g}_{\mu\nu}$; it satisfies

$$\bar{D}_\mu \bar{\xi}_\nu + \bar{D}_\nu \bar{\xi}_\mu = 0. \tag{2.5}$$

* Note that the other conserved local quantity in gravity, the Bel-Robinson tensor, does not lead to useful integrated constants because it lacks an underlying Bianchi identity. It is therefore only conserved with respect to the full (rather than background) metric, except in linearized theory [10].

Then, since

$$T^{\mu\nu}=T^{\nu\mu}, \tag{2.6}$$

we find that

$$\bar{D}_\mu\left(T^{\mu\nu}\bar{\xi}_\nu\right)=\left(\bar{D}_\mu T^{\mu\nu}\right)\bar{\xi}_\nu+\tfrac{1}{2}T^{\mu\nu}\left(\bar{D}_\mu\bar{\xi}_\nu+\bar{D}_\nu\bar{\xi}_\mu\right)=0. \tag{2.7}$$

Now the quantity $T^{\mu\nu}\bar{\xi}_\nu$ is a vector density whose covariant divergence becomes an ordinary one, and gives the desired conservation law,

$$\bar{D}_\mu\left(T^{\mu\nu}\bar{\xi}_\nu\right)=\partial_\mu\left(T^{\mu\nu}\bar{\xi}_\nu\right)=0. \tag{2.8}$$

If $h_{\mu\nu}$ vanishes sufficiently rapidly at spatial infinity then, as usual, we find that

$$\int \mathrm{d}^3x\, T^{0\nu}\bar{\xi}_\nu \tag{2.9}$$

is time dependent. Thus, associated with any Killing vector $\bar{\xi}_\mu$ is a conserved quantity defined by

$$E(\bar{\xi})=\frac{1}{8\pi G}\int \mathrm{d}^3x\, T^{0\nu}\bar{\xi}_\nu. \tag{2.10}$$

If $\bar{\xi}_\mu$ is a timelike vector, this quantity is just what we refer to as the Killing energy.

When $\Lambda=0$, it is well known that the energy-momentum tensor for the gravitational field can be written in terms of a superpotential and that the energy can be expressed as a flux integral over a two-dimensional spatial surface. The same statements apply for $\Lambda\neq 0$. Here, the energy-momentum tensor can be written, using (2.3), as follows:

$$(-\bar{g})^{-1/2}T^{\mu\nu}=\bar{D}_\alpha\bar{D}_\beta K^{\mu\alpha\nu\beta}+X^{\mu\nu}. \tag{2.11}$$

The superpotential $K^{\mu\alpha\nu\beta}$ is

$$K^{\mu\alpha\nu\beta}=\tfrac{1}{2}\left[\bar{g}^{\mu\beta}H^{\nu\alpha}+\bar{g}^{\nu\alpha}H^{\mu\beta}-\bar{g}^{\mu\nu}H^{\alpha\beta}-\bar{g}^{\alpha\beta}H^{\mu\nu}\right], \tag{2.12}$$

where

$$H^{\mu\nu}=h^{\mu\nu}-\tfrac{1}{2}\bar{g}^{\mu\nu}h^{\alpha}{}_{\alpha}. \tag{2.13}$$

It has the algebraic symmetries of the Riemann tensor,

$$K^{\mu\alpha\nu\beta}=K^{\nu\beta\mu\alpha}=-K^{\alpha\mu\nu\beta}=-K^{\mu\alpha\beta\nu}. \tag{2.14}$$

The additional term $X^{\mu\nu}$ can be written in several equivalent ways. The first is the most immediate from the field equation expansion:

$$X^{\mu\nu}=\tfrac{1}{2}\left[\bar{D}_\alpha,\bar{D}^\nu\right]H^{\mu\alpha}-\Lambda H^{\mu\nu}=X^{\nu\mu}. \tag{2.15a}$$

Alternatively, one may replace the commutator by a curvature to obtain a manifestly $(\mu\nu)$ symmetric form,

$$X^{\mu\nu}=\tfrac{1}{2}\left[\bar{R}^\mu{}_{\alpha\beta}{}^\nu H^{\alpha\beta}-\Lambda H^{\mu\nu}\right] \tag{2.15b}$$

Finally, a form involving only K is obtained algebraically from (2.15b) using (2.12) and the background equations $\bar{R}_{\mu\nu}=\Lambda\bar{g}_{\mu\nu}$:

$$X^{\mu\nu}=\tfrac{1}{2}\bar{R}^\nu{}_{\lambda\alpha\beta}K^{\mu\lambda\alpha\beta}. \tag{2.15c}$$

It can be checked explicitly that each term on the right of (2.11) is separately symmetric so that $T^{\mu\nu}=T^{\nu\mu}$, and that $T^{\mu\nu}$ is conserved ($\bar{D}_\mu T^{\mu\nu}=0$), provided both the background field equation and its derivative consequences,

$$\bar{D}_\beta\bar{R}^{\mu\nu\alpha\beta}=\bar{D}^\nu\bar{R}^{\alpha\mu}-\bar{D}^\mu\bar{R}^{\alpha\nu}=0=\bar{D}_\beta\bar{R}^{\mu\nu}, \tag{2.16}$$

are used.

