Nuclear Physics B244 (1984) 454–468

INFRARED DIVERGENCES AND A NON-LOCAL GAUGE FOR SUPERSPACE YANG–MILLS THEORY

L.F. ABBOTT[1] and M.T. GRISARU[2]

Physics Department, Brandeis University, Waltham, MA 02254, USA

D. ZANON[3]

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 24 April 1984

The usual superspace approach to supersymmetric gauge theories suffers from problems with infrared divergences which greatly complicate multiloop calculations. We eliminate these divergences by introducing a non-local gauge-fixing term. In the background field method this term leads to unusual quantum-background interactions. Functional methods are presented for dealing with these interactions. As an example we compute the two-loop Yang–Mills β-function using the background field method in superspace. We also show how a non-local gauge can be used in ordinary, non-supersymmetric Yang–Mills theory.

1. Introduction

Supergraph techniques provide an elegant and powerful approach to perturbative calculations in supersymmetric field theories. In the case of supersymmetric gauge theories, supergraph techniques can be combined with the background-field method resulting in a formulation which is explicitly gauge invariant and supersymmetric. However, along with the usual on-shell infrared divergences of Yang–Mills theory [1], the supergraph approach suffers from severe problems with additional infrared divergences [2, 1] which occur at the one-loop level in an arbitrary Landau-type gauge and at the two-loop level even in the Feynman gauge. Although these divergences are not expected to affect physical quantities they must be separated from ultraviolet divergences before renormalization and this procedure is cumbersome, especially when dimensional regularization is used. In this paper we present a procedure to eliminate such divergences by introducing a non-local gauge-fixing term.

The infrared divergences in the superspace approach to a supersymmetric gauge theory described by the real superfield V arise because of the form of the propagator

[1] Supported in part by Department of Energy contract no. DE-AC02-76-ER03230 and by an Alfred P. Sloan Foundation Fellowship.
[2] Supported in part by NSF grant no. 83-13243.
[3] On leave of absence from Istituto di Fisica di Milano and INFN, Italy. Supported in part by NSF grant no. PHY-82-15249.

in a general covariant gauge. After conventional gauge-fixing with a term $(1/\alpha)D^2V\bar{D}^2V$ the propagator is

$$\Delta = \frac{1+(1-\alpha)(D^2\bar{D}^2+\bar{D}^2D^2)k^{-2}}{k^2}\,\delta^4(\theta-\theta') \tag{1.1}$$

The leading term in $D^2\bar{D}^2+\bar{D}^2D^2$ as k^2 approaches zero is a constant so this propagator goes like $1/k^4$ at small k^2. This causes the infrared divergence problems mentioned above. By comparison, the corresponding term in ordinary gauge theories goes like $k_\mu k_\nu/k^4$.

If we choose the Feynman gauge, $\alpha=1$, the propagator (1.1) is better behaved and the problems with infrared divergences appear to be cured. However, at the two-loop level we find diagrams containing a one-loop corrected propagator as an insertion. The one-loop correction to the inverse propagator has the transverse form

$$D^\alpha\bar{D}^2D_\alpha\Pi(k^2)=(k^2+D^2\bar{D}^2+\bar{D}^2D^2)\Pi(k^2)\,. \tag{1.2}$$

In dimensional regularization in $n=4-2\varepsilon$ dimensions, $\Pi(k^2)\sim(1/\varepsilon)[(k^2)^{-\varepsilon}+O(\varepsilon)]$ so it contains divergent and finite constant terms and a finite term proportional to $\ln k^2$. The correction (1.2) induces a term

$$\frac{\bar{D}^2D^2+D^2\bar{D}^2}{k^4}\,\Pi(k^2) \tag{1.3}$$

into the one-loop corrected propagator. The constant part of $\Pi(k^2)$ in (1.3) can be eliminated by renormalizing the gauge-fixing parameter α in (1.1), i.e. in the Feynman gauge $\alpha=1+O(g^2)$. However this still leaves the $\ln k^2$ term so the propagator goes like $\ln k^2/k^4$ at low k^2 and the infrared divergences reappear in the Feynman gauge at the two-loop level. In a sense, the one-loop correction has unavoidably taken us out of the Feynman gauge and brought back the infrared divergences.

Clearly the key to resolving this problem lies in cancelling the entire correction (1.3), not just the constant part of $\Pi(k^2)$. This can be achieved by a suitable modification of the gauge-fixing term. We choose a gauge-fixing term in $n=4-2\varepsilon$ dimensions of the form

$$D^2V[1+\xi(-\Box_0)^{-\varepsilon}]\bar{D}^2V, \tag{1.4}$$

where $\Box_0$ is the ordinary d'alembertian. This gives a contribution of the form $\xi(\bar{D}^2D^2+D^2\bar{D}^2)(1-\varepsilon\ln k^2)$ to the inverse propagator. The parameter ξ is then chosen so that the entire one-loop correction (1.3) is cancelled. This solves the problem of infrared divergences at the two-loop level. To go beyond two loops additional non-local modifications of the gauge-fixing term can be made. We will restrict ourselves to two loops in this paper.

In the background field approach the gauge-fixing term must be background gauge invariant. This means that the derivatives in eq. (1.4) must be replaced by background covariant derivatives resulting in unconventional interactions between the background field and the quantum fields. One of the purposes of this paper is to describe methods for dealing with these unusual interaction terms.

