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CONSTRAINTS ON GENERALIZED INFLATIONARY COSMOLOGIES

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We consider cosmologies having an inflationary period during which the Robertson–Walker scale factor is an arbitrary function of time satisfying $\ddot{R} > 0$ (not necessarily an exponential). We show that any such inflationary period will produce long-wavelength gravitational waves which can affect present observations of the microwave background. Using present bounds on the quadrupole anisotropy we derive constraints on general inflationary cosmologies. Models with power law inflation ($R \sim t^p$) are considered in detail and the maximum reheating temperature is given as a function of p. Finally, predictions for the anisotropy of the microwave background produced by gravitational waves generated by ordinary exponential inflation are presented.

1. Introduction

The homogeneity and spatial flatness of the present universe strongly suggest that some time in the past a period of inflationary expansion occurred [1]. In the usual inflationary cosmology [1], the Robertson–Walker scale factor grows exponentially during this period. However, exponential inflation is by no means uniquely required. It is straightforward to characterize what sort of behavior is needed to explain the present homogeneity and flatness. The evolution of the Robertson–Walker scale factor is governed by the equations

$$\left(\frac{\dot{R}}{R}\right)^2 = \tfrac{8}{3}\pi G\rho - \frac{k}{R^2}, \tag{1.1}$$

$$\frac{\ddot{R}}{R} = -\tfrac{4}{3}\pi G(\rho + 3p), \tag{1.2}$$

where ρ and p are the energy density and pressure and $k = \pm 1$. The flatness problem is solved if, during the inflationary epoch, the term $\frac{8}{3}\pi G\rho$ in eq. (1.1) grows much larger than $1/R^2$ which characterizes the magnitude of the curvature term in (1.1). From (1.1) we can write the ratio of these terms as $\frac{8}{3}\pi G\rho/(1/R^2) = \dot{R}^2 + k$. Since

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this must grow and become very large, $\dot{R}$ must increase with time, or equivalently $\ddot{R}$ must be positive during inflation. From eq. (1.2) this implies that $\rho + 3p$ must be negative so that the normally decelerating force of gravity in an expanding universe becomes accelerating. This is what uniquely characterizes inflationary expansion. The physical distance between fixed coordinates in an expanding Robertson–Walker universe grows like R. The horizon size is proportional to the inverse of the Hubble constant. Since $\ddot{R} > 0$ during inflation, the ratio of this physical distance to the horizon size, $R(\dot{R}/R) = \dot{R}$, increases with time. This means that fixed coordinate lengths get pushed outside the horizon and, if enough inflation occurs, this solves the horizon problem. Thus, if sufficient growth in R takes place, any model with an inflationary period during which $\ddot{R} > 0$ can solve the horizon and flatness problems.

The fact that fixed coordinate lengths get pushed outside the horizon in any inflationary cosmology has an important consequence. Short-wavelength quantum fluctuations in the gravitational field will get red-shifted out of the horizon during the inflationary period. There they remain with constant amplitude until, much later, they re-enter the horizon during the radiation or matter dominated eras as long-wavelength gravitational waves. Such waves can disturb the isotropy of the microwave background through the Sachs–Wolfe effect [2]. The large-scale anisotropy induced by gravitational waves is dominated by physical wavenumbers of order the present Hubble constant, H_0. We will show in sect. 3 that the amplitude of these waves in any generalized inflationary cosmology is of order H_{HC}/m_P where H_{HC} is the value of the Hubble constant at the time when a wave of present physical wavenumber H_0 crossed the horizon during the inflationary era. These waves produce a $\delta T/T$ of the order of their amplitude, H_{HC}/m_P. Since observations indicate a large-scale isotropy of less than 10^{-4}, we find

$$H_{HC} < 10^{-4} m_P . \tag{1.3}$$

This result applies to any model and it indicates that at least some part of the inflation must occur at scales well below the Planck mass where quantum gravitationarl effects are weak.