To show that (2.10) is actually a flux integral, multiply (2.11) by $\bar{\xi}_\nu$:

$$(-\bar{g})^{-1/2}T^{\mu\nu}\bar{\xi}_\nu=\bar{D}_\alpha\left[\left(\bar{D}_\beta K^{\mu\alpha\nu\beta}\right)\bar{\xi}_\nu-K^{\mu\beta\nu\alpha}\bar{D}_\beta\bar{\xi}_\nu\right]+\left[K^{\mu\alpha\nu\beta}\bar{D}_\beta\bar{D}_\alpha+X^{\mu\nu}\right]\bar{\xi}_\nu. \tag{2.17}$$

The Killing vector identity,

$$\bar{D}_\beta\bar{D}_\alpha\bar{\xi}_\nu=\bar{R}^\lambda{}_{\beta\alpha\nu}\bar{\xi}_\lambda, \tag{2.18}$$

together with (2.12) and (2.15b) then removes the last term in (2.17) leaving a total divergence. Furthermore, the latter is of the form $\bar{D}_\nu F^{\mu\nu}$, where $F^{\mu\nu}$ is an antisymmetric tensor density*, and the divergence, therefore, becomes an ordinary one. This means that $E(\bar{\xi})$ has the flux integral form

$$\begin{aligned}E(\bar{\xi})&=\frac{1}{8\pi G}\int \mathrm{d}^3x\, T^{0\nu}\bar{\xi}_\nu\\&=\frac{1}{8\pi G}\oint \mathrm{d}S_i\sqrt{-\bar{g}}\left[\bar{D}_\beta K^{0i\nu\beta}-K^{0j\nu i}\bar{D}_j\right]\bar{\xi}_\nu.\end{aligned} \tag{2.19}$$

* Antisymmetry follows from that of $\bar{D}_\alpha\bar{\xi}_\nu$ and the properties of (2.14).

When $\Lambda = 0$ and the background is chosen to be flat, introduction of cartesian coordinates simplifies (2.19) to be the usual form for the energy,

$$E = \frac{1}{16\pi G} \oint \mathrm{d}S_i \left[\partial_j h_{ij} - \partial_i h_{jj} \right], \tag{2.20}$$

when $\bar{\xi}_\mu$ is taken to be the timelike Killing vector $(1,\mathbf{0})$.

The expression (2.19) is actually generic for any Killing generator. In particular, for $\Lambda = 0$, it yields the other Poincaré generators $(P_i, J_{\mu\nu})$ as $\bar{\xi}$ ranges over the other nine Killing vectors of flat space, and automatically ensures that they obey the global Poincaré algebra.

We may also use the general form (2.19) with $\Lambda \neq 0$ to express the ten Killing generators corresponding to the symmetries of a background de Sitter or anti-de Sitter space, in which case they will automatically satisfy the appropriate global algebra.

A final remark concerns the relation between (2.19) and the canonical framework of sect. 4. The latter is simply a first order way of obtaining $E(\bar{\xi})$ by taking the parts of the Einstein constraints linear in (h_{ij}, p^{ij}) and multiplying them by $\bar{\xi}^\mu$.

When $\Lambda < 0$, the anti-de Sitter algebra can be graded by introducing spinor charges Q. The resulting local supersymmetry gives rise to supergravity with a cosmological constant [11] and a spin $\frac{3}{2}$ "mass" [12] term. Grading of the de Sitter algebra is not possible. In terms of supergravity, this is because the Rarita-Schwinger lagrangian acquires a "mass" $\sqrt{-\frac{1}{3}\Lambda}$, so when $\Lambda > 0$ the resulting mass term would not be hermitian. Also, in the O(4,1) de Sitter space there are no Majorana fermions, so the real gravitational field cannot have a fermionic partner. In sect. 6 we will use the graded algebra to show that J_{04} is positive and thus to establish stability. However, in preparation, we must demonstrate that the spinor charges of supergravity can be written as surface integrals as in eq. (2.19) so that they will satisfy the graded anti-de Sitter algebra for asymptotically anti-de Sitter metrics. The spinor charge can be written in terms of the vector-spinor density

$$Q^\mu = \varepsilon^{\mu\alpha\beta\nu} \gamma_5 \bar{\gamma}_\alpha \tilde{D}_\beta \psi_\nu , \tag{2.21}$$

which is just the Rarita-Schwinger equation divided into linear and non-linear parts, as in (2.3) for gravity. Here ψ_ν is the spin $\frac{3}{2}$ field, $\bar{\gamma}_\alpha$ are the background covariant γ matrices defined through the background vierbein and

$$\tilde{D}_\beta = \bar{D}_\beta + \tfrac{1}{2} m \bar{\gamma}_\beta , \qquad m^2 = \tfrac{1}{3}|\Lambda| . \tag{2.22}$$

The current Q^μ satisfies $\tilde{D}_\mu Q^\mu = 0$, but as for $T^{\mu\nu}$ we must construct a contravariant vector density from Q^μ which will satisfy ordinary conservation. The remedy is again to introduce an appropriate Killing quantity, in this case a spinor satisfying the equation

$$\tilde{D}_\mu \alpha = 0. \tag{2.23}$$

This equation has non-vanishing solutions because the $\tilde{D}_\mu$ satisfy

$$\left[\tilde{D}_\mu, \tilde{D}_\nu\right]\psi = 0 \tag{2.24}$$

for any spinor ψ. Now from (2.23) and the definition of (2.21) we find that

$$\bar{\alpha}Q^\mu = \bar{\alpha}\varepsilon^{\mu\alpha\beta\nu}\gamma_5\bar{\gamma}_\alpha\tilde{D}_\beta\psi_\nu = \bar{D}_\beta\left(\bar{\alpha}\varepsilon^{\mu\alpha\beta\nu}\gamma_5\bar{\gamma}_\alpha\psi_\nu\right)$$
$$= \partial_\beta\left(\bar{\alpha}\varepsilon^{\mu\alpha\beta\nu}\gamma_5\bar{\gamma}_\alpha\psi_\nu\right). \tag{2.25}$$

This last equality follows from the fact that the quantity in parentheses is an antisymmetric tensor density. Now from (2.25) and the antisymmetry of the ε tensor, it follows immediately that

$$\partial_\mu(\bar{\alpha}Q^\mu) = 0. \tag{2.26}$$

As a result the spinor charge

$$Q(\alpha) = \int \mathrm{d}^3x\, \bar{\alpha}Q^0 \tag{2.27}$$

is a conserved quantity. Furthermore, from (2.25) it can be written as a two-dimensional surface integral:

$$Q(\alpha) = \oint \mathrm{d}S_j \bar{\alpha}\varepsilon^{0ijk}\gamma_5\bar{\gamma}_i\psi_k. \tag{2.28}$$

This is the desired analogue of eq. (2.19) for the bosonic generators. We will choose the parameters α to be commuting spinors (for convenience) in order to preserve the anticommuting character of $Q(\alpha)$.