To describe the computation techniques we will use with the unusual interactions and ghosts coming from our non-local gauge-fixing term and to verify the validty of the method we will consider first ordinary, non-supersymmetric Yang–Mills theory. Here, of course, there are no problems with infrared divergences of the sort we find in the supersymmetric case. However, all of the other issues can still be explored. The general Landau-type gauge propagator is

$$\frac{g_{\mu\nu} - (1-\alpha)k_\mu k_\nu / k^2}{k^2}. \tag{1.5}$$

For the Feynman gauge, $\alpha = 1$, and no $k_\mu k_\nu$ term is present at the tree level. However, the loop correction to the propagator takes the form

$$(g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2)\,, \tag{1.6}$$

and this induces a term $(k_\mu k_\nu / k^4)\Pi(k^2)$ into the loop-corrected propagator. Again, we can eliminate the constant parts of $\Pi(k^2)$ in this term through a renormalization of the gauge-fixing parameter. However, a term $(k_\mu k_\nu / k^4) \ln k^2$ will remain. Though it is harmless near $k^2 = 0$, we can also eliminate this logarithmic term by introducing a non-local gauge-fixing term

$$\partial_\mu A^\mu [1 + \xi(-\Box_0)^{-\epsilon}] \partial_\nu A^\nu\,. \tag{1.7}$$

The background covariant form of (1.7) introduces unusual interactions and a corresponding Nielsen–Kallosh ghost [3]. In sect. 2 we will examine the computation of the two-loop Yang–Mills β-function in this non-local gauge to develop and explain the techniques we will need for the supersymmetric case. Although not needed for eliminating infrared divergences, such gauges might be convenient for certain calculations. In sect. 3 we return to superspace Yang–Mills and illustrate the method by performing a two-loop calculation of the β-function using superfield and background field techniques and a non-local gauge-fixing term. In addition to exhibiting the successful elimination of infrared divergences our calculation shows that the non-linear background-field splitting used in the superfield approach works correctly at the two-loop level. It has never been proven that this is the case.

2. Ordinary Yang–Mills theory in a non-local gauge

In this section we will describe the computation of the two-loop β-function for ordinary, non-supersymmetric Yang–Mills theory with a non-local gauge-fixing term to illustrate the use of such an unconventional gauge. We use a matrix notation for the fields and split the gauge field into a quantum part Q_μ and a background part

A_μ. The usual background-field lagrangian is

$$L = \mathrm{Tr}\left[\frac{1}{4g^2C_A}\{(D_\mu Q_\nu - D_\nu Q_\mu) + [Q_\mu, Q_\nu] + F_{\mu\nu}(A)\}^2 + \frac{1}{2C_A g^2 \alpha}(D^\mu Q_\mu)^2 + c' D^\mu (D_\mu c + [Q_\mu, c])\right]. \tag{2.1}$$

In the Feynman gauge $\alpha = 1$. Here all derivatives are background covariant:

$$D_\mu Q_\nu = \partial_\mu Q_\nu + [A_\mu, Q_\nu] \tag{2.2}$$

and c and c' are the usual ghost fields.

At the one-loop level the effective action contains a correction to the quantum field propagator of the form

$$\Gamma^{(2)} = \frac{\xi}{2g^2C_A}\int \frac{\mathrm{d}^n p}{(2\pi)^n} Q^\mu(-p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu)Q^\nu(p)(p^2)^{-\varepsilon}, \tag{2.3}$$

where, in $n = 4 - 2\varepsilon$ dimensions

$$\xi = \frac{g^2 C_A}{(4\pi)^{2-\varepsilon}} \frac{5}{3\varepsilon}(1 + \tfrac{31}{5}\varepsilon - \gamma\varepsilon). \tag{2.4}$$

In the usual approach one can subtract out the divergent part proportional to $p_\mu p_\nu$ by renormalizing the gauge-fixing parameter α in (2.1), i.e. by choosing $\alpha = 1 + \xi$. This leads to some $O(g^2)$ quantum-background vertices which must be included in a two-loop calculation [4]. However, all of the propagator correction proportional to $p_\mu p_\nu$ can be eliminated by introducing a non-local gauge-fixing counterterm as described in the introduction. This term must be background covariant. In addition, we must introduce a third ghost [3] to cancel the non-trivial determinant produced by the gauge averaging with a non-local weight factor.

We cancel the term in (2.3) proportional to $p_\mu p_\nu$ by replacing the usual gauge averaging by

$$\int \mathscr{D}f\,\mathscr{D}b\,\delta(D^\mu Q_\mu - f)\left\{\exp\left[\tfrac{1}{2}\int \mathrm{d}^4x\, f[1 + \xi(-\Box)^{-\varepsilon}]f\right] \times \exp\left[\tfrac{1}{2}\int \mathrm{d}^4x\, b[1 + \xi(-\Box)^{-\varepsilon}]b\right]\right\},$$

where b is a scalar field with abnormal statistics (the Nielsen–Kallosh ghost). This leads to the modified gauge-fixing and Nielsen–Kallosh lagrangian

$$\Delta L_{GF} + \Delta L_{NK} = \mathrm{Tr}\left\{\frac{1}{2g^2C_A} D_\mu Q^\mu[1 + \xi(-\Box)^{-\varepsilon}]D^\nu Q_\nu + \tfrac{1}{2}b[1 + \xi(-\Box)^{-\varepsilon}]b\right\}, \tag{2.5}$$

which replaces the usual gauge-fixing term in the lagrangian (2.1). In (2.5) $\Box$ is the background covariant d'alembertian. If $(-\Box)^{-\epsilon}$ is replaced by one the third ghost decouples from the background and is therefore usually ignored.

To illustrate the use of the non-local gauge defined by (2.1) and (2.5) we will reexamine the computation of the two-loop Yang–Mills β-function performed in ref. [4]. This calculation was carried out without the factor $(-\Box)^{-\epsilon}$ in the gauge-fixing term, but it did include contributions from the quantum-background vertices extracted from $(\xi/2g^2C_A)(D^\mu Q_\mu)(D^\nu Q_\nu)$. We now have additional vertices from the background fields contained in $(-\Box)^{-\epsilon}$. However, rather than just considering the corresponding graphs, we will recompute the whole contribution of the gauge-fixing term proportional to ξ. For the other contributions we will use the results for all the two-loop graphs involving conventional quantum and quantum-background vertices (the diagrams in figs. 3a–k of ref. 4). The result of these graphs is a contribution of $-\frac{14}{3}gg^4C_A^2/(4\pi)^4$ to the two-loop β-function. Therefore, we only need to compute the contribution coming from the gauge-fixing counterterm (2.5) to complete the calculation.

