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GRAVITON PRODUCTION IN INFLATIONARY COSMOLOGY

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We provide a completely quantum-mechanical derivation of the spectrum of gravitational waves produced in any inflationary cosmology. The gravitational waves result from a sequence of Bogoliubov transformations between creation and annihilation operators defined in de Sitter space and in radiation- and matter-dominated Robertson-Walker spacetimes. We discuss how the results depend on the initial state at the beginning of the inflationary period.

The production of gravitational waves in an inflationary cosmology provides an important constraint on any model of inflation [1-4]. The Hubble constant during the de Sitter phase must be less than 10^{15} GeV or else these gravitational waves will produce unacceptably large anisotropies in the microwave background radiation [1-4]. Similar constraints hold for power law or any other form of inflation [4]. The gravitational waves can also affect pulsar timing measurements [5].

The calculation of the spectrum of gravitational waves produced by inflation is similar to, although simpler than, the calculation for the scalar energy-density fluctuations [6] which give rise to large-scale structure in an inflationary universe. In both cases, the quantum mechanical two-point function is related to a two-point statistical average of an ensemble of classical fields. The evolution of the fields to the present time is then governed by the classical equations of motion. Since the use of such a mixed quantum mechanical-classical formalism raises some subtle issues [7], we present here an alternate derivation of the gravitational wave spectrum which is purely quantum mechanical. Our results are not new, agreeing with those of refs. [1-4]. However, we believe that our presentation will help clarify the physics behind these important results and remove any doubts about their validity. We also address the question of the generality of the results and of their dependence on the initial state of the Universe at the beginning of the inflationary epoch.

In our approach, gravitational waves exist today because the de Sitter vacuum established during the inflationary period looks like a multiparticle state when the definition of particles relevant to the present matter-dominated universe is used. In our computation we will determine the Bogoliubov transformations relating creation

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and annihilation operators in de Sitter space to those of radiation and matter-dominated Robertson–Walker spacetimes. Actually, our calculation could be done ignoring the radiation-dominated era entirely. However, the intermediate results during the radiation-dominated period are relevant to pulsar timing measurements [5] and the three-step calculation we present is more realistic than the two-step calculation which ignores the radiation-dominated period. Our final results are relevant for calculations of large-scale anisotropies of the microwave background radiation [1-4].

We use here a conformal time variable, τ, so that the metric during inflationary, radiation-dominated and matter-dominated eras can be written in the form

$$ds^2 = S^2(\tau)[-d\tau^2 + dx \cdot dx]\,, \tag{1}$$

with

$$S(\tau) = \begin{cases} -1/\chi\tau & \text{during inflation} \\ A\tau & \text{during radiation domination} \\ B\tau^2 & \text{during matter domination}\,. \end{cases} \tag{2}$$

Here χ is the value of the Hubble constant during the de Sitter phase of the inflationary period and A and B are constants. If we choose to make the scale factor S equal to one today and match S and its first derivative at the two transitions, then

$$A = 4\tau_1/\tau_0^2\,,$$
$$B = 1/\tau_0^2\,, \tag{3}$$

where τ_0 is the present conformal time ($\tau_0 = 3t_0$) and τ_1 is the conformal time at the end of the radiation-dominated era ($\tau_1 = 2^{-1/3}3^{2/3}t_m^{1/3}t_0^{2/3}$ where t_m is the proper time when the matter-dominated period begins).

In treating the transitions between different cosmological epochs, we will only require that S and its first derivative be continuous. Thus, for example, the curvature has unphysical discontinuities at the transitions in this approximation which would be avoided in a realistic model. However, this is unimportant as long as we are not interested in details during the transitions or in short wavelengths.

The definition of conformal time used above has the property that it jumps discontinuously at the transitions between different epochs. For instance, if the inflationary era ends at time τ_2 (which by eq. (2) must be negative) then by requiring S and its first derivative to be continuous at the transition, we find that the radiation-dominated era starts at time $-\tau_2$. Likewise, if the radiation-dominated period ends at time τ_1 then the matter-dominated period begins at time $2\tau_1$. It is important to take these discontinuities into account when computing the relevant Bogoliubov coefficients. Finally, we note that the time τ_2 when inflation ends is determined by matching S and its first derivative from eq. (2) giving

$$\tau_2 = -\tau_0/2\sqrt{\chi\tau_1}\,. \tag{4}$$

We work in a transverse, traceless gauge and write the metric variations about the background spacetime in terms of graviton creation and annihilation operators as

