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GAUGE-INVARIANT COSMOLOGICAL FLUCTUATIONS OF UNCOUPLED FLUIDS*

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The gauge-invariant approach to cosmological perturbations introduced by Bardeen is extended to the case of multiple fluids which are uncoupled, except for gravitational interactions. The most general scalar, vector, and tensor fluctuations are considered. The resulting equations are applied to fluctuations involving axions and radiation and axions and baryons in the early universe.

1. Introduction

There is considerable evidence [1] that the energy density of the universe contains a large dark matter component along with the ordinary luminous component we see. Massive neutrinos [2], photinos [3], gravitinos [4] and axions [5] have all been proposed as candidates for the dark matter. The luminous and dark matter appear to be uncoupled, except for gravitational interactions. The dark matter resides in galactic halos and presumably in galactic clusters and superclusters and it is likely that it played an important role in the formation of these structures [5–9]. The problem of the evolution of structure in the universe has thus become a problem of multiple uncoupled gravitating fluids.

The most elegant and powerful formalism for treating cosmological perturbations is the gauge-invariant approach of Bardeen [10, 11]. In its original formulation, this approach applies to a single fluid. Here we extend the gauge-invariant formalism to multiple uncoupled fluids. Scalar, vector and tensor perturbations are considered respectively in sects. 2, 3 and 4. The resulting equations are ideally suited for

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treatments of fluctuations involving both dark and luminous matter, especially when the fluctuations are outside of the horizon. In sect. 5 we apply the equations for the evolution of scalar perturbations to axions and radiation in a radiation-dominated universe and to axions and baryons in an axion-dominated universe [5–9].

We consider fluctuations involving N fluids in a spatially flat Robertson-Walker background spacetime. The letters a and b, running from 1 to N are used to denote the N fluids. The letters i and j, running from 1 to 3 are spatial indices while μ and ν, running from 0 to 3 are spacetime indices. Because the N fluids are uncoupled, except for gravitational interactions, their energy-momentum tensors are individually covariantly conserved:

$$D^{\mu} T^{a}_{\mu\nu} = 0 . \tag{1}$$

This, along with the Einstein gravitational equations

$$G_{\mu\nu} = -8\pi G \sum_{a=1}^{N} T^{a}_{\mu\nu}, \tag{2}$$

determine the dynamics. The background Robertson-Walker metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R(t)^2 \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, \tag{3}$$

and the background stress tensors take the perfect fluid form

$$T_0^{a0} = -\rho_a(t), \qquad T_j^{ai} = p_a(t)\,\delta_j^i, \qquad T_i^{a0} = T_0^{ai} = 0 . \tag{4}$$

It follows from eqs. (1) and (2) that $R(t)$ satisfies the Friedman equations.

$$\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \tfrac{8}{3}\pi G \sum_{a=1}^{N} \rho_a, \qquad \frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}t^2} = -\tfrac{4}{3}\pi G \sum_{a=1}^{N} (\rho_a + 3p_a), \tag{5}$$

and $\rho_a(t)$ and $p_a(t)$ satisfy the conservation equations

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = -\frac{3}{R}\frac{\mathrm{d}R}{\mathrm{d}t}(\rho_a + p_a). \tag{6}$$

Following Bardeen [10], we find it convenient to introduce a conformal time τ satisfying

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{R(t)} \equiv \frac{1}{S(\tau)}, \tag{7}$$

so that the background metric is

$$\mathrm{d}s^2 = S^2(\tau)[-\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}]. \tag{3a}$$

In conformal time eqs. (5) and (6) have the form

$$\left(\frac{\dot{S}}{S}\right)^2 = \tfrac{8}{3}\pi G S^2 \sum_{a=1}^{N} \rho_a, \qquad \left(\frac{\dot{S}}{S}\right)^{\cdot} = -\tfrac{4}{3}\pi G S^2 \sum_{a=1}^{N} (\rho_a + 3p_a), \tag{5a}$$

$$\dot{\rho}_a = -3\left(\frac{\dot{S}}{S}\right)(\rho_a + p_a), \tag{6a}$$

where a super-dot is used to denote differentiation with respect to conformal time.

