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THE BACKGROUND FIELD METHOD AND THE S-MATRIX

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We prove that the S-matrix can be correctly obtained from the gauge-invariant effective action in the background field approach to gauge theories. In addition, we present a computation of the two-loop fermionic contributions to the Yang–Mills β-function.

1. Introduction

It is well established that the background field method [1–6] greatly simplifies calculations of renormalization factors in gauge theories [4, 7–10]. Less appreciated is the fact that the method can also be useful for calculating S-matrix elements. The S-matrix is constructed from trees with vertices and propagators given by the 1PI Green functions. When constructing these trees, it is sometimes convenient to use a gauge, which is particularly well suited to the physical process being considered. This is often a Lorentz non-covariant gauge. In the usual formulation of gauge theories, the 1PI Green functions must also be computed in this non-covariant gauge making loop calculations extremely difficult. In the background field approach the gauge used for calculating 1PI graphs and the gauge used for constructing trees out of 1PI parts are unrelated and can be chosen independently to best suit each phase of the complete calculation.

To take advantage of the background field technique in S-matrix calculations we must know that the S-matrix can be correctly constructed in the background field approach. In the conventional approach the effective action is used to generate 1PI Green functions and from these the S-matrix is constructed in the usual manner. In ref. [4], the gauge-invariant effective action calculated in the background field method was related to the usual effective action. This relation was sufficient for proving that renormalization factors could be correctly obtained using the background field method. However, the 1PI Green functions generated by the gauge-invariant effective action are not the same as those of the conventional approach. In sect. 2, we prove that, nevertheless, the correct S-matrix can be obtained*. DeWitt has discussed the S-matrix in his formulation of the background field method

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* The need for such a proof was emphasized to us by P. van Nieuwenhuizen.

[5] and this discussion has been extended by Hart [11]. Here, we prove that the correct S-matrix is constructed by the simple graphical procedure presented in ref. [4]. In addition, we describe in sect. 3 a background field calculation of the fermionic contributions to the two-loop Yang–Mills β-function. When added to the pure Yang–Mills result computed in ref. [4] this gives the complete two-loop β-function. The background field method makes this calculation much easier than previous calculations [12, 13] using the conventional approach.

2. The S-matrix in the background field method

The background field method is a technique for computing a gauge-invariant effective action, $\tilde{\Gamma}[A]$, which is used to generate 1PI Green functions. The Feynman rules in this approach (given in ref. [4]) distinguish between lines which appear inside loops and external lines. Initially, a gauge is chosen (the background field gauge [1]) for internal lines but the gauge-invariance of external lines is left intact. An effective action, $\tilde{\Gamma}[A]$, is then computed which is a gauge-invariant functional of its argument, A. Once the 1PI Green functions have been determined the S-matrix is constructed from trees of 1PI parts in the usual way. To define the propagator used to connect the 1PI pieces a gauge-fixing term is added to $\tilde{\Gamma}[A]$. As stated in the introduction this gauge-fixing term is not related to the original term used to fix the gauge inside loop diagrams.

In ref. [4], a relation between the gauge-invariant effective action, $\tilde{\Gamma}$, and the conventional effective action, Γ, was derived. To understand this relation consider the usual generating functional in a Yang–Mills theory,

$$Z[J]=\int \delta A\, \delta\theta\, \delta\theta^{\dagger} \exp\left(i\int \mathrm{d}^4x\left[-\tfrac{1}{4}F^{a2}_{\mu\nu}+L_{\mathrm{ghost}}-\frac{1}{2\alpha}G_a^2+J^a_\mu A^a_\mu\right]\right), \quad (2.1)$$

computed with an unconventional gauge-fixing term

$$G_a=\partial_\mu A^a_\mu-\partial_\mu V^a_\mu+gf^{abc}V^b_\mu A^c_\mu\,, \quad (2.2)$$

where f^{abc} are the group structure constants and V^a_μ is an arbitrary function. In eq. (2.1) θ is the ghost field and L_{ghost} is the ghost lagrangian corresponding to the gauge-fixing term (2.2). Although we have introduced the function V^a_μ, (2.2) is still a perfectly acceptable gauge-fixing term. In particular, the gauge-independence of the S-matrix assures us that the correct S-matrix can be computed in this gauge using the conventional construction. The effective action, Γ, is obtained from $Z[J]$ by Legendre transformation. Because of the presence of V^a_μ in the gauge-fixing term, Γ will depend on V^a_μ as well as on its usual argument, the vacuum expectation value of the gauge field in the presence of the source J^a_μ. We will therefore write it as $\Gamma[A;\,V]$. The relation derived in ref. [4] between the gauge-invariant action, $\tilde{\Gamma}$, and Γ is