$$h_{ij}=\sqrt{8\pi G}\sum_{\lambda}\int\frac{d^3k}{(2\pi)^{3/2}S(\tau)\sqrt{2k}}[a_{\lambda}(\boldsymbol{k})\varepsilon_{ij}(\boldsymbol{k},\lambda)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\xi(k\tau)$$
$$+a_{\lambda}^{\dagger}(\boldsymbol{k})\varepsilon_{ij}^{*}(\boldsymbol{k},\lambda)\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\xi^{*}(k\tau)]\,,\tag{5}$$

where λ runs over the two polarizations and $\varepsilon_{ij}(\boldsymbol{k},\lambda)$ are polarization tensors. The function $\xi(k\tau)$ is given by

$$\xi(k\tau)=e^{-ik\tau}(1-i/k\tau)\tag{6}$$

during the inflationary- and matter-dominated eras and by

$$\xi(k\tau)=e^{-ik\tau}\tag{7}$$

during the radiation-dominated period.

The graviton creation and annihilation operators during the radiation-dominated era are related to those of the inflationary epoch by the Bogoliubov transformations

$$a_{\lambda}^{\rm rad}(\boldsymbol{k})=\alpha_1(k)a_{\lambda}^{\rm inf}(\boldsymbol{k})+\beta_1^{*}(k)(a_{\lambda}^{\rm inf}(-\boldsymbol{k}))^{\dagger}\,.\tag{8}$$

Likewise, the creation and annihilation operators in the matter-dominated era are given by

$$a_{\lambda}^{\rm mat}(\boldsymbol{k})=\alpha_2(k)a_{\lambda}^{\rm rad}(\boldsymbol{k})+\beta_2^{*}(k)(a_{\lambda}^{\rm rad}(-\boldsymbol{k}))^{\dagger}\,,\tag{9}$$

or equivalently

$$a_{\lambda}^{\rm mat}(\boldsymbol{k})=(\alpha_1\alpha_2+\beta_1\beta_2^{*})a_{\lambda}^{\rm inf}(\boldsymbol{k})+(\beta_1^{*}\alpha_2+\alpha_1^{*}\beta_2^{*})(a_{\lambda}^{\rm inf}(-\boldsymbol{k}))^{\dagger}\,.\tag{10}$$

Before proceeding to the calculation of the Bogoliubov coefficients α_1, β_1, α_2 and β_2 we will discuss the significance of such transformations in general. Early in the inflationary period we will characterize the state of the universe $|\psi\rangle$ in terms of numbers of gravitons defined by the creation and annihilation operators $a_{\lambda}^{\rm inf}(\boldsymbol{k})^{\dagger}$ and $a_{\lambda}^{\rm inf}(\boldsymbol{k})$. This may seem a bit arbitrary, but recall that the wavelengths we are interested in are of the order of the size of the observed universe, that is $k\tau_0\approx 1$. Early in the inflationary epoch $|k\tau|$ was much greater than one for these waves since inflation had not yet red shifted them to long wavelengths. Note that $|k\tau|$ is just the ratio of the horizon size to the physical wavelength S/k. Thus $|k\tau|\gg 1$ tells us that originally the waves were well inside the horizon, while $k\tau_0=1$ tells us that these waves are presently entering the horizon. For $|k\tau|\gg 1$ the modes of eq. (6) reduce to plane waves and thus we are just using the ordinary Minkowski definition of particles to characterize the state $|\psi\rangle$. For the long-wavelength waves we are discussing, the transitions between different cosmological epochs are effectively sudden and therefore the universe will remain in the state $|\psi\rangle$. However, the graviton creation and annihilation operators will change. Consider the gravitational field h_{ij} defined by eq. (5) during some epoch with the creation and annihilation operators $a_\lambda^\dagger(\boldsymbol{k})$ and $a_\lambda(\boldsymbol{k})$ related to those of inflation by

$$a_\lambda(\boldsymbol{k}) = \alpha(k)a_\lambda^{\text{inf}}(\boldsymbol{k}) + \beta^*(k)(a_\lambda^{\text{inf}}(-\boldsymbol{k}))^\dagger . \tag{11}$$

A convenient quantity with which we can characterize fluctuations in h_{ij} at time τ is

$$\Delta h^2(\boldsymbol{k}) \equiv \frac{k^3}{(2\pi)^3}\frac{1}{2}\int \mathrm{d}^3x\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\langle\psi|h_{ij}(\boldsymbol{x},\tau)h_{ij}(\boldsymbol{o},\tau)|\psi\rangle \tag{12}$$

with a sum on i and j. The factor of $\frac{1}{2}$ serves to average over polarizations. If τ occurs during the epoch in which the operators of eq. (11) are relevant then from (11) and (5)

$$\Delta h^2(\boldsymbol{k}) = 8\pi G\frac{k^2}{(2\pi)^3 2S^2(\tau)}\frac{1}{2}\sum_\lambda (N_\lambda(\boldsymbol{k}) + N_\lambda(-\boldsymbol{k}) + 1)$$
$$\times[(|\alpha(k)|^2 + |\beta(k)|^2)|\xi(k\tau)|^2 + 2\,\mathrm{Re}\,\{\alpha(k)\beta^*(k)\xi^2(k\tau)\}] . \tag{13}$$

In evaluating expression (12) one encounters an infrared divergence which we remove by cutting off wavelengths which are much larger than the present observed universe and hence which are unobservable.