2. Scalar fluctuations

The physical spacetime consists of the Robertson-Walker metric in eq. (3) and the perfect fluid energy-momentum tensor in eq. (4), plus small fluctuations. Scalar fluctuations are expanded in scalar harmonics (i.e. plane waves) $Q(x)$ satisfying the Helmholtz equation

$$\nabla^2 Q + k^2 Q = 0, \tag{8}$$

and in derivatives of Q,

$$Q_i \equiv -\frac{1}{k}\partial_i Q, \tag{9a}$$

$$Q_{ij} \equiv \frac{1}{k^2}\partial_i \partial_j Q + \tfrac{1}{3}\delta_{ij} Q. \tag{9b}$$

The homogeneity and isotropy of the Robertson-Walker background ensures that, in linear perturbation theory, scalar fluctuations corresponding to different wavenumbers, k, evolve independently. The indices on Q_i and Q_{ij} will be raised and lowered with the ordinary Kronecker delta.

For scalar fluctuations of wave number k, the physical metric and energy-momentum tensors are

$$g_{00} = -S^2(\tau)[1 + 2A(\tau)Q(x)], \tag{10a}$$

$$g_{0i} = -S^2(\tau)B(\tau)Q_i(x), \tag{10b}$$

$$g_{ij} = S^2(\tau)\left\{[1 + 2H_L(\tau)Q(x)]\,\delta_{ij} + 2H_T(\tau)Q_{ij}(x)\right\}, \tag{10c}$$

$$T_0^{a0} = -\rho_a(\tau)[1 + \delta_a(\tau)Q(x)], \tag{11a}$$

$$T_0^{ai} = -(\rho_a(\tau) + p_a(\tau))v_a(\tau)Q^i(x), \tag{11b}$$

$$T_i^{a0} = (\rho_a(\tau) + p_a(\tau))(v_a(\tau) - B(\tau))Q_i(x), \tag{11c}$$

$$T_j^{ai} = p_a(\tau)\left\{[1 + \pi_L^a(\tau)Q(x)]\,\delta_j^i + \pi_T^a(\tau)Q_j^i(x)\right\}. \tag{11d}$$

The quantities A, B, H_{L}, H_{T}, δ_a, v_a, π_{L}^a and π_{T}^a which parameterize the time dependences of the perturbations are not invariant under scalar coordinate transformations of wavenumber, k, in the physical spacetime. Their use can lead to confusion in treating fluctuations outside the horizon [12] and makes the ultimate equations for the evolution of fluctuations needlessly complicated.

Under a scalar coordinate redefinition of wavenumber k,

$$\tilde{\tau} = \tau + T(\tau)Q(x), \tag{12a}$$

$$\tilde{x}_i = x_i + L(\tau)Q_i(x), \tag{12b}$$

the quantities A, B, H_{L}, H_{T}, δ_a, v_a, π_{L}^a and π_{T}^a become

$$\tilde{A} = A - \dot{T} - (\dot{S}/S)T, \tag{13a}$$

$$\tilde{B} = B + \dot{L} + kT, \tag{13b}$$

$$\tilde{H}_{\mathrm{L}} = H_{\mathrm{L}} - (\tfrac{1}{3}k)L - (\dot{S}/S)T, \tag{13c}$$

$$\tilde{H}_{\mathrm{T}} = H_{\mathrm{T}} + kL, \tag{13d}$$

$$\tilde{\delta}_a = \delta_a + (3(\rho_a + p_a)/\rho_a)(\dot{S}/S)T, \tag{14a}$$

$$\tilde{v}_a = v_a + \dot{L}, \tag{14b}$$

$$\tilde{\pi}_{\mathrm{L}}^a = \pi_{\mathrm{L}}^a - (\dot{p}_a/p_a)T, \tag{14c}$$

$$\tilde{\pi}_{\mathrm{T}}^a = \pi_{\mathrm{T}}^a. \tag{14d}$$

Bardeen noted [10] that linear combinations of the above quantities can be formed which are invariant under coordinate transformations. These gauge-invariant quantities are

$$\Phi_A = A + \frac{1}{k}\dot{B} + \frac{1}{k}\left(\frac{\dot{S}}{S}\right)B - \frac{1}{k^2}\left(\ddot{H}_{\mathrm{T}} + \left(\frac{\dot{S}}{S}\right)\dot{H}_{\mathrm{T}}\right), \tag{15a}$$