$$\tilde{\Gamma}[A]=\Gamma[A;\,V]|_{V=A}\,. \quad (2.3)$$

The S-matrix in the conventional approach is defined by generating 1PI Green functions from $\Gamma[A; V]$ and using them to construct tree graphs. The same procedure is used in the background field method. However, the two approaches are not identical because the 1PI Green functions they use are different. In the conventional approach the 1PI functions are defined by taking A-derivatives of $\Gamma[A; V]$ at fixed V. The background field 1PI Green functions are defined by taking A-derivatives of $\tilde{\Gamma}[A]$ which from eq. (2.3) is equivalent to taking A-derivatives *plus* V-derivatives of $\Gamma[A; V]$ and then setting $V=A$. To prove that the correct S-matrix can be obtained in the background field approach we must show that the presence of the extra V-derivative terms in the background field 1PI functions does not affect the S-matrix.

We can think of $\Gamma[A; V]$ as generating Green functions with two types of fields. Taking A-derivatives of $\Gamma[A; V]$ generates the usual 1PI Green functions with external A-fields. Differentiating with respect to V gives graphs with external V-fields. The usual construction of the S-matrix uses only the A-field 1PI functions. On the other hand, in the background field approach, as discussed above, the A-derivatives of $\tilde{\Gamma}[A]$ correspond to A- and V-derivatives of $\Gamma[A; V]$ with $V=A$ so in this approach the S-matrix is constructed from 1PI parts with both A and V fields. This is shown in fig. 1 where a typical contribution to the S-matrix is drawn. We indicate A-derivatives of Γ by a circle and V-derivative by a square. The S-matrix is obtained in the background field approach by summing over all trees with all possible combinations of circles and squares. The graphs with only circles (A-derivatives) are the same as those of the conventional approach. Thus, to prove our claim we must show that trees of 1PI parts with one or more squares (V-derivatives) do not contribute any additional terms to the S-matrix.

Consider a 1PI Green function with one external V-field and an arbitrary number of A-fields. Any S-matrix graph with only one square is obtained from this 1PI function by building trees on the external lines as in fig. 2. The part of this graph to the right of the dashed line is the V-derivative of a connected Green function. To obtain a contribution to the S-matrix from fig. 2 the external propagators are

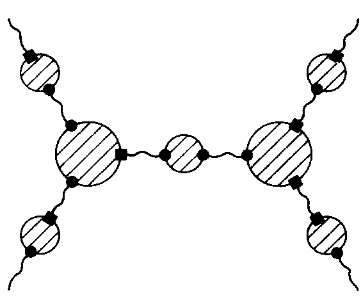

Fig. 1. A typical contribution to the S-matrix in the background field approach. Solid circles represent A-derivatives and squares V-derivatives.

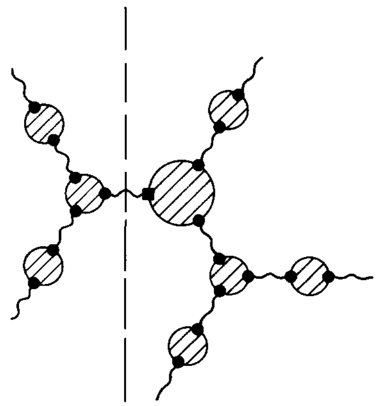

Fig. 2. A contribution to the S-matrix with one square or V-derivative. The part to the right of the dashed line vanishes when the external lines are amputated and put on shell as discussed in the text.

amputated and the external lines are put on shell with physical polarizations. The part to the right of the dashed line then becomes the V-derivative of an on-shell amputed Green function. We will show below that the V-derivative of an on-shell Green function vanishes for *any* value of V. Thus, this graph and graphs with more than one square like fig. 1 which are proportional to higher V-derivatives of connected Green functions will not contribute to the S-matrix. This proves that the S-matrix is given by graphs without squares and is equal to the S-matrix in the conventional construction.

Since the field V_μ^a only enters into the gauge theory through the gauge-fixing term (2.2), the V independence of amputated, on-shell, Green functions is related to the gauge-independence of the S-matrix. We prove this result using Ward identities [14] derived from BRS invariance [15] applied to the generating functional (2.1).