It is essential to note that the spinor α plays the same role as the Killing vectors play for the bosonic generators. Just as there are ten independent Killing vectors, $\bar{\xi}_\mu^{(ab)}$, each of which defines a particular generator $J_{(ab)}$, there are four independent Killing spinors $\alpha^{(\beta)}$ which define four fermionic charges $Q_{(\beta)}$, $\beta = 1,\dots,4$. The labels (ab) on $J_{(ab)}$ and (β) on $Q_{(\beta)}$ refer to the particular Killing vector or spinor motions along which they generate. It is these functions which satisfy the global graded algebra. In particular, we have

$$\left\{Q_{(\beta)}, \bar{Q}_{(\beta')}\right\} = \tfrac{1}{2}\gamma^{(\mu)}_{(\beta\beta')}J_{(\mu4)} + \sigma^{(\mu\nu)}_{(\beta\beta')}J_{(\mu\nu)}, \tag{2.29}$$

where, in this relation, the appropriate correspondence between the Killing spinors on the left and the Killing vectors on the right is to be made. In (2.29), $J_{(04)}$ is just the Killing energy and all γ matrices are the ordinary numerical ones.

We note finally that the same "Coulomb" component of ψ_i is involved in Q^0 here as for $\Lambda = 0$. There, an arbitrary vector spinor ψ_i may be uniquely given an orthogonal decomposition into projections:

$$\psi_i = \psi_i^{\mathrm{T}} + \kappa_i \varepsilon + \partial_i \eta, \qquad \kappa_i \equiv \sqrt{\tfrac{1}{2}} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \gamma^j, \tag{2.30}$$

where

$$\kappa_i \psi_i^{\mathrm{T}} = \partial_i \psi_i^{\mathrm{T}} = \gamma_i \psi_i^{\mathrm{T}} = 0, \qquad \kappa_i \partial_i = \partial_i \kappa_i = 0, \qquad \kappa^2 = 1. \tag{2.31}$$

As a result, the Rarita-Schwinger constraint $Q^0 = 0$ may be written in the Coulomb form $\not\nabla \varepsilon = Q^0_{\mathrm{N}}$, where Q^0_{N} is the non-linear part of the constraint involving deviations of the metric and affinity from their flat values. Here, exactly the same development can be made, with all ∂_i (and ∇^2) replaced by $\tilde{D}_i$ (and $\tilde{D}^2$). Since $[\tilde{D}_i, \tilde{D}_j] = 0$, all required properties (2.31) may be verified to hold and $Q^0 \sim \tilde{\not{D}} \varepsilon$, so that in the surface integral for Q only the ε component of ψ_i actually occurs.

We have now defined all the conserved quantities which we need. The next step is to find the appropriate Killing vectors to use in eq. (2.19). This is done in sect. 3.

3. Properties of de Sitter and anti-de Sitter spaces

3.1. DE SITTER SPACE

De Sitter space corresponds to a four-dimensional surface in a flat five-dimensional space with metric $(-,+,+,+,+)$ described by

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3/\Lambda, \qquad \Lambda > 0. \tag{3.1}$$

The symmetries of this space are then the ten rotations and boosts of this five-dimensional embedding space. Rotations among the z_1–z_4 clearly result in spacelike Killing vectors. However, boosts which mix z_1–z_4 with z_0 can lead to Killing vectors which are timelike. For example, the Killing vector which corresponds to a mixing of z_4 and z_0 is

$$\bar{\xi}_a = (-z_4, 0, 0, 0, z_0). \tag{3.2}$$

Now

$$\bar{\xi}^2 = -z_4^2 + z_0^2, \tag{3.3}$$

so

$$\bar{\xi}^2 < 0, \tag{3.4}$$

if and only if

$$|z_4| > |z_0|. \tag{3.5}$$

This is a distinctive feature of de Sitter space which implies the existence of an event horizon. Clearly, the resulting restriction on where the Killing vector corresponding to J_{04} is timelike will restrict the region of spacetime where the Killing energy can be used to establish stability.

To be more specific, consider the de Sitter space metric in the form

$$d\tau^2 = -dt^2 + f^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{3.6}$$

where

$$f(t) = \exp\sqrt{\tfrac{1}{3}\Lambda}\, t. \tag{3.7}$$

The coordinates $(t, \boldsymbol{x})$ only cover half the space defined by eq. (3.1), but this metric will exhibit the event horizon problem and has the advantage of simplicity. There also exist static forms of the de Sitter metric, but they become singular at the event horizon and so will not be used here. If we wish to make a time translation for the metric (3.6), we must simultaneously dilate the spatial coordinates to generate a symmetry because of the factor $f^2(t)$. The Killing vector for this symmetry is

$$\bar{\xi}^\mu = \left(-1, \sqrt{\tfrac{1}{3}\Lambda}\, \boldsymbol{x}\right). \tag{3.8}$$