$$\Phi_H = H_{\mathrm{L}} + \tfrac{1}{3}H_{\mathrm{T}} + \frac{1}{k}\left(\frac{\dot{S}}{S}\right)B - \frac{1}{k^2}\left(\frac{\dot{S}}{S}\right)\dot{H}_{\mathrm{T}}, \tag{15b}$$

$$\varepsilon_a = \delta_a + 3\frac{(\rho_a + p_a)}{\rho_a}\frac{1}{k}\left(\frac{\dot{S}}{S}\right)(v_a - B), \tag{15c}$$

$$v_s^a = v_a - \frac{1}{k}\dot{H}_{\mathrm{T}}, \tag{15d}$$

$$\eta_a = \pi_{\mathrm{L}}^a - c_a^2\left(\frac{\rho_a}{p_a}\right)\delta_a, \tag{15e}$$

$$\pi_{\mathrm{T}}^a, \tag{15f}$$

where $c_a^2 \equiv \mathrm{d}p_a/\mathrm{d}\rho_a$ is the square of the speed of sound in the ath fluid.

The program is to re-express the dynamics of fluctuations in terms of the gauge invariant variables defined in eqs. (15). In terms of these variables the Einstein equations (2) imply

$$\frac{2k^2}{S^2}\Phi_H = 8\pi G \sum_{a=1}^{N} \varepsilon_a \rho_a , \tag{16a}$$

$$\frac{k^2}{S^2}(\Phi_A + \Phi_H) = -8\pi G \sum_{a=1}^{N} \pi_T^a p_a , \tag{16b}$$

and the conservation eqs. (1) imply

$$(\rho_a \varepsilon_a S^3)^{\cdot} + \frac{3S^3(\rho_a + p_a)}{k}\left\{\left[\left(\frac{\dot S}{S}\right)^2 - \left(\frac{\dot S}{S}\right)^{\cdot}\right] v_s^a + k\dot\Phi_H + \tfrac{1}{3}k^2 v_s^a\right\}$$
$$-3S^3\left(\frac{\dot S}{S}\right)\left[(\rho_a + p_a)\Phi_A - \tfrac{2}{3}p_a\pi_T^a\right] = 0 , \tag{17}$$

$$\dot v_s^a + \left(\frac{\dot S}{S}\right)v_s^a - k\Phi_A - \frac{k}{\rho_a + p_a}\left\{ p_a\eta_a + c_a^2\rho_a\varepsilon_a - \tfrac{2}{3}p_a\pi_T^a\right\} = 0 . \tag{18}$$

Combining these equations some straightforward, but tedious, algebra yields N coupled equations for the variables ε_a. Defining

$$E_a \equiv \rho_a \varepsilon_a R^3 , \qquad \Pi_a \equiv p_a \pi_T^a R^3 , \qquad N_a \equiv p_a \eta_a R^3 , \tag{19a}$$

$$\rho \equiv \sum_{a=1}^{N} \rho_a , \qquad p \equiv \sum_{a=1}^{N} p_a , \qquad q \equiv k/R , \qquad H = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} , \tag{19b}$$

these equations are (in terms of the ordinary time variable t):

$$\frac{\mathrm{d}^2E_a}{\mathrm{d}t^2} + (2 + 3c_a^2)H\frac{\mathrm{d}E_a}{\mathrm{d}t} + \frac{12\pi GH}{q^2 + 12\pi G(\rho + p)}$$
$$\times\left\{\left(\frac{\mathrm{d}E_a}{\mathrm{d}t}\right)\sum_{b=1}^{N}\left[(1+3c_b^2)(\rho_b + p_b)\right] - (\rho_a + p_a)\sum_{b=1}^{N}\left[(1+3c_b^2)\frac{\mathrm{d}E_b}{\mathrm{d}t}\right]\right\}$$
$$-4\pi G(\rho_a + p_a)\sum_{b=1}^{N}[E_b] + q^2c_a^2E_a$$
$$+12\pi G\left\{(\rho + p)c_a^2E_a - (\rho_a + p_a)\sum_{b=1}^{N}\left[c_b^2E_b\right]\right\}$$