A simple consequence of the anti-commuting nature of the ghost field θ is the identity

$$\int \delta A\, \delta\theta\, \delta\theta^\dagger\, \theta_b^\dagger(y) \exp\left(i \int \mathrm{d}^4 x \left[-\tfrac{1}{4}F_{\mu\nu}^{a^2} + L_{\text{ghost}} - \frac{1}{2\alpha} G_a^2 + J_\mu^a A_\mu^a\right]\right) = 0\,. \quad (2.4)$$

We perform the BRS transformation

$$\delta A_\mu^a = D_\mu^{ab}(A)\theta_b\lambda\,,$$
$$\delta\theta_a = \tfrac{1}{2} g f^{abc}\theta_b\theta_c\lambda\,,$$
$$\delta\theta_a^\dagger = \frac{1}{\alpha} G_a\lambda\,, \quad (2.5)$$

where $D_\mu^{ab}(A) = \delta^{ab}\,\partial_\mu + g f^{acb} A_\mu^c$ and λ is an anti-commuting parameter. Using the

BRS invariance of the lagrangian we derive the Ward identity

$$\int \delta A\, \delta\theta\, \delta\theta^{\dagger} \left\{ \frac{1}{\alpha} G_b(y) + \theta_b^{\dagger}(y) \int \mathrm{d}^4 z i J_\mu^c(z) D_\mu^{cd}(A) \theta_d(z) \right\}$$

$$\times \exp\left(i \int \mathrm{d}^4 x \left[-\tfrac{1}{4} F_{\mu\nu}^{a2} + L_{\text{ghost}} - \frac{1}{2\alpha} G_a^2 + J_\mu^a A_\mu^a \right] \right) = 0 . \qquad (2.6)$$

On the other hand, by directly differentiating $Z[J]$ in eq. (2.1) with respect to $V_\mu^a(y)$ and using

$$L_{\text{ghost}} = -\theta_a^{\dagger} D_\mu^{ac}(V) D_\mu^{cb}(A) \theta_b , \qquad (2.7)$$

we find

$$\frac{\delta Z}{\delta V_\mu^a(y)} = \int \delta A\, \delta\theta\, \delta\theta^{\dagger} \left\{ -\frac{1}{\alpha} D_\mu^{ab}(A) G_b(y) + g f^{abc} \theta_b^{\dagger}(y) D_\mu^{cd}(A) \theta_d(y) \right\}$$

$$\times \exp\left(i \int \mathrm{d}^4 x \left[-\tfrac{1}{4} F_{\mu\nu}^{a2} + L_{\text{ghost}} - \frac{1}{2\alpha} G_a^2 + J_\mu^a A_\mu^a \right] \right) . \qquad (2.8)$$

Now, operating on eq. (2.6) with $D_\mu^{ab}(-i\,\delta/\delta J(y))$ we obtain the identity

$$\int \delta A\, \delta\theta\, \delta\theta^{\dagger} \left\{ \frac{1}{\alpha} D_\mu^{ab}(A) G_b(y) - g f^{abc} \theta_b^{\dagger}(y) D_\mu^{cd}(A) \theta_d(y) \right.$$

$$\left. + i[D_\mu^{ab}(A) \theta_b^{\dagger}(y)] \int \mathrm{d}^4 z\, J_\nu^c(z) D_\nu^{cd}(A) \theta_d(z) \right\}$$

$$\times \exp\left(i \int \mathrm{d}^4 x \left[-\tfrac{1}{4} F_{\mu\nu}^{a2} + L_{\text{ghost}} - \frac{1}{2\alpha} G_\alpha^2 + J_\mu^a A_\mu^a \right] \right) = 0 . \qquad (2.9)$$

Finally, (2.8) and (2.9) can be combined to give our final result

$$\frac{1}{i} \frac{\delta Z}{\delta V_\mu^a(y)} = \int \delta A\, \delta\theta\, \delta\theta^{\dagger} \left\{ [D_\mu^{ab}(A) \theta_b^{\dagger}(y)] \int \mathrm{d}^4 z\, J_\nu^c(z) D_\nu^{cd}(A) \theta_d(z) \right\}$$

$$\times \exp\left(i \int \mathrm{d}^4 x \left[-\tfrac{1}{4} F_{\mu\nu}^{a2} + L_{\text{ghost}} - \frac{1}{2\alpha} G_a^2 + J_\mu^a A_\mu^a \right] \right) . \qquad (2.10)$$

To obtain an identity for the V-derivative of an N-point Green function from (2.10) we expand both sides in powers of the source J. Then, the connected term with N powers of J will give the N-point Green function. The implications of the identity (2.10) on the N-point function is shown diagrammatically in fig. 3. Recall that we are interested in V-derivatives of on-shell, amputated Green functions with physical polarizations. The condition of physical polarizations is met by using a transverse source satisfying $\partial \cdot J = 0$. Thus, for the Green functions with transverse polarizations the terms a and b in fig. 3 do not contribute. Next, we remove the J factors from the external lines, amputate the external propagators and put these