If we know R as a function of time during the inflationary period we can turn the result (1.3) into a bound on the reheating temperature after inflation. We will consider the cases $R \sim t^p$ with $p > 1$ in sect. 2 However, it is useful to present an order of magnitude estimate of the result here before proceeding to the detailed analysis. The gravitational waves which dominate the anisotropy have a physical wavelength of order $1/H_0$ today and had a physical wavelength $1/H_{HC}$ at the time of horizon crossing during the inflationary epoch. Using our knowledge of the scale factors and the fact that $t \sim 1/H$ to relate these wavelengths we find

$$\frac{1}{H_{HC}} \left(\frac{H_{HC}}{H_{rh}}\right)^p \left(\frac{H_{rh}}{H_m}\right)^{1/2} \left(\frac{H_m}{H_0}\right)^{2/3} = \frac{1}{H_0} . \tag{1.4}$$

where H_{rh} and H_m are the values of the Hubble constant at the time of reheating and at the time of matter domination. Solving (1.4) for H_{rh} and using the bound (1.3) we find

$$H_{rh} < [(10^{-4} m_P)^{p-1} H_0^{1/3} H_m^{1/6}]^{1/(p-1/2)} . \qquad (1.5)$$

Finally, using conservation of energy we can relate this Hubble constant to a maximum reheating temperature,

$$T_{rh}^{max} = \left(\frac{45}{4\pi^3}\right)^{1/4} (m_P H_{rh}^{max})^{1/2} . \qquad (1.6)$$

Although this analysis has been quite crude, eqs. (1.5) and (1.6) provide a fairly good approximation to our results in sect. 2.

In the limit $p \to \infty$ power law inflation becomes equivalent to exponential inflation and the results (1.5) and (1.6) give us the bound on the reheating temperature, or equivalently on the scale of the potential [3, 4]:

$$T_{rh}^{max} = 10^{17}\,\mathrm{GeV} .$$

For completeness we present the predictions of exponential inflation for the gravitational wave contribution to moments of the microwave background temperature in sect. 4. These have previously been given in ref. [4].

2. Bounds on power law inflation

We consider cosmologies in which

$$R = h^{p-1} t^p \qquad (2.1)$$

during inflation. Here $p > 1$ and h is a constant. Using the conformal time variable $\tau = t^{1-p}/(1-p)h^{p-1}$ we can write the metric in the form

$$ds^2 = S^2(\tau)[-d\tau^2 + d\boldsymbol{r} \cdot d\boldsymbol{r}] , \qquad (2.2)$$

where

$$S(\tau) = \frac{1}{h}[(1-p)\tau]^{p/(1-p)} . \qquad (2.3)$$

To compute the amplitude of gravitational waves generated from quantum fluctuations during the inflationary period we follow the methods used in ref. [5]. We define fluctuations in the metric by writing the spatial metric as

$$g_{ij} = S^2(\tau)[\delta_{ij} + 2h_{ij}] , \qquad (2.4)$$

The computation of the fluctuation spectrum for h_{ij} can be reduced to a calculation for a massless scalar field φ with a factor of $\sqrt{8\pi G}$ relating the amplitude of the fluctuations in the two cases. We define the amplitude of fluctuations in φ at coordinate wavenumber k in terms of the scalar two-point function by [5]

$$|\Delta\varphi_k(\tau)|^2 \equiv k^3 \int \frac{d^3x}{(2\pi)^3} e^{ik\cdot x} \langle\varphi(x,\tau)\varphi(0,\tau)\rangle . \tag{2.5}$$

It is straightforward to calculate the scalar two-point function in the spaces described by (2.2) and (2.3). The result is*

$$\langle\varphi(x,\tau)\varphi(0,\tau)\rangle = \frac{h^2\tau^{2\nu}}{32\pi^2(1-p)^{1-2\nu}} \int d^3k\, e^{-ik\cdot x}|H_\nu(k\tau)|^2 , \tag{2.6a}$$

where

$$\nu = \frac{3p-1}{2(p-1)} , \tag{2.6b}$$

and H_ν is the Hankle function. We want to know the fluctuation amplitude for wavelengths well outside the horizon so we take the limit $k\tau \to 0$. Then, using

$$H_\nu(k\tau) \xrightarrow[k\tau\to 0]{} \frac{-i}{\pi}\Gamma(\nu)(\tfrac{1}{2}k\tau)^{-\nu} , \tag{2.7}$$

we find from (2.6) and (2.7) that

$$|\Delta\varphi_k(\tau)|^2 \xrightarrow[k\tau\to 0]{} \frac{h^2\Gamma^2(\nu)2^{2\nu-5}}{k^{2\nu-3}|1-p|^{1-2\nu}\pi^4} . \tag{2.8}$$