$$= -q^2 N_a + 12\pi G\left\{(\rho_a + p_a)\sum_{b=1}^{N}[N_b] - (\rho + p)N_a\right\} + \tfrac{2}{3}q^2\Pi_a$$

$$+ 8\pi G\Pi_a\left(\tfrac{2}{3}\rho + 2p - 2c_a^2\rho\right) - 2H\frac{\mathrm{d}\Pi_a}{\mathrm{d}t} + \frac{24\pi G H^2}{q^2 + 12\pi G(\rho + p)}$$

$$\times\left\{(\rho_a + p_a)\sum_{b=1}^{N}\left[(1 + 3c_b^2)\Pi_b\right] - \Pi_a\sum_{b=1}^{N}\left[(1 + 3c_b^2)(\rho_b + p_b)\right]\right\}. \tag{20}$$

Note that the terms in eq. (20) in brace brackets cancel for a single fluid so for $N = 1$ our result agrees with the single fluid result of Bardeen [10] (expressed in ordinary time). The quantity q is the physical wave number of the fluctuation while k is the coordinate wavenumber. The fluctuation crosses the horizon when $q = H$. Well inside the horizon $\varepsilon_a \simeq \delta_a = \delta\rho_a/\rho_a$ so eq. (20) becomes an equation for $\delta\rho_a/\rho_a$. For fluctuations taking the form of a perfect fluid the entire right-hand side of eq. (20) vanishes. Eq. (20) is our main result and in sect. 5 we apply it to the evolution of scalar perturbations for axions and radiation and axions and baryons in the early universe.

3. Vector fluctuations

Vector fluctuations are expanded in vector harmonics $Q_j^{(1)}(x)$ satisfying

$$\nabla^2 Q_j^{(1)} + k^2 Q_j^{(1)} = 0, \qquad \partial_j Q_j^{(1)} = 0, \tag{21}$$

and in derivatives of $Q_j^{(1)}$,

$$Q_{kj}^{(1)} \equiv -\frac{1}{2k}\left(\partial_k Q_j^{(1)} + \partial_j Q_k^{(1)}\right). \tag{22}$$

Indices on $Q_j^{(1)}$ and $Q_{jk}^{(1)}$ will be raised and lowered with the ordinary Kronecker delta. The divergenceless condition, $\partial_j Q_j^{(1)} = 0$, ensures that vector fluctuations evolve independently of scalar fluctuations in linear perturbation theory. Again the homogeneity and isotropy of the Robertson-Walker background implies that, in linear perturbation theory, vector fluctuations corresponding to different wavenumbers, k, evolve independently.

For vector fluctuations of wavenumber k, the physical metric and energy-momentum tensor are

$$g_{00}=-S^2(\tau), \tag{23a}$$

$$g_{0j}=-S^2(\tau)B^{(1)}(\tau)Q_j^{(1)}(\boldsymbol{x}), \tag{23b}$$

$$g_{ij}=S^2(\tau)\left[\delta_{ij}+2H_T^{(1)}(\tau)Q_{ij}^{(1)}(\boldsymbol{x})\right], \tag{23c}$$

$$T_0^{a0}=-\rho_a(\tau), \tag{24a}$$

$$T_j^{a0}=(\rho_a(\tau)+p_a(\tau))(v_a^{(1)}(\tau)-B^{(1)}(\tau))Q_j^{(1)}(\boldsymbol{x}), \tag{24b}$$

$$T_0^{ai}=-(\rho_a(\tau)+p_a(\tau))v_a^{(1)}(\tau)Q^{(1)i}(\boldsymbol{x}), \tag{24c}$$

$$T_j^{ai}=p_a(\tau)\left[\delta_j^i+\pi_T^{a(1)}(\tau)Q_j^{(1)i}(\boldsymbol{x})\right]. \tag{24d}$$