The non-local term (2.5) involves some unusual interactions between the quantum and background fields. In order to handle them we find it convenient to use functional techniques rather than a direct Feynman graph computation. This approach will also be useful in the supersymmetric case.

Since the correction of (2.5) is already of order g^2 relative to the tree-level lagrangian (2.1) we need only compute one-loop corrections to the effective action which are first order in ξ. By the usual rules for functional integration they are obtained from the quadratic part of the first line of (2.1) together with (2.5):

$$\Gamma_\xi = -\tfrac{1}{2}\,\mathrm{Tr}\ln\{\Box\delta_{\mu\nu} + 2F_{\mu\nu} + \xi D_\mu(-\Box)^{-\epsilon}D_\nu\} + \tfrac{1}{2}\,\mathrm{Tr}\ln\{1+\xi(-\Box)^{-\epsilon}\}\,. \qquad (2.6)$$

Expanding the logarithms to order ξ we get

$$\Gamma_\xi = -\tfrac{1}{2}\xi\,\mathrm{Tr}\{\Box^{-1}D^\mu(-\Box)^{-\epsilon}D_\mu - 2\Box^{-1}F^{\mu\nu}\Box^{-1}D_\nu(-\Box)^{-\epsilon}D_\mu + 4\Box^{-1}F^{\mu\nu}\Box^{-1}F_\nu{}^\lambda\Box^{-1}D_\lambda(-\Box)^{-\epsilon}D_\mu - (-\Box)^{-\epsilon}\}\,. \qquad (2.7)$$

We wish to compute the term in (2.7) which is quadratic in the background field. From this we can determine the gauge-fixing contribution to the background-field renormalization factor which in turn gives us the β-function. Note that the next-to-last term in (2.7) already has two $F_{\mu\nu}$'s so to compute its contribution to the quadratic part of the effective action we can replace all the D_μ and $\Box$ factors with ordinary derivatives. Our procedure in the other terms is to commute the D_μ factors through the $\Box^{-1}$ factors until they are beside each other. Then their product either cancels a $\Box^{-1}$ or produces an $F_{\mu\nu}$. Commutator terms $[D_\mu, D_\nu]$ likewise produce $F_{\mu\nu}$'s. Using

$$[\Box^{-1}, D_\mu] = \Box^{-1}(F_{\mu\nu}D^\nu + D^\nu F_{\mu\nu})\Box^{-1}\,, \qquad (2.8)$$

we find that the quadratic part of (2.7) is just

$$\Gamma_{\xi}^{(2)} = -\tfrac{1}{2}\xi \,\mathrm{Tr}\int \mathrm{d}^4x\,\{(\partial_\nu F^{\mu\nu} - F^{\mu\nu}\partial_\nu)\Box_0^{-1}(\partial^\lambda F_{\mu\lambda} - F_{\mu\lambda}\partial^\lambda)(-\Box_0)^{-\varepsilon-2} + F^{\mu\nu}D_\mu[(-\Box)^{-\varepsilon-2}, D_\nu]\}\,. \tag{2.9}$$

The commutator $[(-\Box)^{-\varepsilon-2}, D_\nu]$ is evaluated in appendix A. The resulting momentum space expression for $\Gamma_\xi^{(2)}$ is

$$\Gamma_{\xi}^{(2)} = -\tfrac{1}{2}\xi\,\mathrm{Tr}\int \frac{\mathrm{d}^4k}{(2\pi)^4} F^{\mu\lambda}(-k)F_\lambda{}^\nu(k) \times \int \frac{\mathrm{d}^nq}{(2\pi)^n}\left[\frac{k_\mu k_\nu}{(q+k)^2(q^2)^{\varepsilon+2}} + (\varepsilon+2)\int_0^1 \mathrm{d}\alpha \frac{(q+k)_\mu(2q+k)_\nu}{[(k+q)^2\alpha + q^2(1-\alpha)]^{\varepsilon+3}}\right]. \tag{2.10}$$

Performing the integrals and using the value of ξ from (2.4) we find

$$\Gamma_\xi^{(2)} = \frac{\mathrm{Tr}}{4g^2C_\mathrm{A}}\int \frac{\mathrm{d}^4k}{(2\pi)^4} F_{\mu\nu}(-k)F^{\mu\nu}(k)\left(-\frac{10}{3\varepsilon}\frac{g^4C_\mathrm{A}^2}{(4\pi)^4}\right). \tag{2.11}$$

From this we determine that the contribution of the gauge-fixing term to the β-function is $-\frac{20}{3}gg^4C_\mathrm{A}^2/(4\pi)^4$, which when added to the ordinary two-loop contribution gives the two-loop β-function $-\frac{34}{3}gg^4C_\mathrm{A}^2/(4\pi)^4$.

We note that if we replace $(-\Box)^{-\varepsilon}$ by one, we should reproduce the contribution of the graphs fig. 3 l and m in ref. [4]. In (2.7) this amounts to keeping only the second term with $(-\Box)^{-\varepsilon}$ dropped. Indeed, we obtain the same result. Thus, the non-local term plays no role here as expected since no infrared divergences are present. In the supersymmetric case, however, where the conventional two-loop graphs do contain infrared divergences that give a spurious contribution to the β-function the non-local gauge-fixing term is essential to cancel these contributions.