Now

$$\bar{\xi}^2 = -1 + \tfrac{1}{3}\Lambda f^2 |\boldsymbol{x}|^2, \tag{3.9}$$

which is timelike in the region

$$\tfrac{1}{3}\Lambda f^2 |\boldsymbol{x}|^2 < 1. \tag{3.10}$$

This $\bar{\xi}_\mu$ generates a Killing energy through eq. (2.19). However, the restriction (3.10) limits the region of applicability of this quantity. In order for $E(\bar{\xi})$ to act like an energy, the surface of integration in eq. (2.19) must lie inside the event horizon defined by

$$\tfrac{1}{3}\Lambda f^2 |\boldsymbol{x}|^2 = 1. \tag{3.11}$$

This restriction will be exhibited clearly below in terms of a simple field model and in sect. 4. It makes sense physically since an observer fixed at the origin will only detect or be affected by events occurring inside the event horizon. Thus, we are testing whether the de Sitter spacetime is stable to fluctuations visible to this observer when we use the Killing energy $E(\bar{\xi})$. Finally, we mention that the spacelike translation Killing vectors have the flat space form, $\bar{\xi}^{\mu}_{(j)} = (0, \delta^{i}_{\ j})$, so that the translation generators are defined as usual in terms of the momentum density:

$$T^{0}_{\ \mu}\bar{\xi}^{\mu}_{(j)} = T^{0}_{\ j}. \tag{3.12}$$

If we use the Killing vector of eq. (3.8) and the metric (3.6) in eq. (2.19), we find explicitly that

$$\begin{aligned} E(\bar{\xi}) = \frac{f^3}{8\pi G}\oint \mathrm{d}S_i \Big\{ \tfrac{1}{2}\Big(-\partial_i h^{jj} + \partial_j h^{ij} + 2\sqrt{\tfrac{1}{3}\Lambda}\, h^{0i} \Big) \\ + \tfrac{1}{2} f^2 \sqrt{\tfrac{1}{3}\Lambda}\, x_j \Big[f^{-2}\delta^{ij}\partial_0 \big(f^2 h^{kk} \big) + f^{-2}\big(\delta^{ij}\partial_k h^{0k} - \partial_i h^{0j} \big) - \partial_0 h^{ij} \Big] \\ + \tfrac{1}{3}\Lambda\Big(h^{00} x_i - f^2 h^{ij} x_j \Big) \Big\}. \end{aligned} \tag{3.13}$$

As a reassurance that our $E(\bar{\xi})$ makes sense as a physical quantity we will evaluate (3.13) for the de Sitter-Schwarzschild solution [13]. Let us consider this metric with Schwarzschild radius $f^{-1} r_0$ much smaller than the radius of the event horizon, $f^{-1}\sqrt{3/\Lambda}$. Thus, we require that

$$r_0 \ll \sqrt{\frac{3}{\Lambda}} \tag{3.14}$$

In this case, one can perform the surface integral in eq. (3.13) far away from the Schwarzschild radius and yet still be inside the event horizon. To evaluate (3.13) in this limit we only need to know the terms of $h_{\mu\nu}$ of order r_0/r, which are

$$\begin{aligned} h^{ij} &= \delta^{ij} f^{-2}\big[(1+\psi)^4 - 1 \big] = 4 f^{-3}\delta^{ij}\frac{r_0}{r} + \cdots, \\ h^{00} &= 1 - \left(\frac{1-\psi}{1+\psi} \right)^2 = 4 f^{-1}\frac{r_0}{r} + \cdots, \\ \psi &= r_0/fr. \end{aligned} \tag{3.15}$$

For a metric of the form

$$h^{ij} = A\delta^{ij}, \qquad h^{00} = B, \tag{3.16}$$

eq. (3.13) reduces to

$$E(\bar{\xi})=\frac{f^3}{8\pi G}\oint \mathrm{d}S_i\left\{-\partial_i A+x_i\left[\sqrt{\tfrac{1}{3}\Lambda}\,f^2\partial_0 A+\tfrac{1}{3}\Lambda\left(2f^2A+B\right)\right]\right\}. \tag{3.17}$$

Substituting in eq. (3.15) one immediately finds that

$$E(\bar{\xi})=\frac{2r_0}{G}=M, \tag{3.18}$$

since

$$r_0=\tfrac{1}{2}GM. \tag{3.19}$$

There are various types of corrections to the result (3.18). First, there are terms of order $M(r_0/fR)$, where R is the radius at which the surface integral of eq. (3.17) is performed. These are completely analogous to corrections in flat space and merely tell us that we must choose a radius such that $fR \gg r_0$ if we are to include all of the energy M inside the boundary of our integration region. If fR is not much greater than r_0, this also tells us that we should use a comoving volume element to evaluate the Killing energy in order to prevent a loss of energy out through the surface of integration. [For an integration surface far away from the Schwarzschild radius, $fR \gg r_0$, it should not and does not matter whether we use a comoving or fixed volume element.] In addition, there are corrections to eq. (3.18) of order $M\frac{1}{3}\Lambda(fR)r_0$. Inside the event horizon $\frac{1}{3}\Lambda(fR)^2$ is always less than one so these terms are always smaller than Mr_0/fR, i.e., smaller than those we have just discussed. However, such corrections signal a problem if one were to try to move the surface of integration outside the event horizon. Similar comments clearly hold for higher order effects. A potentially dangerous term would have been of order $M(\frac{1}{3}\Lambda f^2R^2)$ which is r_0 independent, and therefore would not vanish even for small mass. However, these cancel upon explicit evaluation.

An illuminating application of the Killing energy in de Sitter space is to a scalar field theory in a background de Sitter metric given by (3.6). The action for this theory is

$$I=\int \mathrm{d}^4x\, f^3\left[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}f^{-2}(\nabla\phi)^2-V(\phi)\right], \tag{3.20}$$

and it has an energy-momentum tensor density

$$-T^0{}_0=\tfrac{1}{2}f^{-3}\pi^2+\tfrac{1}{2}f(\nabla\phi)^2+f^3V(\phi), \tag{3.21}$$

$$T^0{}_i=-\pi\partial_i\phi, \tag{3.22}$$

where

$$\pi = f^3\dot{\phi}. \tag{3.23}$$

Note that T_0^0 is explicitly time dependent through the various f factors. For this theory, the Killing energy is, of course, not expressible as a flux integral but rather as the volume integral

$$E(\bar{\xi}) = \int \mathrm{d}^2x\, T^{0\nu}\bar{\xi}_\nu. \tag{3.24}$$

Conservation of $E(\bar{\xi})$ can easily be checked from the field equations. Using the Killing vector of eq. (3.8) we see that the integrand of eq. (3.17) is

$$T^{0\nu}\bar{\xi}_\nu = \tfrac{1}{2}f^{-3}\pi^2 + \tfrac{1}{2}f(\nabla\phi)^2 + f^3V(\phi) - \sqrt{\tfrac{1}{3}\Lambda}\,\boldsymbol{x}\cdot\pi\nabla\phi. \tag{3.25}$$