Under a vector coordinate redefinition of wavenumber, k,

$$\tilde{\tau}=\tau, \tag{25a}$$

$$\tilde{x}^j=x^j+L^{(1)}(\tau)Q^{(1)j}(\boldsymbol{x}), \tag{25b}$$

the quantities $B^{(1)}$, $H_T^{(1)}$, $v_a^{(1)}$ and $\pi_T^{a(1)}$ transform like

$$\tilde{B}^{(1)}=B^{(1)}+\dot{L}^{(1)}, \tag{26}$$

$$\tilde{H}_T^{(1)}=H_T^{(1)}+kL^{(1)}, \tag{26b}$$

$$\tilde{v}_a^{(1)}=v_a^{(1)}+\dot{L}^{(1)}, \tag{27a}$$

$$\tilde{\pi}_T^{a(1)}=\pi_T^{a(1)}. \tag{27b}$$

Gauge invariant linear combinations of $B^{(1)}$, $H_T^{(1)}$, $v_a^{(1)}$, and $\pi_T^{a(1)}$ are [10]

$$\psi=B^{(1)}-\frac{1}{k}\dot{H}_T^{(1)}, \tag{28a}$$

$$v_a^c=v_a^{(1)}-B^{(1)}, \tag{28b}$$

$$\pi_T^{a(1)}. \tag{28c}$$

In terms of these variables Einstein equations (2) imply

$$\tfrac{1}{2}k^2\psi=8\pi GS^2\sum_{a=1}^{N}(\rho_a+p_a)v_a^c, \tag{29}$$

and the conservation equations (1) imply

$$\dot{v}_a^c + \left(\frac{\dot{S}}{S}\right)(1 - 3c_a^2) v_a^c + \tfrac{1}{2}k \frac{p_a}{\rho_a + p_a} \pi_T^{a(1)} = 0 . \tag{30}$$

Note that (unlike scalar fluctuations) vector fluctuations in the N stress tensors $T_\nu^{a\mu}$ evolve independently.

4. Tensor fluctuations

Tensor fluctuations are expanded in the tensor harmonics $Q_{jk}^{(2)}(\boldsymbol{x})$ which satisfy

$$\nabla^2 Q_{jk}^{(2)} + k^2 Q_{jk}^{(2)} = 0 , \tag{31a}$$

$$Q_{jk}^{(2)} = Q_{kj}^{(2)} , \tag{31b}$$

$$\partial_j Q_{jk}^{(2)} = 0 , \qquad Q_{jj}^{(2)} = 0 . \tag{31c}$$

Eq. (31c) implies that tensor fluctuations evolve independently of scalar and vector fluctuations and the isotropy and homogeneity of the Robertson-Walker background ensures that tensor fluctuations with different wave numbers k evolve independently. Indices on $Q_{kj}^{(2)}$ are raised and lowered with the ordinary Kronecker delta.

For tensor fluctuations of wave number, k, the physical metric and energy momentum tensors are

$$g_{00} = -S^2(\tau) , \tag{32a}$$

$$g_{0j} = 0 , \tag{32b}$$

$$g_{ij} = S^2(\tau)\left[\delta_{ij} + 2H_T^{(2)}(\tau) Q_{ij}^{(2)}(\boldsymbol{x})\right] , \tag{32c}$$

$$T_0^{a0} = -\rho_a(\tau) , \tag{33a}$$

$$T_j^{a0} = T_0^{aj} = 0 , \tag{33b}$$

$$T_j^{ai} = p_0(\tau)\left[\delta_j^i + 2\pi_T^{a(2)}(\tau) Q_j^{(2)i}(\boldsymbol{x})\right] . \tag{33c}$$

There are no tensor coordinate redefinitions so $H_T^{(2)}$ and $\pi_T^{a(2)}$ are gauge invariant. Einsteins eqs. (2) imply that

$$\ddot{H}_T^{(2)} + 2(\dot{S}/S)\dot{H}_T^{(2)} + k^2 H_T^{(2)} = 8\pi G S^2 \sum_{a=1}^{N} p_a(\tau) \pi_T^{a(2)} . \tag{34}$$

The conservation eqs. (1) are automatically satisfied by tensor fluctuations.

5. Axions

Models which break a U(1) Peccei-Quinn symmetry at an energy scale of about 10^{11-12} GeV produce a large density of axions when the universe is about 10^{-6} seconds old [5]. These axions are non-relativistic, although they are very light, and they begin to dominate the energy density of the universe at about 1000 years.