We now make the transition from a quantum to a classical analysis by equating the result (2.8) (multiplied by $\sqrt{8\pi G}$) with the square root of the correlation function for a classical gravitational wave of coordinate wavenumber k and polarization λ. Outside the horizon such a wave has a constant amplitude which we write as $A(\boldsymbol{k})\hat{a}_\lambda(\boldsymbol{k})$, where $\hat{a}_\lambda(\boldsymbol{k})$ is a random variable with statistical expectation value

$$\langle a_\lambda^*(\boldsymbol{k})a_{\lambda'}(\boldsymbol{q})\rangle = \frac{1}{k^3}\delta^3(\boldsymbol{k}-\boldsymbol{q})\delta_{\lambda\lambda'} . \tag{2.9}$$

This assures us that the correlation function of the wave with amplitude $A(\boldsymbol{k})\hat{a}_\lambda(\boldsymbol{k})$ defined in analogy with (2.5) will be $A^2(\boldsymbol{k})$. If we match this result with the quantum result (2.8) and include the normalization factor $\sqrt{8\pi G}$ we find that the amplitude for gravitational waves satisfies

$$A^2(\boldsymbol{k}) = \frac{h^2\Gamma^2(\nu)2^{2\nu-2}}{k^{2\nu-3}(p-1)^{1-2\nu}\pi^3 m_{\mathrm{P}}^2} . \tag{2.10}$$

To summarize, quantum fluctuations in the gravitational field get pushed outside the horizon during inflation where they remain as classical gravitational waves with amplitudes given by a random variable $\hat{a}_\lambda(\boldsymbol{k})$ satisfying (2.9) times a factor $A(\boldsymbol{k})$ satisfying (2.10). Note that as $p\to\infty$, $\nu\to\frac{3}{2}$, so A in (2.10) is independent of k. This

* We determine the vacuum by considering times when the modes k have wavelengths small compared to the horizon length. Then the curved nature of spacetime is unimportant and the vacuum is defined by having the usual particle modes unoccupied.

is the scale-invariant spectrum predicted by exponential inflation. For finite p we do not get a scale-invariant spectrum. The gravitational waves thus produced remain outside the horizon until the radiation or matter dominated eras. The waves of relevance to the large-scale anisotropy of the microwave background re-enter the horizon during the matter dominated era. During the matter dominated period gravitational waves have amplitudes which go like $j_1(k\tau)/k\tau$ where j_1 is a spherical Bessel function. Matching this to the constant amplitude predicted by (2.10) when the wavelength is well outside the horizon (i.e. in the limit $k\tau \to 0$) we obtain our prediction for the behavior of the gravitational waves generated by inflation, during the matter dominated era

$$h_{ij}(\mathbf{x}, \tau) = \sum_{\lambda} \int \mathrm{d}^3k \, 3A(k)\hat{a}_{\lambda}(\mathbf{k}) \frac{j_1(k\tau)}{k\tau} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} e_{ij}(\mathbf{k}, \lambda) , \tag{2.11}$$

where e_{ij} is a polarization tensor, $\hat{a}_{\lambda}(\mathbf{k})$ satisfies (2.9) and $A(k)$ is given by (2.10).