For a positive definite potential V, this will be positive provided that

$$\tfrac{1}{2}\left[f^{-3}\pi^2 + f(\nabla\phi)^2\right] > \sqrt{\tfrac{1}{3}\Lambda}\,\boldsymbol{x}\cdot\pi\nabla\phi. \tag{3.26}$$

Now define the vectors

$$\boldsymbol{A} = f^{-3/2}\pi\hat{\boldsymbol{x}},$$

$$\boldsymbol{B} = f^{1/2}\nabla\phi. \tag{3.27}$$

Then eq. (3.19) can be written as

$$\tfrac{1}{2}\left[|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2\right] > \sqrt{\tfrac{1}{3}\Lambda}\, f|\boldsymbol{x}|\boldsymbol{A}\cdot\boldsymbol{B}. \tag{3.28}$$

But, by the triangle inequality,

$$\tfrac{1}{2}\left[|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2\right] > \boldsymbol{A}\cdot\boldsymbol{B}. \tag{3.29}$$

Thus, (3.28) and (3.26) will be satisfied and as a result the Killing energy (3.24) will be positive provided that

$$\sqrt{\tfrac{1}{3}\Lambda}\, f|\boldsymbol{x}| < 1. \tag{3.30}$$

By eq. (3.10) this is just the condition that the integration volume of (3.24) lie inside the event horizon. Thus, the contributions to the Killing energy are positive inside the event horizon but not outside, so this scalar field theory is stable to fluctuations inside the event horizon. Furthermore, we learn that the stable vacuum state is given by $\pi = \nabla\phi = 0$ and ϕ chosen to minimize V just as in flat space, with no added Λ dependence. Note the correlation between the event horizon and positivity which is expected from results on the Hawking effect [5].

3.2. ANTI-DE SITTER SPACE

Anti-de Sitter space is the covering space for the four-dimensional surface, in a flat five-dimensional space with metric $(-,+,+,+,-)$, described by

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 - z_4^2 = 3/\Lambda, \qquad \Lambda < 0. \tag{3.31}$$

Once again the symmetries of this space are just the rotations and boosts in the five-dimensional embedding space. Here, however, there is a global timelike Killing vector corresponding to the rotation mixing z_0 and z_4,

$$\bar{\xi}_a = (-z_4, 0, 0, 0, z_0), \tag{3.32}$$

and

$$\bar{\xi}^2 = -z_4^2 - z_0^2 < 0. \tag{3.33}$$

Thus, $\bar{\xi}_a$ is timelike everywhere [note that the condition (3.31) excludes the point $z_4 = z_0 = 0$], and there is no event horizon.

A complete metric for anti-de Sitter space is, for example,

$$\mathrm{d}\tau^2 = -\cosh^2\sqrt{\tfrac{1}{3}|\Lambda|}\, r\,\mathrm{d}t^2 + \mathrm{d}r^2 + \sinh^2\sqrt{\tfrac{1}{3}|\Lambda|}\, r\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{3.34}$$

This has the timelike Killing vector

$$\bar{\xi}^\mu = (-1, \mathbf{0}), \tag{3.35}$$

which can be used in eq. (2.10) to define a Killing energy, this time in terms of T_0^0 alone since $\bar{\xi}^i$ vanishes. The properties of the resulting $E(\bar{\xi})$ are given in sects. 4 and 6. We mention also that the Schwarzschild–anti-de Sitter metric gives exactly $E(\bar{\xi}) = M$ since one may integrate at spatial infinity in this case.

We conclude with some comments on the bearing of the absence of a global Cauchy surface in anti-de Sitter space on our analysis. In this space, specification of initial data on a complete spacelike surface does not lead to a unique prediction of the future state of a dynamical system (including gravity itself). This is because radiation, not specified by the initial conditions, can propagate in from infinity. Whenever a conserved quantity is defined, one must impose boundary conditions excluding incoming radiation. Normally, it suffices to make this restriction at the initial time and it will then hold for all time. Here, however, one will only be safe from incoming radiation within an ever more restricted region of space (the "Cauchy development"). Therefore, if at a later time we wish to extend the integration volume in the energy integral outside this region we must impose the further (timelike) boundary condition at infinity that no incoming radiation appears. In any case, at the initial time, the energy is perfectly well-defined by the initial data.

4. Analysis of small fluctuations

We will now evaluate the Killing energy, which was shown to be well-defined in the last two sections, for small fluctuations about the de Sitter or anti-de Sitter background metrics. We will use the canonical approach to gravitational dynamics [14] to derive an expression for the Killing energy valid to second order in the fluctuation $h_{\mu\nu}$. An expression for T_ν^0 has previously been derived for de Sitter space, using these methods, by Nariai and Kimura [15]. Our approach is different from theirs, but our results agree. We will begin by determining T_ν^0 and then construct $E(\bar{\xi})$ using the appropriate Killing vectors.

In the canonical approach, the action for the gravitational field with a non-vanishing cosmological constant is written as

$$I=\int \mathrm{d}^4x\left\{\pi^{ij}\partial_0 g_{ij}+Ng^{-1/2}\left[g({}^3R-2\Lambda)+\tfrac{1}{2}\pi^2-g_{ik}g_{jl}\pi^{ij}\pi^{kl}\right]+2N_i\left[\pi^{ij}{}_{|j}\right]\right\}, \tag{4.1}$$

where

$$N\equiv(-g^{00})^{-1/2},\qquad N_i\equiv g_{0i},$$

$$g\equiv\det(g_{ij}),\qquad \pi\equiv g_{ij}\pi^{ij},$$

$$\pi^{ij}\equiv N\sqrt{g}\left[\Gamma^0_{kl}-g_{kl}g^{mn}\Gamma^0_{mn}\right]g^{ik}g^{jl}.$$

Here ${}^3R$ is the three-dimensional curvature scalar and $\pi^{ij}{}_{|j}$ is the three-dimensional divergence of the density π^{ij}. Our background decomposition is now with respect to the basic set (g_{ij},π^{ij},N,N_i):

$$g_{ij}=\bar{g}_{ij}+h_{ij},\qquad \pi^{ij}=\bar{\pi}^{ij}+p^{ij},$$

$$N=\bar{N}+n,\qquad N_i=\bar{N}_i+h_{0i}, \tag{4.2}$$

and we wish to evaluate the action to second order. The two terms in brackets in eq.

