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THE COLLAPSE OF AN ANTI-DE SITTER BUBBLE

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We prove that the ultimate fate of a bubble of negative energy density which forms in a metastable universe of zero energy density is gravitational collapse. We improve on previous treatments in that we allow for departures from O(3, 1) symmetry in the initial state, so long as they are not too great.

It is an intriguing possibility that the universe is currently in a metastable phase, a false vacuum which will eventually decay into a phase of lower energy density. In the semiclassical approximation, such decays occur through the formation and growth of bubbles of the new phase inside the old phase [1]. The formation of the bubbles is a quantum tunneling effect, but their subsequent growth is governed by the classical equations of motion. If the current vacuum has zero-energy density, the bubble interiors will have negative energy density; their geometries will resemble that of anti-de Sitter space. Some years ago, Frank de Luccia and one of us argued that gravitational effects rendered these bubble interiors themselves unstable, that the ultimate fate of a zero-energy metastable vacuum was not merely a transition to a new phase but also a catastrophic gravitational collapse [2].

Our arguments were highly suggestive, but not conclusive. Quite apart from deficiencies in rigor, they rested upon the exact O(3, 1) invariance of the solution to the classical field equations describing the evolution of the bubble after its formation. It remained a possibility that the slightest departure from this symmetry would be sufficient to avoid gravitational collapse, that the perturbation introduced by the presence of a paramecium would be enough to save the universe. [This is not an extraordinary possibility. A familiar example is a particle moving in an inverse-square

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force in three dimensions. If we choose initial conditions invariant under rotation about some axis, the particle will inevitably encounter the central singularity at some finite (positive or negative) time. However, arbitrarily small asymmetry (nonzero angular momentum) is sufficient to avoid the singularity for all time.]

In this note, we use a theorem of Penrose [3] to eliminate this possibility. We consider the initial conditions (values of the fields and their time derivatives) describing the state of the universe at the moment of bubble formation. We show that for all initial conditions sufficiently close to the O(3,1) symmetric ones, either (i) time evolution leads to a curvature singularity, or (ii) the initial-value surface is not Cauchy, that is to say, the initial conditions are not sufficient to determine the future evolution of the universe.

Although we are not able to eliminate rigorously the second alternative, we favor the first. Firstly, it is what happens in the case of O(3,1) symmetry. Secondly, although anti-de Sitter space itself notoriously has no Cauchy surface, the geometry just after bubble formation is not that of a time slice of anti-de Sitter space but that of a piece of anti-de Sitter space (the bubble interior) surrounded by infinite flat space; this should constitute a Cauchy surface. (It would be very amusing though, if we were wrong in this guess. This would mean that an inhabitant of an unstable zero-energy vacuum would face a future genuinely uncertain, not just one subject to quantum coherence.)

The remainder of this note gives our detailed arguments.

We begin by briefly reviewing the relevant parts of the analysis of ref. [2]. We consider the theory of a single scalar field with nonderivative self-couplings, minimally coupled to einsteinian gravity:

$$\mathcal{L} = -\frac{R}{16\pi G} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi)\,. \tag{1}$$

Here U is a function with two local minima ("vacua"), a false vacuum at $\phi = \phi_+$ with $U(\phi_+) = 0$, and a true vacuum at $\phi = \phi_-$, with $U(\phi_-) \equiv U_- < 0$. Vacuum decay is especially easy to describe if U_- is close to zero. (This is the so-called thin-wall approximation.) In the leading thin-wall approximation, the world after the materialization of the bubble of true vacuum is the union of two regions. One is the bubble exterior, that part of Minkowski space obeying

$$|\boldsymbol{x}|^2 - t^2 \geqslant \bar{\rho}^2\,, \tag{2}$$

where $\bar{\rho}$ is a length determined by the exact form of U. (We have chosen our coordinates such that the bubble materializes at $t = 0$, with its center at the origin of coordinates.) In the bubble exterior, ϕ is ϕ_+. The other region is the bubble interior, where $\phi = \phi_-$. The geometry of the bubble interior is that of a portion of anti-de Sitter space. If we describe anti-de Sitter space as the hyperboloid

$$w^2 + t^2 - |\boldsymbol{x}|^2 = -\frac{3}{8\pi G U_-} \equiv \Lambda^2, \tag{3}$$

in a five-dimensional space with metric

$$ds^2 = dw^2 + dt^2 - |d\boldsymbol{x}|^2, \tag{4}$$

then the bubble interior consists of that portion of the space with

$$w \geqslant -\sqrt{\Lambda^2 + \bar{\rho}^2}\,. \tag{5}$$

The interior and the exterior are joined together by identifying points on their respective boundaries with the same values of $\boldsymbol{x}$ and t. The O(3, 1) symmetry is the usual Lorentz group acting on $\boldsymbol{x}$ and t, in both the interior and the exterior.

The bubble interior can also be described as an expanding-and-contracting open Robertson-Walker universe. That is to say, there exist coordinates for the interior such that

$$ds^2 = d\tau^2 - \rho^2(\tau)\, d\sigma^2, \tag{6}$$

where $d\sigma^2$ is the metric for a three space of unit negative curvature,

$$d\sigma^2 = dr^2 + \sinh^2 r\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{7}$$

and

$$\rho = \Lambda \sin(\tau/\Lambda)\,. \tag{8}$$

In these coordinates, the O(3, 1) group is the isometry group of the universe at fixed τ. These coordinates are not without their deficiencies. For one thing, they do not cover the entire bubble interior; we must continue to imaginary τ to reach the boundary. For another, the apparent singularities at $\tau = n\pi\Lambda$, where the radius of the universe, ρ, vanishes are mere coordinate artifacts. Nevertheless, these coordinates will be very useful for our purposes.