The result (2.11) can be applied directly to the Sachs–Wolfe formula which predicts that deviations in the isotropy of the microwave background temperature measured along the unit vector $\mathbf{e}$ are given by

$$\frac{\delta T_0}{T_0} = -\int_0^{\tau_0 - \tau_E} \mathrm{d}y \dot{h}_{ij}(y\mathbf{e}, \tau_0 - y) e^i e^j . \tag{2.12}$$

Here the dot denotes a τ derivative and τ_0 and τ_E are the conformal times today and at the time of recombination respectively. Using eq. (2.12) we can derive predictions for multipole moments of $\delta T_0/T_0$ by projecting out a given multipole in (2.12), squaring and taking a statistical expectation value. If we write $\delta T_0/T_0$ in a multipole expansion

$$\frac{\delta T_0}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\mathbf{e}) , \tag{2.13}$$

and define

$$a_l^2 \equiv \sum_{m=-l}^{l} |a_{lm}|^2 , \tag{2.14}$$

then we find

$$\begin{aligned} \langle a_l^2 \rangle = {} & 72\pi^2 l(l-1)(l+1)(l+2)(2l+1) \int_0^{k_{\max}\tau_0} \mathrm{d}\omega \, \omega A^2\left(\frac{\omega}{\tau_0}\right) \\ & \times \left[\int_0^{1-\tau_E/\tau_0} \mathrm{d}z \left(\frac{3\cos[\omega(1-z)]}{(\omega(1-z))^3} - \frac{3\sin[\omega(1-z)]}{(\omega(1-z))^4} + \frac{\sin[\omega(1-z)]}{(\omega(1-z))^2} \right) \right. \\ & \times \left(\frac{2}{(2l-1)(2l+3)} j_l(\omega z) + \frac{1}{(2l-1)(2l+1)} j_{l-2}(\omega z) \right. \\ & \left. \left. + \frac{1}{(2l+1)(2l+3)} j_{l+2}(\omega z) \right) \right]^2 . \end{aligned} \tag{2.15}$$

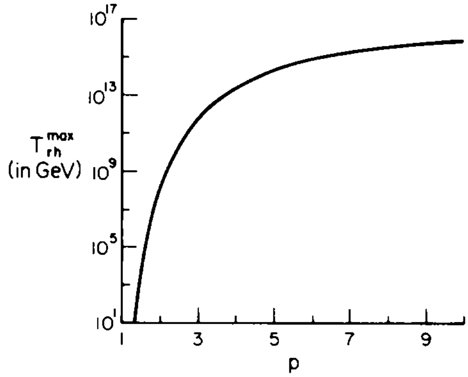

Fig. 1. The maximum reheating temperature as a function of p for power law inflation.

The result* (2.15) is an expectation value but since the individual a_{lm}'s have gaussian fluctuations about zero we can compute the statistical probabilities for fluctuations about the mean (2.15). The ultraviolet cutoff $k_{\max}$ restricts the integral to waves that entered the horizon in the matter dominated era. For $l=2$ we find that at 90% confidence level a_2^2 will be greater than $0.3\langle a_2^2\rangle$ and we use this in computing our bounds. Requiring that the expected quadrupole moment be no larger than the experimental bound [8]:

$$a_2^2 < 5.2\times 10^{-8}\,. \tag{2.16}$$

gives us a limit on the parameter h which can easily be converted into a limit on the value of the Hubble constant at the time of reheating through the relation

$$H_{\rm rh} = [p^p h^{p-1} H_0^{-2/3} H_{\rm m}^{1/6}]^{1/(p-1/2)}\,. \tag{2.17}$$

Finally, this can be related to a maximum reheating temperature through eq. (1.6). The resulting values are plotted in fig. 1. Any inflationary model must produce sufficient reheating to reintroduce baryons into the universe. Depending on the energy scale at which baryon number vioilation is allowed this restricts the value of p allowed through the results in fig. 1. For example, $p=2$ as predicted by domain wall driven inflation can only have reheating to a maximum temperature of 10^8 GeV so baryon violation would have to occur below this scale if such a model is to be viable.

3. Results for general $R(t)$

In the introduction we stated that for any $R(t)$ satisfying $\ddot{R}>0$, gravitational waves will be produced with amplitude of order $H_{\rm HC}/m_{\rm P}$ outside the horizon. This

* For $p<\frac{3}{2}$ this expression diverges in the infrared. We cut off the infrared divergence by restricting the integral to wavelengths that crossed the horizon in the inflationary epoch when the Hubble constant was less than $m_{\rm P}$.

resulted in a model-independent bound of $10^{-4} m_P$ on the value of the Hubble constant when wavelengths which are now of order $1/H_0$ crossed the horizon during inflation. In this section we derive this result.