(4.1) are constraints:

$$g({}^3R-2\Lambda)+\tfrac{1}{2}\pi^2-g_{ik}g_{jl}\pi^{ij}\pi^{kl}=0=\pi^{ij}{}_{|j}. \tag{4.3}$$

Eqs. (4.3) must, of course, be satisfied by the background metric $\bar{g}_{\mu\nu}$. In addition, they are imposed as constraints on the h_{ij} and p^{ij} by requiring that they hold to linear order in these variables. For convenience we will impose the gauge conditions

$$p^i{}_i=0=\bar{D}^j h_{ij}, \tag{4.4}$$

where $\bar{D}^j$ is the three-dimensional background covariant divergence. For the cases we are considering, namely the de Sitter metric of eq. (3.6) and the anti-de Sitter metric of eq. (3.34), the linear parts of the constraint equations (4.3) imply that

$$h^i{}_i=0=\bar{D}_j p^{ij}. \tag{4.5}$$

The gauges (4.4) and constraints (4.5) together ensure that the excitations are transverse-traceless with respect to the background metric*. Also, for these metrics the background part of N_i vanishes, $\bar{N}_i\equiv\bar{g}_{i0}=0$. Then, from eq. (4.1) we can immediately determine that the hamiltonian is

$$\mathcal{H}=-T^0{}_0=-\bar{N}\bar{g}^{-1/2}\left[g({}^3R-2\Lambda)+\tfrac{1}{2}\pi^2-g_{ik}g_{jl}\pi^{ij}\pi^{kl}\right], \tag{4.6}$$

with the quantity in brackets to be evaluated to second order in h_{ij} and p^{ij}, subject to the conditions (4.4) and (4.5). Because the zeroth order and linear constraints have been satisfied, the complete lagrangian up to second order in h_{ij} and p^{ij} is just given by

$$\mathcal{L}=p^{ij}\partial_0 h_{ij}-\mathcal{H}, \tag{4.7}$$

with $\mathcal{H}$ defined as above.

4.1. DE SITTER SPACE

To evaluate the hamiltonian for the de Sitter space metric (3.6),

$$\bar{g}_{ij}=f^2\delta_{ij}, \qquad \bar{g}_{00}=-1, \tag{4.8}$$

note that the three-space is flat so ${}^3\bar{R}=0$ and the first constraint of eqs. (4.3)

* The covariant decomposition of a spatial tensor with respect to a constant curvature metric is discussed in ref. [16]. Of course, in the de Sitter case the three-space is flat, but not in our form of the anti-de Sitter metric.

requires that

$$\tfrac{1}{2}\bar{\pi}^2 - f^4\bar{\pi}^{ij}\bar{\pi}^{ij} = 2f^6\Lambda . \tag{4.9}$$

Thus,

$$\bar{\pi}^{ij} = -2\sqrt{\tfrac{1}{3}\Lambda}\, f\delta^{ij} . \tag{4.10}$$

Using conditions (4.4) and (4.5), we find that to second order

$$g = f^6 - \tfrac{1}{2}f^2(h_{ij})^2 ,$$

$$^3R = -\tfrac{1}{4}f^{-6}(\nabla h_{ij})^2 . \tag{4.11}$$

The π terms in (4.6) are evaluated using (4.4), (4.5) and (4.10) to second order in p^{ij}. The resulting hamiltonian from (4.6) is

$$\mathcal{H} = -T^0{}_0 = \tfrac{1}{4}f^{-3}(\nabla h_{ij})^2 + f\left(p^{ij} - \sqrt{\tfrac{1}{3}\Lambda}\, f^{-1}h_{ij}\right)^2 . \tag{4.12}$$

This gives the expression for $-T^0{}_0$, but before proceeding to evaluate the Killing energy it is useful to make a canonical change of variables. Nariai and Kimura [15] have noted that in terms of

$$Q_{ij} = \sqrt{\tfrac{1}{2}}\, f^{-2}h_{ij} ,$$

$$P^{ij} = \sqrt{2}\, f^2\left(p^{ij} - 2\sqrt{\tfrac{1}{3}\Lambda}\, f^{-1}h_{ij}\right), \tag{4.13}$$

the hamiltonian takes the particularly simple form

$$\mathcal{H} = -T^0{}_0 = \tfrac{1}{2}\left[f^{-3}(P^{ij})^2 + f(\nabla Q_{ij})^2\right]. \tag{4.14}$$

Note that this is similar to the scalar field hamiltonian found in sect. 3. The form of $T^0{}_i$, whose correctness is also justified below by conservation of $T^{0\nu}\bar{\xi}_\nu$ is just

$$T^0{}_i = -P^{jk}\partial_i Q_{jk} . \tag{4.15}$$

Notice that the hamiltonian of (4.14) is not time-independent. Using the field equations*,

$$\partial_0 P^{ij} = f\,\nabla^2 Q_{ij} , \qquad \partial_0 Q_{ij} = f^{-3}P^{ij} , \tag{4.16}$$

* These equations, incidentally, are equivalent to the covariant wave equation $\Box h^i_j = 0$, where $\Box$ is the scalar d'alembertian, which is correct since h^i_j is effectively a background scalar in our gauge.