Gravitational collapse emerges when we consider the corrections to the leading thin-wall approximation. The exact bubble solution is O(3, 1) invariant, just like the approximate one, but ϕ in the interior of the bubble differs by a small but nonzero amount from its equilibrium value, ϕ_-. During most of the evolution described by eq. (6), this has only a tiny effect. ϕ oscillates about ϕ_-, but, as the universe expands, the magnitude of the oscillation diminishes. When the expansion becomes contraction, at $\tau = \frac{1}{2}\pi\Lambda$, the oscillations begin to grow, but for a long while they remain small. However, it was shown in ref. [2] that as τ approaches $\pi\Lambda$, the oscillations become very large, the energy density in the scalar field grows without bound, and gravitational collapse ensues, turning the coordinate singularity in eq. (8) into a genuine curvature singularity.

This analysis rests heavily on exact O(3,1) invariance. If we imagine some departure from O(3,1) invariance in the initial state of the universe, no matter how small it is to begin with, it will eventually grow large as gravitational collapse approaches. A linear approximation will then no longer be adequate; we must consider the full nonlinear coupling of the symmetric and asymmetric modes. It is by no means clear whether this coupling will hasten or abort collapse.

Fortunately, we do not have to analyze this formidable problem, for a theorem of Penrose [3] can be used to resolve the question. Penrose's theorem states that a spacetime can not be null geodesically complete if

(A) $R_{\mu\nu}K^{\mu}K^{\nu} \geqslant 0$ for all null vectors K^{μ},

(B) it has a noncompact Cauchy surface, and

(C) it has a closed trapped surface.

Null geodesic incompleteness means that it is impossible to indefinitely prolong a null geodesic; it is a sufficient condition for the existence of a singularity.

For our theory, Einstein's equations imply that for null K^{μ},

$$R_{\mu\nu}K^{\mu}K^{\nu} = 8\pi G\left(K^{\mu}\partial_{\mu}\phi\right)^{2}. \tag{9}$$

Thus, condition A is satisfied. [There are stronger successors to Penrose's theorem [3], which an informed reader might be tempted to apply to this problem. Unfortunately, they involve stronger restrictions on the Ricci tensor, which do not apply in our case. (We do not even have positivity of the energy density.)]

We have explained in the introductory portion of this note why we believe the initial state of the universe at the moment of bubble formation satisfies condition B.

Condition C requires a little more work. A closed trapped surface is a closed two-dimensional surface from which light can not escape. More precisely, it is a surface for which both the inward-pointing and outward-pointing normal null geodesics are convergent. This can be expressed analytically by the statement that certain geometric objects (the null second fundamental forms) are negative on the surface. Thus, given a spacetime with a closed trapped surface, any spacetime with metric sufficiently close to that of the given one in the region of the surface will also have a closed trapped surface.

We shall now show that the exact O(3,1) invariant solution has a closed trapped surface, for some time earlier than the moment of collapse. Since we can always choose initial-value perturbations sufficiently small so that they remain small at this time (although they may grow large later), this implies that gravitational collapse occurs even in the presence of these perturbations.

The exact metric still obeys eqs. (6) and (7). However eq. (8) is replaced by the solution to the differential equation

$$\dot{\rho}^{2} = 1 + \tfrac{8}{3}\pi G\rho^{2}\left(\tfrac{1}{2}\dot{\phi}^{2} + U\right), \tag{10}$$

where the dot indicates differentiation with respect to τ. The quantity in parentheses is the energy density of the scalar field. As we have said, in the exact solution this grows without bound as we approach collapse. This will be important to us shortly.

Now let us consider the emission of light from a sphere of radius r at time τ. The area of this sphere is

$$A = 4\pi\rho^2 \sinh^2 r. \tag{11}$$

After a time $\mathrm{d}\tau$ light emitted from this sphere will reach radii

$$r_\pm = r \pm \frac{\mathrm{d}\tau}{\rho}, \tag{12}$$

and define two new spheres with areas

$$A_\pm = A\left[1 + \frac{2}{\rho}(\dot{\rho} \pm \coth r)\,\mathrm{d}\tau\right]. \tag{13}$$

The initial sphere is a closed trapped surface if both these areas are less than A. From eq. (10), when the scalar-field energy density becomes positive, on its way to infinity, $\dot{\rho}$ becomes less than minus one. At this time, for sufficiently large r, the initial sphere is a closed trapped surface. This completes the argument.

References

[1] We follow here the treatment of S. Coleman, Phys. Rev. D15 (1977) 2929; 16 (1977) 1248 (E); C.G. Callan and S. Coleman, Phys. Rev. D16 (1977) 1762. These contain references to the earlier literature

[2] S. Coleman and F. De Luccia, Phys. Rev. D21 (1980) 3305

[3] R. Penrose, Phys. Rev. Lett. 14 (1965) 57. A more detailed treatment with information about related theorems is in:
S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, 1973)