3. Superspace background field calculation

We now return to the main objective of our paper, a description of superspace techniques using the background-field method with a non-local gauge-fixing term to eliminate infrared divergences. As an example, we will compute the two-loop contribution to the β-function for supersymmetric Yang–Mills theory. As in sect. 2 our starting point is the lagrangian expressed in terms of a background superfield and a quantum superfield V. We use the conventions and definitions of ref. [5].

After quantum-background splitting, supersymmetric Yang–Mills theory is described by the superspace action

$$S = -\frac{1}{4g^2C_\mathrm{A}}\mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,(\mathrm{e}^{-V}\nabla^\alpha\,\mathrm{e}^V)\bar\nabla^2(\mathrm{e}^{-V}\nabla_\alpha\,\mathrm{e}^V)\,, \tag{3.1}$$

where V is the quantum field and the spinor derivatives are background covariant. Conventional gauge-fixing adds the terms

$$S_{\mathrm{GF}}+S_{\mathrm{ghost}}=-\frac{1}{2\alpha}\frac{1}{g^2C_{\mathrm{A}}}\mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\nabla^2V\bar{\nabla}^2V$$
$$+\mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\{\bar{c}'c-c'\bar{c}+\tfrac{1}{2}(c'+\bar{c}')[V,c+\bar{c}]+\cdots+\bar{b}b\}\,, \qquad (3.2)$$

with FP ghosts c, c' and NK ghost b, all background covariantly chiral. The ellipsis represents higher-order ghost-V interactions. The gauge-fixing term and NK ghost in (3.2) have been introduced by the usual gauge-averaging

$$\int \mathscr{D}f\mathscr{D}\bar{f}\mathscr{D}b\,\mathscr{D}\bar{b}\,\delta(\nabla^2V-\bar{f})\delta(\bar{\nabla}^2V-f)$$
$$\times\exp\left[-\frac{1}{2\alpha}\frac{1}{g^2C_{\mathrm{A}}}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\bar{f}f\right]\times\exp\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\bar{b}b\,, \qquad (3.3)$$

where the second exponential, with covariantly chiral ghost b, is introduced to normalize the averaging. More generally, we could consider instead an averaging with exponential factors $\bar{f}Mf$, $\bar{b}Mb$, where M is any operator.

In the background-field method the ultraviolet divergences of the theory are removed by an overall wave function renormalization Z_V of the action in (3.1) or equivalently coupling constant renormalization (with $Z_g=Z_V^{-1/2}$). The renormalization constant Z_V can be obtained by computing the two-point function with external background fields. However, if one works with the propagator in the Feynman gauge $\alpha=1$, a further renormalization of the gauge parameter $\alpha\to\alpha Z_\alpha$ is needed to maintain this gauge. Z_α is obtained by computing the two-point function with external *quantum* lines. As explained in ref. [4] this renormalization introduces additional quantum-background vertices proportional to $Z_\alpha-1$.

As discussed in the introduction, a renormalization of the gauge parameter is not sufficient to keep the exact VV propagator in the Feynman gauge and a modification of the gauge-fixing term is required. Otherwise, infrared divergences appear beyond one loop, which, in dimensional regularization, are difficult to separate from the ultraviolet divergences. We describe now the details of the procedure at the two-loop level.

The V self-energy has the form of eq. (1.2). For example, at one loop, in the Feynman gauge, from the graphs of fig. 1 we obtain a correction

$$\Gamma^{(2)}=\frac{\xi}{4g^2C_{\mathrm{A}}}\mathrm{Tr}\int\frac{\mathrm{d}^np}{(2\pi)^n}\mathrm{d}^4\theta\,[V(-p)D^\alpha\bar{D}^2D_\alpha V(p)(p^2)^{-\varepsilon}]\,, \qquad (3.4)$$

where

$$\xi=-\frac{3}{\varepsilon}(1+2\varepsilon-\gamma\varepsilon)\frac{g^2C_{\mathrm{A}}}{(4\pi)^{2-\varepsilon}}\,. \qquad (3.5)$$

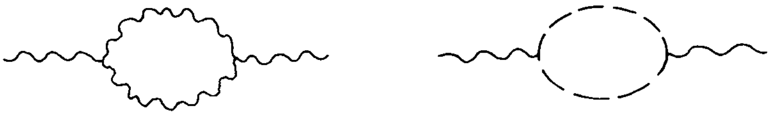


Fig. 1. One-loop corrections to the quantum propagator.

If we write

$$D^{\alpha}\bar{D}^2 D_{\alpha} = p^2 + D^2\bar{D}^2 + \bar{D}^2 D^2\,, \tag{3.6}$$

the last two terms give a contribution that takes us out of the Feynman gauge and introduces infrared divergences in higher-loop calculations.

In order to cancel the harmful piece of the correction we modify the gauge-fixing term and the NK ghost term of (3.2) by replacing them with

$$\begin{aligned} S_{\mathrm{GF}} + S_{\mathrm{NK}} = & -\frac{1}{2g^2 C_{\mathrm{A}}}\,\mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\nabla^2 V[1+\xi(-\Box_+)^{-\epsilon}]\bar{\nabla}^2 V \\ & + \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\bar{b}(1+\xi(-\Box_+)^{-\epsilon})b\,, \end{aligned} \tag{3.7}$$

(cf. our remarks following (3.3)). Here $\Box_+$ is the appropriate d'alembertian when acting on covariantly chiral quantities, $\Box_+\phi \equiv \bar{\nabla}^2\nabla^2\phi$ (see ref. [5], sect. 6.5):

$$\Box_+ = \Box - iW^{\alpha}\nabla_{\alpha} - \tfrac{1}{2}i(\nabla^{\alpha}W_{\alpha})\,, \tag{3.8}$$

and W^{α} is the background-field strength. With this modification the one-loop corrected propagator will be in the Feynman gauge and the infrared divergences will be absent when we perform two-loop calculations. For higher-loop calculations (3.7) must be suitably corrected.