we find instead that

$$\partial_0 \mathcal{H} = -\partial_0 T^0{}_0 = \sqrt{\tfrac{1}{3}\Lambda}\left[-\tfrac{3}{2} f^{-3}(P^{ij})^2 + \tfrac{1}{2} f(\nabla Q_{ij})^2\right] + \text{3-divergence}. \quad (4.17)$$

According to the discussion of sect. 2, however, the conserved quantity is

$$\begin{aligned} T^{0\nu}\bar{\xi}_\nu &= -T^0{}_0 + \sqrt{\tfrac{1}{3}\Lambda}\, x_i T^0{}_i \\ &= \tfrac{1}{2}\left[f^{-3}(P^{ij})^2 + f(\nabla Q_{ij})^2\right] - \sqrt{\tfrac{1}{3}\Lambda}\, x_i P^{jk}\partial_i Q_{jk}, \end{aligned} \quad (4.18)$$

using the $\bar{\xi}_\mu$ of (3.8). Now taking the time derivative of the second term in (4.18) and using the field equations gives

$$\begin{aligned} \partial_0\left[\sqrt{\tfrac{1}{3}\Lambda}\, x_i P^{jk}\partial_i Q_{jk}\right] &= \sqrt{\tfrac{1}{3}\Lambda}\, x_i\left[f\,\nabla^2 Q_{jk}\partial_i Q_{jk} + f^{-3}P^{jk}\partial_i P^{jk}\right] \\ &= \sqrt{\tfrac{1}{3}\Lambda}\left[-\tfrac{3}{2} f^{-3}(P^{ij})^2 + \tfrac{1}{2} f(\nabla Q_{ij})^2\right] + \text{3-divergence}. \end{aligned} \quad (4.19)$$

Comparing (4.17) and (4.19) we see immediately that the time derivative of $T^0\nu\bar{\xi}_\nu$ is equal to a total three-divergence so that

$$E(\bar{\xi}) = \frac{1}{8\pi G}\int d^3x\, T^{0\nu}\bar{\xi}_\nu \quad (4.20)$$

with $T^0{}_i$ given by (4.15) has been explicitly shown to be conserved. Finally, $E(\bar{\xi})$ can be shown to be positive for small fluctuations inside the event horizon. If we define

$$A^{ij} = f^{-3/2}\hat{x}P^{ij}, \qquad B_{ij} = f^{1/2}\nabla Q_{ij}, \quad (4.21)$$

then, exactly as in the scalar field case

$$T^{0\nu}\bar{\xi}_\nu = \tfrac{1}{2}\left[|A^{ij}|^2 + |B_{ij}|^2\right] - \sqrt{\tfrac{1}{3}\Lambda}\, f|x| A^{ij}\cdot B_{ij}. \quad (4.22)$$

Once again from the triangle inequality we see that $T^0\nu\bar{\xi}_\nu$ is positive provided that

$$\tfrac{1}{3}\Lambda f^2|x|^2 < 1, \quad (4.23)$$

which is just the condition that the excitations be inside the event horizon. Thus we have shown that de Sitter space is stable against small fluctuations inside the event horizon. Notice that the change in sign of $E(\bar{\xi})$ at the horizon occurs exactly as for the scalar example. This supports the universality of the Hawking effect [5].

Indeed, one would expect all systems with positive energy to behave in the same way: the free part of the energy is always $\sim \tfrac{1}{2} f(\pi^2 + (\nabla\phi)^2)$, while the momentum density is $\sim \pi\nabla\phi$.

For physical matter, the non-linear parts of the energy are [like $V(\phi)$ in the scalar case] positive, so the critical condition for the effect arises primarily because of the free field part, where excitations beyond the horizon can give "negative" Killing energy contributions. In particular, if the higher terms in $T^0{}_0(h)$ are effectively positive (as in the $\Lambda = 0$ case), then the only negative contributions would be those due to the horizon.

4.2. ANTI-DE SITTER SPACE

For the anti-de Sitter space metric of (3.34), ${}^3\bar{R} = 2\Lambda$ so the constraint equation (4.3) requires that

$$\bar{\pi}^{ij} = 0. \tag{4.24}$$

To second order in h_{ij} and p^{ij}, the hamiltonian of (4.6) is straightforward to evaluate, taking the manifestly positive form

$$\mathcal{H} = -T^0{}_0 = \bar{N}\bar{g}^{-1/2}\left[p_{ij}p^{ij} + \bar{g}\left(\tfrac{1}{4}\left(\bar{D}_k h_{ij}\right)^2 + \tfrac{1}{6}|\Lambda|\left(h_{ij}\right)^2\right)\right]. \tag{4.25}$$

The Killing energy for the anti-de Sitter case, being an integral over $-T^0{}_0$, since $\bar{\xi}_i = 0$ in our choice of $\bar{g}_{\mu\nu}$, is thus positive, and anti-de Sitter space is stable against small fluctuations. Although the hamiltonian (4.25) contains a mass-like term, the field h_{ij} still corresponds to a massless graviton. This is most easily seen by noting that the transverse-traceless h_{ij} has only two degrees of freedom, so the spin 2 mass term is an artifact in the same way as that of the spin $\frac{3}{2}$ field [12] in supergravity.

5. Semiclassical stability for $\Lambda > 0$

We have proved that de Sitter space is stable against small fluctuations about the vacuum within the event horizon. Another test for stability is to look for euclidean "bounce" solutions which can signal an instability to quantum-mechanical tunnelling [6]. Of course, if one knew that the total Killing energy was positive there would be no need to check for such decays. However, in the absence of a complete positivity proof, a check of semiclassical stability is useful. It is known that unusual topologies can lead to stability problems in gravity [17] and semiclassical instability has been found for gravity at finite temperature [7] and for the Kaluza-Klein theory [8].