There are two periods during which the multi-fluid eqs. (20) are useful. Before 1000 years, when the universe is radiation dominated, fluctuations in ε_γ grow linearly with time and then begin to oscillate. The axion fluctuations ε_A follow this growth and then grow logarithmically. After 1000 years, when the universe is axion dominated, the axion fluctuations ε_A grow like $t^{2/3}$ and after recombination, baryon fluctuations ε_B follow the axion perturbations leading to the structures we now see. Thus, we will first consider radiation and axion fluctuations in a radiation dominated universe and then axion and baryon fluctuations in an axion dominated universe. Some of the results we will obtain have previously been derived by other methods [5–9], however, we present them here as an example of the gauge invariant approach to the evolution of fluctuations.

We treat the axions, radiation and baryons as perfect fluids. In a radiation dominated universe the radiation fluctuations obey the equation

$$\frac{d^2}{dt^2}E_\gamma + \left(2 + 3c_\gamma^2\right)H\frac{dE_\gamma}{dt} - 4\pi G(\rho_\gamma + p_\gamma)E_\gamma + q^2c_\gamma^2E_\gamma = 0, \qquad (35)$$

where $c_\gamma^2 = \frac{1}{3}$ and $p_\gamma = \frac{1}{3}\rho_\gamma$. Defining the variable

$$x = \frac{qc_\gamma}{H}, \qquad (36)$$

which is the ratio of the sound horizon length for 2π times the fluctuation wavelength, the solution to eq. (35) is

$$\varepsilon_\gamma = \alpha_\gamma\left[\frac{\sin x - x\cos x}{x}\right] + \beta_\gamma\left[\frac{\cos x + x\sin x}{x}\right], \qquad (37)$$

where α_γ and β_γ are constants. Note that eq. (36) implies that for small x the terms in the brackets of eq. (37) behave like t and $1/\sqrt{t}$, since x goes like $t^{1/2}$. For a galactic-size fluctuation $x \approx 2\times 10^3\sqrt{t(\textit{in years})}$ so that $x \approx 10^{-5}$ when the non-relativistic axions are produced at $t \approx 10^{-6}$ sec. The value $x \approx 1$ corresponds to $t \approx 10^{-4}$ years and axions begin to dominate the universe when $x \approx 3000$. When $x \gtrsim 1$ the radiation fluctuations given by eq. (37) oscillate. Between the time the non-relativistic axions are produced and the time $x \approx 1$, ε_γ grows by about a factor of 10^9.

In a radiation dominated universe eq. (20) implies that axion fluctuations obey

$$\frac{d^2}{dt^2}\varepsilon_A + 2H\left[1 + \frac{1}{1 + q^2/12\pi G(\rho_\gamma + p_\gamma)}\right]\frac{d\varepsilon_A}{dt}$$

$$= 8\pi G\rho_\gamma\varepsilon_\gamma + \tfrac{3}{2}H\left[\frac{1}{1 + q^2/12\pi G(\rho_\gamma + p_\gamma)}\right]\left(\frac{d\varepsilon_\gamma}{dt} - H\varepsilon_\gamma\right), \tag{38}$$

or in terms of the variable x of eq. (36)

$$\frac{d^2\varepsilon_A}{dx^2} + \frac{1}{x}\left[1 + \frac{2}{1 + \frac{1}{2}x^2}\right]\frac{d\varepsilon_A}{dx} = \frac{3}{x^2}\varepsilon_\gamma + \frac{3}{2x}\left[\frac{1}{1 + \frac{1}{2}x^2}\right]\left(\frac{d\varepsilon_\gamma}{dx} - \frac{\varepsilon_\gamma}{x}\right). \tag{39}$$

For small x we find that

$$\varepsilon_A = \tfrac{9}{16}\varepsilon_\gamma + \alpha_A + \frac{\beta_A}{t}, \tag{40}$$

where α_A and β_A are constants. Since ε_γ is growing linearly with time, ε_A will rapidly reach the value $\frac{9}{16}\varepsilon_\gamma$ regardless of its initial value and then grow linearly in time following ε_γ. When x gets of order one, and ε_γ begins to oscillate, ε_A starts to level off and logarithmic growth starts. The behavior of ε_γ and ε_A as a function of x is shown in fig. 1.