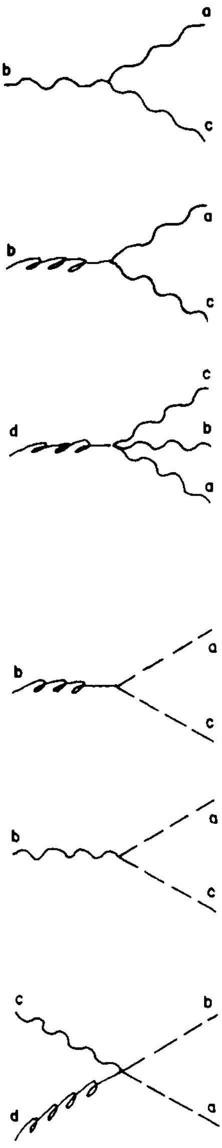
In principle we should do two-loop calculations by using corrected self-energy insertions which are infrared finite. In practice it is simpler to work with uncorrected two-loop graphs which will be individually infrared divergent and separately compute contributions that are proportional to the parameter ξ. When added together, the result will be free of infrared divergences.

As an illustration of the procedure we describe now the calculation of the β-function at the two-loop level. We determine it from the background-field renormalization factor which in turn is determined from the background-field two-point function.

The Feynman rules are obtained by expanding the exponential in (3.1) and expressing the background covariant derivative in (3.1), (3.7) in terms of ordinary derivatives and connections:

$$\nabla_{\mathrm{A}} = D_{\mathrm{A}} - i\Gamma_{\mathrm{A}}\,. \tag{3.9}$$

In addition we must express the covariantly chiral ghosts in terms of ordinary chiral



$$\tfrac{1}{2} i f^{abc} V^a D^\alpha V^b \bar{D}^2 D_\alpha V^c$$

$$\tfrac{1}{2} i f^{abc} V^a [i\Gamma^b \cdot \partial + i W^{b\alpha} D_\alpha + i \bar{W}^{b\dot{\alpha}} \bar{D}_{\dot{\alpha}}] V^c$$

$$\tfrac{1}{2} i f^{abk} f^{cdk} [- V^a D^\alpha V^b \bar{D}_{\dot{\alpha}} D_\alpha V^c \bar{\Gamma}^{d\dot{\alpha}} - V^a D^\alpha V^b \bar{D}^2 V^c \Gamma_\alpha{}^d + V^a \bar{D}^2 D^\alpha V^b V^c \Gamma_\alpha{}^d]$$

$$i f^{abc} [\bar{c}'^a U^b c^c + c'^a U^b \bar{c}^c + \bar{b}^a U^b b^c]$$

$$\tfrac{1}{2} i f^{abc} (\bar{c}' + c')^a V^b (c + \bar{c})^c$$

$$-\tfrac{1}{4} f^{abk} f^{cdk} [\bar{c}'^a \bar{c}^b V^c U^d - c'^a c^b V^c U^d] - \tfrac{1}{4} (f^{ack} f^{dbk} + f^{adk} f^{cbk}) [\bar{c}'^a c^b V^c U^d - c'^a \bar{c}^b V^c U^d]$$

Fig. 2. Feynman rules for background field supergraph calculation in supersymmetric Yang–Mills theory.

fields by

$$c \to e^{U/2} c\, e^{-U/2}, \tag{3.10}$$

where U is the background gauge field in vector representation with $\Omega = \bar{\Omega} = \frac{1}{2}U$ [5]. The vertices needed for low-order calculations are given in fig. 2. Additional vertices arise from the terms in (3.7) proportional to ξ, but we will treat the

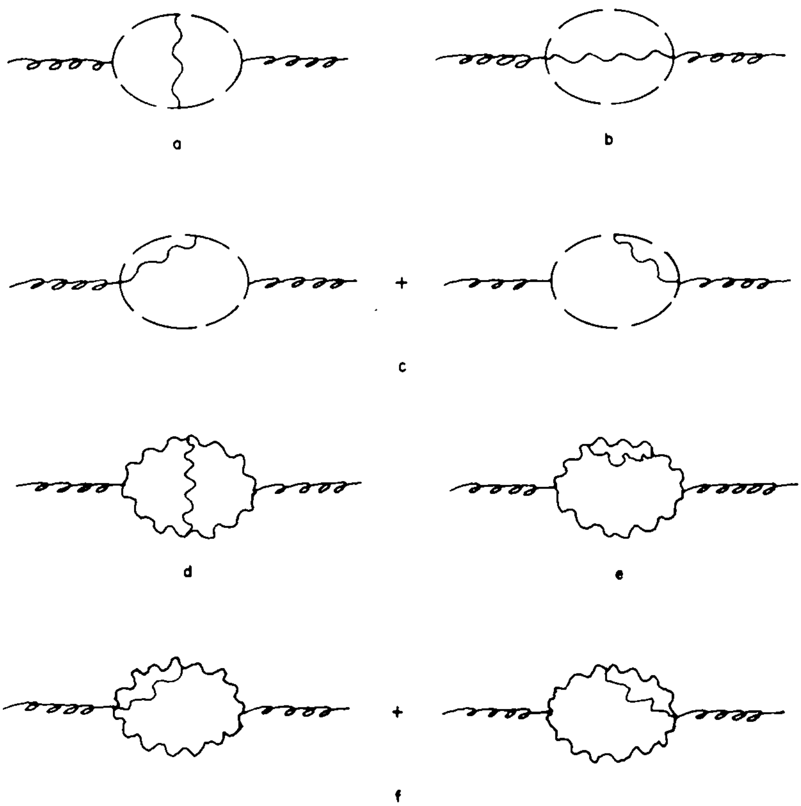


Fig. 3. Two-loop corrections to the background propagator.

corresponding contributions differently. The relevant two-loop supergraphs are shown in fig. 3. Other two-loop graphs are trivially zero. They are evaluated using conventional D-algebra (one can use instead covariant D-algebra [6] which leads to considerable simplifications of the calculation).