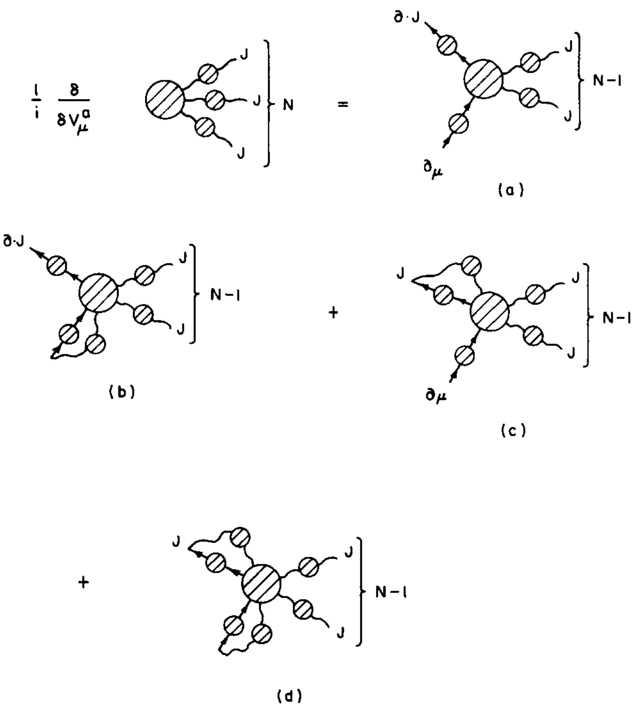

Fig. 3. Diagrammatic representation of the identity (2.10) derived in the text. Wiggly lines are gauge propagators and straight lines with arrows are ghost propagators.

lines on shell. In the terms c and d in fig. 3 one of the J factors is attached to a loop involving a ghost and a gauge propagator, rather than being attached to a single gauge propagator. This loop does not have any on-shell pole so when we amputate and put the momentum on shell diagrams (c) and (d) in fig. 3 will also vanish. Thus, we have shown that amputated, on-shell N-point functions with physical polarizations are V-independent. As discussed above, this means that the extra V-terms introduced in the background field method will not contribute to the S-matrix and completes our proof. In conclusion, the S-matrix can be obtained in the background field method from the gauge-invariant effective action, $\tilde{\Gamma}[A]$, in the standard fashion [4].

3. Computation of the fermionic contribution to the β-function

In ref. [4] the β-function for a pure Yang–Mills theory was computed to two-loop order as an illustration of the background field technique. We have now extended this calculation to include fermions. We use the same procedure and notation as in ref. [4]. In the background field approach, we only need to calculate the background field renormalization constant Z_A to obtain the β-function. This greatly simplifies the computation.

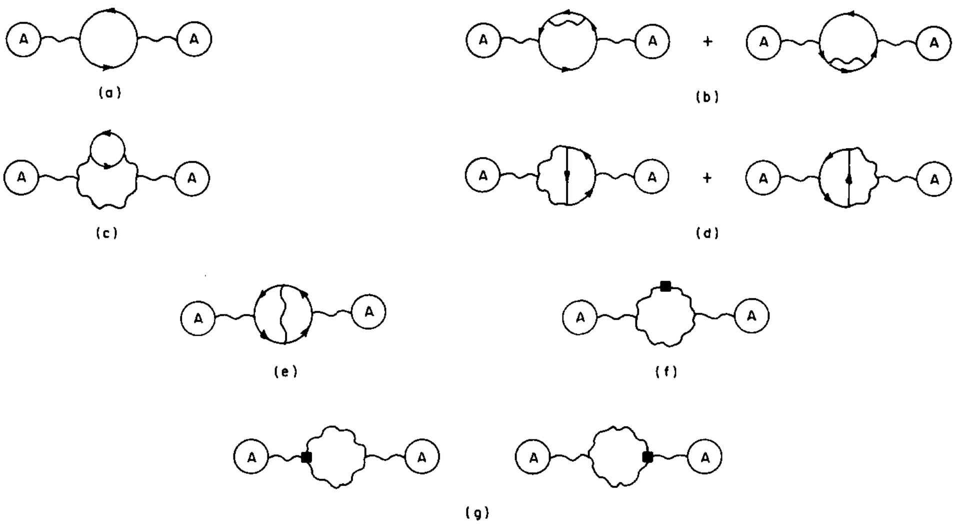

Fig. 4. Fermionic one- and two-loop contributions to the background field propagator. Straight lines with arrows are fermion propagators.