After axions dominate the energy density of the universe at $t \approx 1\,000$ years axion perturbations obey

$$\frac{d^2\varepsilon_A}{dt^2} + 2H\frac{d\varepsilon_A}{dt} - 4\pi G\rho_A\varepsilon_A = 0, \tag{41}$$

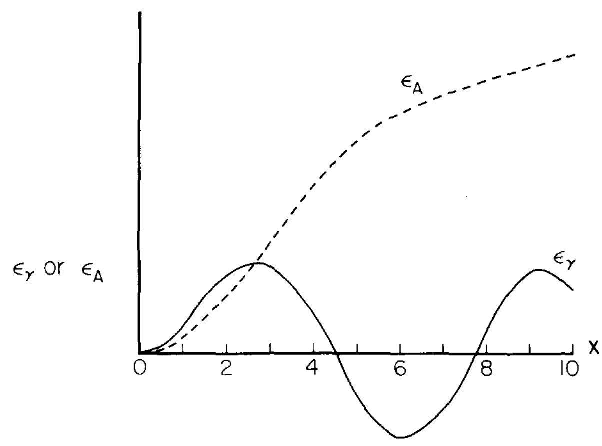

Fig. 1. ε_γ and ε_A as a function of x during the radiation dominated era.

which has the solution

$$\varepsilon_A = \alpha_A t^{2/3} + \beta_A t^{-2}. \tag{42}$$

The axion perturbations grow like $t^{2/3}$ once axions dominate the energy density of the universe. After recombination, baryons form a non-relativistic perfect fluid. Fluctuations in ε_B larger than the baryon Jeans mass (i.e., about 10^5 solar masses) obey

$$\frac{d^2\varepsilon_B}{dt^2} + 2H\frac{d\varepsilon_B}{dt} = 4\pi G\rho_A\varepsilon_A. \tag{43}$$

Eq. (43) has the solution

$$\varepsilon_B = \varepsilon_A + \alpha_B + \beta_B t^{-1/3}, \tag{44}$$

where α_B and β_B are constants. Regardless of its initial value, ε_B, will rapidly approach ε_A and follow it growing like $t^{2/3}$ [6, 7].

References

[1] S. Faber and J.S. Gallagher, Ann. Rev. Astr. and Ast. 17 (1979) 135;
J.R. Primack and G.R. Blumenthal, Invited talk, 1983 Rencontre de Moriond, physics and astrophysics, March 16, 1983. USC-TH-161-83
[2] R. Cowsik and J. McClelland, Phys. Rev. Lett. 29 (1972) 669; Astr. J. 180 (1973) 7;
R. Cowsik, Phys. Rev. Lett. 39 (1977) 784;
A. De Rujula and S. Glashow, Phys. Rev. Lett. 45 (1980) 942
[3] N. Cabibbo, G. Farrar and L. Maiani, Phys. Lett. 105B (1981) 155
[4] H. Pagels and J. Primack, Phys. Rev. Lett. 48 (1982) 223
[5] J. Preskill, M. Wise and F. Wilczek, Phys. Lett. 120B (1983) 127;
L. Abbott and P. Sikivie, Phys. Lett. 120B (1983) 133;
M. Dine and W. Fischler, Phys. Lett. 120B (1983) 137
[6] J. Ipser and P. Sikivie, Phys. Rev. Lett. 50 (1983) 925;
F. Stecker and Q. Shafi, Phys. Rev. Lett. 50 (1983) 928
[7] M. Turner, F. Wilczek and A. Zee, Phys. Lett. 125B (1983) 35
[8] M. Axenides, R. Brandenberger, and M. Turner, Harvard preprint HUTP-83/A021 (1983)
[9] M. Yoshimura, KEK preprint KEK-63 (1983);
M. Fukugita and M. Yoshimura, KEK preprint KEK-TH61 (1983)
[10] J. Bardeen, Phys. Rev. D22 (1980) 1882
[11] R. Brandenberger, R. Kahn and W. Press, Center for Astrophysics preprint 1812 (1983)
[12] W. Press and E. Vishniac, Astr. J. 239 (1980) 1