The divergent ghost contributions corresponding to figs. 3a, b, c (with a factor

$$2g^2 C_A \int \frac{d^n k\, d^n q}{(2\pi)^{2n}} d^4\theta \frac{1}{k^2 q^2 (k+q)^2 (k+p)^2 (q-p)^2} \tag{3.11}$$

removed from each term) are

(a) $\mathrm{Tr}\,[-\frac{1}{2}U(-p)D^\alpha \bar{D}^2 D_\alpha U(p) q^2 + \frac{1}{4}k^2 (q-p)^2 U(-p) U(p)]$,

(b) $\mathrm{Tr}\,[\frac{1}{4}k^2 (q-p)^2 U(-p) U(p)]$,

(c) $\mathrm{Tr}\,[\frac{1}{2}U(-p)D^\alpha \bar{D}^2 D_\alpha U(p) q^2 - \frac{1}{2}k^2 (q-p)^2 U(-p) U(p)]$. (3.12)

We make the interesting observation that *the total two-loop ghost contribution vanishes.*

The remaining graphs in fig. 3 give the following divergent contributions (with a factor $2g^2C_A/(4\pi)^4\int d^np\, d^4\theta/(2\pi)^n$)

(d) $\frac{3}{8}\frac{1}{\varepsilon}\mathrm{Tr}\,[\Gamma^{\alpha\dot\alpha}(-p)\Gamma_{\beta\dot\beta}(p)\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}]$,

(e) $\frac{3}{8}\frac{1}{\varepsilon^2}\mathrm{Tr}\,[\Gamma^{\alpha\dot\alpha}(-p)\Gamma_{\beta\dot\beta}(p)]\left[2\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}(1+\frac{9}{2}\varepsilon-2\rho\varepsilon)-\frac{p_{\alpha\dot\alpha}p^{\beta\dot\beta}}{p^2}(1+5\varepsilon-2\rho\varepsilon)\right]$,

$-\frac{3}{\varepsilon}\mathrm{Tr}\,[W^\alpha(-p)\bar W^{\dot\alpha}(p)]\frac{p_{\alpha\dot\alpha}}{p^2}$,

(f) $\frac{3}{2\varepsilon^2}(1+5\varepsilon-2\rho\varepsilon)\,\mathrm{Tr}\,(\Gamma^\alpha(-p)W_\alpha(p)+\bar\Gamma^{\dot\alpha}(-p)\bar W_{\dot\alpha}(p))$, (3.13)

where $\rho=\gamma_E-\ln 4\pi+\ln p^2/\mu^2$.

We are using regularization by dimensional reduction and $\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}$ represents a Kronecker-delta in $n<4$ dimensions. The sum of these contributions gives

$$\frac{3}{8\varepsilon^2}(1+5\varepsilon-2\rho\varepsilon)\,\mathrm{Tr}\,[\Gamma^{\alpha\dot\alpha}(-p)\Gamma_{\beta\dot\beta}(p)]\left[2\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}-\frac{p_{\alpha\dot\alpha}p^{\beta\dot\beta}}{p^2}\right]$$
$$+\frac{3}{2\varepsilon^2}(1+5\varepsilon-2\rho\varepsilon)\,\mathrm{Tr}\,[\Gamma^\alpha(-p)W_\alpha(p)+\bar\Gamma^{\dot\alpha}(-p)\bar W_{\dot\alpha}(p)]$$
$$-\frac{3}{\varepsilon}\mathrm{Tr}\,[W^\alpha(-p)\bar W^{\dot\alpha}(p)]\frac{p_{\alpha\dot\alpha}}{p^2}, \tag{3.14}$$

which is manifestly transverse. In vector representation for the background fields, we have

$$W_\alpha=i\bar D^2D_\alpha U,\qquad \Gamma_\alpha=\tfrac{1}{2}iD_\alpha U,$$
$$\Gamma_{\alpha\dot\alpha}=-\tfrac{1}{2}[D_\alpha,\bar D_{\dot\alpha}]U=(\bar D_{\dot\alpha}D_\alpha-\tfrac{1}{2}i\partial_{\alpha\dot\alpha})U. \tag{3.15}$$

From the first term in (3.14) after integration by parts we obtain

$$\int \Gamma^{\alpha\dot\alpha}(-p)\Gamma_{\beta\dot\beta}(p)\left[2\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}-\frac{p_{\alpha\dot\alpha}p^{\beta\dot\beta}}{p^2}\right]$$
$$=\int D^\alpha U\bar D^2D_\beta U\,\delta_{\dot\beta}{}^{\dot\alpha}\left(2\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}-\frac{p_{\alpha\dot\alpha}p^{\beta\dot\beta}}{p^2}\right)$$
$$=(3-2\varepsilon)\int D^\alpha U\bar D^2D_\alpha U, \tag{3.16}$$

where we have used the dimensional reduction rules

$$\hat\delta_{\alpha\dot\alpha}{}^{\beta\dot\beta}\delta_{\dot\beta}{}^{\dot\alpha}=\tfrac{1}{2}n\delta_\alpha{}^\beta,\qquad p_{\alpha\dot\alpha}p^{\beta\dot\alpha}=\delta_\alpha{}^\beta p^2, \tag{3.17}$$

(see eq. (6.6.2) in ref. [5]).