Z_A is determined by renormalizing the background field inverse propagator. To compute the fermionic contributions to two-loop order we must calculate the Feynman graphs in fig. 4. Graphs 4f and 4g are due to gauge-fixing parameter renormalization. To regularize and renormalize the divergences we use dimensional regularization and minimal subtraction. In this method Z_A can be written as a series of poles in ε. If Z_A is written up to two-loop order as

$$Z_A = 1 + \frac{1}{\varepsilon}\left[\beta_0\left(\frac{g}{4\pi}\right)^2 + \beta_1\frac{1}{2}\left(\frac{g}{4\pi}\right)^4\right],$$

then the β-function is given by

$$\beta(g) = -g\left[\beta_0\left(\frac{g}{4\pi}\right)^2 + \beta_1\left(\frac{g}{4\pi}\right)^4\right].$$

The calculation of graph 4a leads to the well-known result $\beta_0(\text{fermionic}) = -\frac{4}{3}T$ where if T^a is a group generator then $T\delta^{ab} \equiv \mathrm{Tr}\,\{T^aT^b\}$. To simplify the presentation of the results for graphs 4b–4g we express them in the form $i(g/4\pi)^4\delta^{ab}\{Ag_{\mu\nu}k^2 - Bk_\mu k_\nu\}$. The values of A and B for individual graphs* as well as their sums are given in table 1. Note that the $1/\varepsilon^2$ divergences are transverse graph by graph as was found previously in the pure Yang–Mills theory [4, 8]. From these results we see that $\beta_1(\text{fermionic}) = -T(\frac{20}{3}C_A + 4C_F)$ with $C_F I = T^aT^a$ and $C_A\delta^{ab} = f^{acd}f^{bcd}$. When added to the beta function for pure Yang–Mills theory from ref. [4] we get

* The algebriac manipulation program MACSYMA was used in the evaluation of fig. 4e.

TABLE 1

Results from the calculation of the graphs in fig. 4

Graph	A	B
b	$TC_F\frac{4}{3\varepsilon^2}(1+4\varepsilon-2\rho\varepsilon)$	$TC_F\frac{4}{3\varepsilon^2}(1+\frac{7}{2}\varepsilon-2\rho\varepsilon)$
c	$-TC_A\frac{10}{3\varepsilon^2}(1+\frac{39}{10}\varepsilon-2\rho\varepsilon)$	$-TC_A\frac{10}{3\varepsilon^2}(1+4\varepsilon-2\rho\varepsilon)$
d	$TC_A\frac{8}{3\varepsilon^2}(1+\frac{13}{4}\varepsilon-2\rho\varepsilon)$	$TC_A\frac{8}{3\varepsilon^2}(1+\frac{7}{2}\varepsilon-2\rho\varepsilon)$
e	$-T(C_F-\frac{1}{2}C_A)\frac{4}{3\varepsilon^2}(1+\frac{11}{2}\varepsilon-2\rho\varepsilon)$	$-T(C_F-\frac{1}{2}C_A)\frac{4}{3\varepsilon^2}(1+5\varepsilon-2\rho\varepsilon)$
f	$-TC_A\frac{20}{9\varepsilon^2}(1+\frac{46}{15}\varepsilon-\rho\varepsilon)$	$-TC_A\frac{20}{9\varepsilon^2}(1+\frac{46}{15}\varepsilon-\rho\varepsilon)$
g	$TC_A\frac{20}{9\varepsilon^2}(1+\frac{28}{15}\varepsilon-\rho\varepsilon)$	$TC_A\frac{20}{9\varepsilon^2}(1+\frac{28}{15}\varepsilon-\rho\varepsilon)$
total	$-\frac{1}{\varepsilon}(\frac{10}{3}TC_A+2TC_F)$	$-\frac{1}{\varepsilon}(\frac{10}{3}TC_A+2TC_F)$

Here $\rho=\gamma_E-\ln 4\pi+\ln k^2/\mu^2$, $f^{acd}f^{bcd}=C_A\,\delta^{ab}$, $T^aT^a=C_FI$, and $T\delta^{ab}=\mathrm{Tr}\{T^aT^b\}$.

the well-known result [12, 13]

$$\beta(g)=-g\left[\left(\frac{g}{4\pi}\right)^2(\tfrac{11}{3}C_A-\tfrac{4}{3}T)+\left(\frac{g}{4\pi}\right)^4(\tfrac{34}{3}C_A^2-\tfrac{20}{3}TC_A-4TC_F)\right].$$

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