For the present case, a bounce solution is a solution to the euclidean space Einstein equations which is asymptotically de Sitter. The Schwarzschild–de Sitter metric might seem to be a possible candidate. When continued into euclidean space, this solution has two potential singularities, at the cosmological event horizon and at the Schwarzschild radius. One, but not both, of these can be removed by the usual trick of making the euclidean time variable periodic [18]. This discussion has a simple physical interpretation in terms of Hawking radiation [5]. De Sitter space contains Hawking radiation at a temperature determined by the cosmological constant. If a black hole could form in this space with an intrinsic temperature less than that of the surrounding radiation, it could accrete this radiation and grow forever. However, for the Schwarzschild–de Sitter black hole, the black hole and cosmological temperatures are [5]

$$T_{\mathrm{BH}} = C\left[1 + \frac{D}{r_0}\right], \qquad T_{\mathrm{cc}} = C\left[1 + \frac{D}{r_{\mathrm{H}}}\right], \tag{5.1}$$

where C, D are positive constants, r_0 is the Schwarzschild radius and r_{H} that of the de Sitter horizon. Then, clearly since $r_{\mathrm{H}} > r_0$,

$$T_{\mathrm{BH}} > T_{\mathrm{cc}}, \tag{5.2}$$

so the black hole can never be cooler than the surrounding radiation. Thus, the space is stable against catastrophic black hole growth.

When $\Lambda = 0$ it is known [4, 19] that there are no bounce solutions. Although it is doubtful that such solutions exist for $\Lambda > 0$, it would be desirable to extend the proofs to this case.

6. Stability for $\Lambda < 0$

In this section, we will see that the Killing energy is positive not only for small fluctuations, but for all excitations of the $\Lambda < 0$ theory which are asymptotically anti-de Sitter. This will establish stability of the anti-de Sitter vacuum metric. As shown in sect. 2, the generators of the graded anti-de Sitter group can be written as flux integrals at spatial infinity, eq. (2.28). Thus for asymptotically anti-de Sitter metrics they will obey the graded global anti-de Sitter algebra

$$\left\{Q_{(\beta)}, \bar{Q}_{(\beta')}\right\} = \tfrac{1}{2}\gamma^{(\mu)}_{(\beta\beta')}J_{(\mu 4)} + \sigma^{(\mu\nu)}_{(\beta\beta')}J_{(\mu\nu)}. \tag{6.1}$$

The explicit relations between the Killing spinors and Killing vectors required to give (6.1) are not difficult to derive in analogy with the $\Lambda = 0$ case*. Although the "spinor" labels in eq. (6.1) really refer to the particular Killing spinor defining the corresponding charge Q, they can be treated as normal flat space spinor indices. In particular, we can multiply (6.1) by $\gamma^{(0)}$ and trace to obtain the desired operator

* Indeed, because the $\tilde{D}$ behave essentially like ordinary derivatives, one can reduce the equation $\tilde{D}_\mu \alpha = 0$ to $\partial_\mu \eta = 0$ by making the transformation [20] $\alpha = s\eta$. A basis for the η's is then given by $\eta_{\beta}{}' = \delta_{\beta}^{(\beta)}$

relation

$$J_{04}=\sum_{\beta} Q^2_{(\beta)} \geqslant 0. \tag{6.2}$$

The argument now proceeds just as in the $\Lambda=0$ case [2]. The quantum operator relation (6.2) can be applied at the tree approximation level ($\hbar \to 0$) with no on-shell fermions. This implies that the Killing energy $E(\bar{\xi})$ which is the conserved tree limit expectation value of J_{04} is positive for classical gravity with $\Lambda<0$.

Although we have not tried to carry out all the details, it also appears likely that the recent proof by Witten [4] that the energy is positive for gravity with $\Lambda=0$ can be extended to the present case. This proof, inspired by supergravity arguments, is based on a consideration of solutions of the spinor equation $\gamma^i D_i \varepsilon=0$, where D_i is the spatial component of D_μ in an asymptotically flat background metric satisfying the Einstein constraints $G_{\mu\nu}=0$. Having determined the asymptotic form of $\varepsilon(x)$ which satisfies this equation, he then shows that $0=\varepsilon^*(\gamma\cdot D)^2\varepsilon \equiv \varepsilon^*(D^2+G_{0\mu}\gamma^0\gamma^\mu)\varepsilon=\varepsilon^* D^2\varepsilon$, and consequently that $\oint \mathrm{d}S_i \varepsilon^* D^i \varepsilon=\int |\nabla \varepsilon|^2 \mathrm{d}^3x \geqslant 0$. But the surface integral is just proportional to E, which is therefore positive. The same arguments should apply with D_i replaced by $\tilde{D}_i \equiv D_i+\frac{1}{2}m\gamma_i$ and the metric now satisfying $G_{0\mu}+\Lambda g_{0\mu}=0$, provided (as is likely) the surface term is proportional to $E(\bar{\xi})$. Similarly, it would be of interest to generalize the classical geometrical proof of Schoen and Yau [3] to the $\Lambda<0$ case.

In any case, the formal supergravity-based proof is sufficient to reassure us that this model is stable, since positive energy also ensures semiclassical stability. Indeed, it might be possible to establish classical stability in the $\Lambda>0$ case as well, for excitations lying entirely within the horizon, perhaps by using the static form of the O(4, 1) metric which covers the interior region and continuing from $\Lambda<0$ to $\Lambda>0$.

7. Conclusions

We have established linearized stability of Einstein theory, with a cosmological term of either sign, for fluctuations about the vacuum. In both cases, the presence of a timelike Killing vector made possible the definition of a physically relevant, conserved, positive flux integral energy expression, although for $\Lambda>0$ the event horizon caused a restriction. The hamiltonian form we have derived for the scalar and gravitational theories makes the peculiar properties of the event horizon for $\Lambda>0$ quite clear and signals the potential for Hawking radiation [5]. For $\Lambda<0$, positivity of the Killing energy for all asymptotically anti-de Sitter metrics was demonstrated in terms of the graded extension of the algebra. It appears that the type of instabilities which might have ruled out the cosmological term does not occur, at least classically.

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