The total divergent contribution from the graphs in fig. 3 to the effective action is therefore

$$g^2 \frac{C_A}{(4\pi)^4}\left(-\frac{3}{8\varepsilon^2}+\frac{3}{8\varepsilon}\right)\int d^4x\, d^4\theta\, D^\alpha U \bar{D}^2 D_\alpha U\,. \tag{3.18}$$

However this is not the correct ultraviolet divergence of the two-loop self-energy, for two reasons: as in component Yang–Mills the renormalization of the gauge-parameter leads to additional one-loop graphs, containing one quantum-background vertex from $\xi\nabla^2 V\bar{\nabla}^2 V$ [4]. In addition, the graph in fig. 3e contains the uncorrected one-loop self-energy leading to an infrared divergence which gives a spurious $1/\varepsilon$ contribution to (3.18). If we use a conventional gauge-fixing term this divergence must be separated out and discarded. However, we have chosen a gauge-fixing term which produces the necessary correction to remove the infrared divergence, and we must now include its contribution. At $O(g^4)$, the addition to (3.18) can be obtained from a calculation of a *one-loop* insertion of $\xi\nabla^2 V(-\square_+)^{-\varepsilon}\bar{\nabla}^2 V$, with two external background field lines. We describe now an operator approach to this calculation.

The quadratic action that follows from (3.1) and (3.7) leads to the one-loop effective action

$$\begin{aligned}\Gamma = &-\tfrac{1}{2}\operatorname{Tr}\ln\left[\hat{\square}+\xi(\nabla^2(-\square_+)^{-\varepsilon}\bar{\nabla}^2+\bar{\nabla}^2(-\square_-)^{-\varepsilon}\nabla^2)\right]\\ &+\operatorname{Tr}\ln\left[\square_- +\xi\nabla^2(-\square_+)^{-\varepsilon}\bar{\nabla}^2\right],\end{aligned} \tag{3.19}$$

corresponding to contributions from V and the covariantly chiral NK ghost b. Here

$$\begin{aligned}\hat{\square} &= \square - i\mathrm{W}^\alpha\nabla_\alpha - i\bar{\mathrm{W}}^{\dot\alpha}\bar{\nabla}_{\dot\alpha}\,,\\ \square_- &= \square - i\bar{W}^{\dot\alpha}\bar{\nabla}_{\dot\alpha} - \tfrac{1}{2}i(\bar{\nabla}^{\dot\alpha} W_{\dot\alpha})\,,\end{aligned} \tag{3.20}$$

where $\square_-$ is defined as $\nabla^2\bar{\nabla}^2\bar{\phi}=\square_-\bar{\phi}$. Expanding to first order in ξ we find

$$\begin{aligned}\Gamma_\xi &= \tfrac{1}{2}\xi\operatorname{Tr}\left(\frac{1}{\square_-}-\frac{1}{\hat{\square}}\right)\nabla^2(-\square_+)^{-\varepsilon}\bar{\nabla}^2+\text{h.c.}\\ &= \tfrac{1}{2}\xi\operatorname{Tr}\frac{1}{\hat{\square}}\left(-iW^\alpha\nabla_\alpha+\tfrac{1}{2}i(\bar{\nabla}^{\dot\alpha}\bar{W}_{\dot\alpha})\right)\frac{1}{\square_-}\nabla^2(-\square_+)^{-\varepsilon}\bar{\nabla}^2+\text{h.c.}\end{aligned} \tag{3.21}$$

The term $-iW^\alpha\nabla_\alpha$ gives zero, since it is acting on an antichiral quantity.

The evaluation of this expression is a simple exercise in covariant D-algebra [6]. We describe the steps in appendix B. To second order in the background fields we obtain a contribution

$$\Gamma_\xi = -\frac{\xi}{8\varepsilon}\left(\frac{1}{4\pi}\right)^2(1-\varepsilon-\gamma\varepsilon)\operatorname{Tr}\int d^4x\, d^4\theta\, D^\alpha U\bar{D}^2 D_\alpha U\,. \tag{3.22}$$

Substituting the value of ξ from (3.5) and adding this result to (3.18), we obtain the final result for the ultraviolet divergence of the two-loop effective action

$$\Gamma_\infty = \frac{3}{4}\frac{1}{(4\pi)^4\varepsilon} g^2 C_A \operatorname{Tr} \int d^4x\, d^4\theta\, D^\alpha U \bar{D}^2 D_\alpha U\,, \qquad (3.23)$$

giving the familiar value $\beta_{2\text{-loop}} = -g\, 6g^4 C_A^2/(4\pi)^4$.

We wish to thank M. Roček and W. Siegel for very helpful suggestions and comments. D.Z. thanks the Physics Departments of Brandeis and Harvard University for their hospitality and INFN for financial support.

Appendix A

We evaluate here the commutator $[(-\Box)^{-\varepsilon-2}, D_\nu]$ that appears in (2.9). Using proper time representation we write first

$$(-\Box)^{-\varepsilon-2} = \frac{1}{(i)^{\varepsilon+2}\Gamma(\varepsilon+2)} \int_0^\infty d\tau\, \tau^{\varepsilon+1}\, e^{-i\tau\Box}\,. \qquad (A.1)$$

Defining $D_\nu(\tau) = e^{-i\tau\Box} D_\nu\, e^{i\tau\Box}$ we obtain in standard fashion

$$\dot{D}_\nu = -i\, e^{-i\tau\Box}[\Box, D_\nu]\, e^{i\tau\Box}\,, \qquad (A.2)$$

so that

$$D_\nu(\tau) = D_\nu - i\int_0^\tau d\tau'\, e^{-i\tau'\Box}[\Box, D_\nu]\, e^{i\tau'\Box}\,, \qquad (A.3)$$

or

$$[e^{-i\tau\Box}, D_\nu] = -i\int_0^\tau d\tau'\, e^{-i\tau'\Box}[\Box, D_\nu]\, e^{-i(\tau-\tau')\Box}\,. \qquad (A.4)$$

Therefore

$$\begin{aligned}[(-\Box)^{-\varepsilon-2}, D_\nu] &= -\frac{1}{(i)^{\varepsilon+1}\Gamma(\varepsilon+2)}\int_0^\infty d\tau \int_0^\tau d\tau'\, \tau^{\varepsilon+1}\, e^{-i\tau'\Box}[\Box, D_\nu]\, e^{-i(\tau-\tau')\Box} \\ &= -\frac{1}{(i)^{\varepsilon+1}\Gamma(\varepsilon+2)}\int_0^\infty d\tau' \int_0^\infty d\tau\, (\tau+\tau')^{\varepsilon+1}\, e^{-i\tau'\Box}[\Box, D_\nu]\, e^{-i\tau\Box}\,.\end{aligned} \qquad (A.5)$$

Writing $\tau = \rho\alpha$, $\tau' = \rho(1-\alpha)$ and going to momentum space leads to the last term in (2.10).