As in sect. 2 we can relate the gravitational problem to a scalar field problem. The expectation value of the square of the amplitude for a gravitational wave well outside the horizon with coordinate wavenumber k is $8\pi G$ times $|\Delta\varphi_k(\tau)|^2$ as defined in eq. (2.5) in the limit $k\tau \to 0$. Thus, we must compute the scalar two-point function in a general inflationary metric in this limit.

We write the scalar field in terms of creation and annihilation operators in the usual way

$$\varphi(x, \tau) = \int d^3k \, \{a(k)\, e^{ik\cdot x} \varphi_k(\tau) + a^+(k)\, e^{-ik\cdot x} \varphi_k^*(\tau)\} . \tag{3.1}$$

The functions $\varphi_k(\tau)$ are solutions to the scalar wave equation, which in the metric $ds^2 = S^2(\tau)[-d\tau^2 + |dx|^2]$ is

$$\ddot{\varphi}_k + \frac{2\dot{S}}{S}\dot{\varphi}_k + k^2\varphi_k = 0 . \tag{3.2}$$

They are normalized so that the canonical commutation relation

$$[\varphi(x, \tau), \dot{\varphi}(y, \tau)] = \frac{i}{S^2}\delta^3(x-y) \tag{3.3}$$

is satisfied. When the wavelength is well within the horizon the solution to (3.2) and (3.3) is

$$\varphi_k(\tau) = \frac{e^{-ik\tau}}{S(\tau)\sqrt{2k}\,(2\pi)^{3/2}} . \tag{3.4}$$

When the wavelength is well outside the horizon, φ_k is constant. We can estimate the value of this constant by matching the solution (3.4) to this constant solution at the time of horizon crossing. This gives

$$\varphi_k = \frac{e^{-ik\tau_{HC}}}{S(\tau_{HC})\sqrt{2k}\,(2\pi)^{3/2}} , \tag{3.5}$$

outside the horizon. Multiplying $|\varphi_k|^2$ in eq. (3.5) by $8\pi G k^3$ we find that the amplitude of gravitational waves outside the horizon satisfies

$$A^2 = \frac{k^2 8\pi G}{S^2(\tau_{HC}) 2(2\pi)^3} = \frac{1}{2\pi^2}\left(\frac{H_{HC}}{m_P}\right)^2 . \tag{3.6}$$

Here we have used that fact that the time of horizon crossing is defined by $k/S(\tau_{HC}) = H_{HC}$. Eq. (3.6) gives that A is of order H_{HC}/m_P.

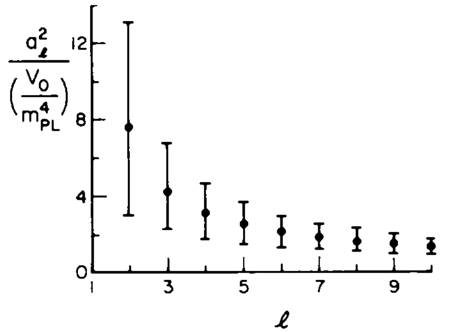

Fig. 2. Predictions for the gravitational contribution to the multipole moments of the microwave background for exponential inflation. V_0 is the value of the vacuum energy density driving the inflation.

4. Predictions for exponential inflation

In the limit $p\to\infty$ the results of sect. 2 are equivalent to results for exponential inflation [3, 4]. We can use eq. (2.15) to generate predictions for the gravitational wave contribution to the expectation value of the moments a_l^2 in exponentially inflating cosmologies. This has previously been done in ref. [4]. In fig. 2 we plot the results including 68% confidence level error bars coming from an analysis of statistical fluctuations about the mean. These predictions do not differ in their l dependence very signficantly from the predictions of microwave anisotropies coming from scalar density perturbations [7]. However, scalar perturbations contribute a large amount to the dipole anisotropy [8]. This contribution is completely absent for gravitational waves. In light of the small value of the dipole anisotropy, an observation of quadrupole and higher moments near the present quadrupole bound could signal the presence of long-wavelength gravitational waves.

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