Eq. (A.1) may also be used for a perturbative evaluation of $(-\Box)^{-\varepsilon}$ with $\Box = \Box_0 + Y$. To first order in Y we have

$$\begin{aligned}(-\Box_0 - Y)^{-\varepsilon} &= \frac{1}{i^\varepsilon\Gamma(\varepsilon)}\int_0^\infty d\tau\, \tau^{\varepsilon-1}\, e^{-i\tau\Box_0} \\ &\quad - \frac{1}{i^{\varepsilon-1}\Gamma(\varepsilon)}\int_0^\infty d\tau \int_0^\tau d\tau'\, \tau^{\varepsilon-1}\, e^{-i(\tau-\tau')\Box_0} Y\, e^{-i\tau'\Box_0}\,.\end{aligned} \qquad (A.6)$$

The second term may be rewritten as

$$-\frac{1}{i^{\varepsilon-1}\Gamma(\varepsilon)}\int_0^\infty d\tau \int_0^\infty d\tau'\,(\tau+\tau')^{\varepsilon-1}\,e^{-i\tau\Box_0}\,Y\,e^{-i\tau\Box_0}\,, \tag{A.7}$$

and treated in the same fashion as (A.5).

Appendix B

We describe here the evaluation of the expression in (3.21). The trace operation includes integrations over $d^4\theta$ and as usual we will need two D's and two $\bar{D}$'s to get a non-vanishing contribution. These must come from the explicit spinor covariant derivatives in (3.8), (3.20), (3.21) (recall that $\nabla_\alpha = D_\alpha - i\Gamma_\alpha$). Furthermore, since we are looking for a contribution to the two-point function, in any term that has two explicit factors of the background field we can replace the covariant derivatives by ordinary derivatives. In (3.21) we already have one $\bar{W}_{\dot\alpha}$ so all other factors need be evaluated only to first order in the background fields. The expression in (3.21) may be rewritten as

$$\Gamma_\xi = \mathrm{Tr}\left(-\tfrac{1}{4}i\xi\frac{1}{\hat{\Box}}(\bar{\nabla}^{\dot\alpha}\bar{W}_{\dot\alpha})[\nabla^2(-\Box_+)^{-\varepsilon-1}\bar{\nabla}^2-\bar{\nabla}^2(-\Box_-)^{-\varepsilon-1}\nabla^2)\right). \tag{B.1}$$

We now use the expressions for $\hat{\Box}$, $\Box_+$, $\Box_-$, in (3.8), (3.20) and also expand

$$\Box = \Box_0 - i\Gamma^a\partial_a - \tfrac{1}{2}i(\partial^a\Gamma_a)\,. \tag{B.2}$$

The Γ^a terms do not contribute due to cancellations between the two terms in (B.1). The W-terms from the expansion of $\hat{\Box}$ also add up to zero using integration by parts and the Bianchi identity $\nabla^\alpha W_\alpha + \bar{\nabla}^{\dot\alpha}\bar{W}_{\dot\alpha} = 0$. The only non-vanishing contributions come from the expansion of $\Box_+$ and $\Box_-$ to first order in W and $\bar{W}$, respectively. Using (A.7),

$$\begin{aligned}\Gamma_\xi &= \tfrac{1}{4}\xi\,\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,[\bar{\nabla}^{\dot\alpha}\bar{W}_{\dot\alpha}(-p)][\nabla^\alpha W_\alpha(p)]\\ &\quad\times\frac{1}{(i)^\varepsilon\Gamma(\varepsilon+1)}\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2}\int_0^\infty d\tau\int_0^\infty d\tau'\,(\tau+\tau')^\varepsilon\,e^{i\tau q^2}\,e^{i\tau'(p+q)^2}\\ &= \tfrac{1}{4}\xi\,\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,p^2D^\alpha U(-p)\bar{D}^2D_\alpha U(p)\\ &\quad\times\int\frac{d^nq}{(2\pi)^n}\left[-\frac{1}{q^2[(p+q)^2]^{\varepsilon+2}}+(\varepsilon+2)\int_0^1 d\alpha\frac{1}{[q^2\alpha+(p+q)^2(1-\alpha)]^{\varepsilon+3}}\right].\end{aligned} \tag{B.3}$$

Performing the integrals in this expression gives (3.22).

References

[1] J.W. Juer and D. Storey, Nucl. Phys. B216 (1983) 185
[2] T.E. Clark, O. Piguet and K. Sibold, Ann. of Phys. 109 (1977) 418;
O. Piguet and K. Sibold, Nucl. Phys. B197 (1982) 272;
P. Howe and P.C. West, The two-loop β-function in models with extended rigid supersymmetry, Imperial college preprint
[3] N.K. Nielsen, Nucl. Phys. B140 (1978) 499;
R.E. Kallosh, Nucl. Phys. B141 (1978) 141
[4] L.F. Abbott, Nucl. Phys. B185 (1981) 189
[5] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, Superspace (Benjamin-Cummings, Reading, MA, 1983)
[6] M.T. Grisaru and D. Zanon, to be published