In eq. (13), $N_\lambda(\boldsymbol{k})$ is the expectation value of the de Sitter particle number operator $(a_\lambda^{\text{inf}}(\boldsymbol{k}))^\dagger a_\lambda^{\text{inf}}(\boldsymbol{k})$. Notice that the dependence of the result (13) on the state $|\psi\rangle$ is exclusively through the phase space densities $N_\lambda(\boldsymbol{k})$ and $N_\lambda(-\boldsymbol{k})$. If $N_\lambda(\boldsymbol{k})$ and $N_\lambda(-\boldsymbol{k})$ are much less than one, the result becomes essentially independent of the state $|\psi\rangle$. Since we have chosen to characterize $|\psi\rangle$ by values of $N_\lambda(\boldsymbol{k})$ early during the inflationary epoch when the physical wavelength is much shorter than the horizon size $1/\chi$, it seems extremely reasonable to assume that $N_\lambda(\boldsymbol{k})$ and $N_\lambda(-\boldsymbol{k})$ are much smaller than one. A phase space density of order one would represent an enormous energy density in what are at this time high frequency modes. In conclusion, if $N_\lambda(\boldsymbol{k})$ and $N_\lambda(-\boldsymbol{k})$ are much less than one for the values of $\boldsymbol{k}$ relevant for our calculation, the result (13) becomes independent of the state $|\psi\rangle$ which can therefore be taken to be the de Sitter invariant vacuum with $N_\lambda(\boldsymbol{k}) = N_\lambda(-\boldsymbol{k}) = 0$.

In order to compute the Bogoliubov coefficients defined in eqs. (8) and (9) we match the field h_{ij} and its first τ-derivative at the moment of the transition from inflation to radiation domination for (8) and from radiation domination to matter domination for (9). In doing this, we must take into account the discontinuities in τ during these transitions. The result for α_1 and β_1 is

$$\alpha_1 = \mathrm{e}^{-2ik\tau_2}\left[1 - \frac{1}{2(k\tau_2)^2} - \frac{i}{(k\tau_2)}\right],$$
$$\beta_1 = \frac{1}{2(k\tau_2)^2} . \tag{14}$$

Recall that τ_2 is the conformal time when inflation ends and that the quantity $k\tau_2$ is minus the ratio of the horizon size to the physical wavelength at the end of the inflationary era. By time τ_2 the modes in question are well outside the horizon, $|k\tau_2| \gg 1$ and, using the value of τ_2 given in eq. (4), the result (14) reduces to

$$\beta_1 \approx -\alpha_1 \approx \frac{2\tau_1 \chi}{(k\tau_0)^2}. \tag{15}$$

If we evaluate Δh^2 from eq. (13) at some time τ during the radiation-dominated era we find, using (15)

$$\Delta h^2(k) = 8\pi G \frac{k^2}{(2\pi)^3 S^2(\tau)} |\beta_1(k)|^2 [|\xi(k\tau)|^2 - \mathrm{Re}\{\xi^2(k\tau)\}]. \tag{16}$$

Evaluating (16) using eqs. (2), (3) and (7) we obtain

$$\Delta h^2(k) = \frac{G\chi^2}{2\pi^2}\left[\frac{\sin k\tau}{k\tau}\right]^2. \tag{17}$$

For modes which enter the horizon during the radiation-dominated period this is essentially the end of the story. These modes are unaffected by the subsequent Bogoliubov transformation into the matter-dominated era because for them this transition is adiabatic rather than sudden. However, the wavelengths which affect the microwave anisotropy are still well outside the horizon at the end of the radiation-dominated era and we must include the effects of this second Bogoliubov transformation. Matching the field h_{ij} and its τ-derivative at the end of the radiation-dominated period we find, with τ_1 equal to the conformal time at the end of the radiation-dominated epoch,

$$\alpha_2 = e^{ik\tau_1}\left[1 - \frac{1}{8(k\tau_1)^2} + \frac{i}{2(k\tau_1)}\right],$$

$$\beta_2 = -e^{-3ik\tau_1}\frac{1}{8(k\tau_1)^2}. \tag{18}$$

The coefficients we need are those relating the creation and annihilation operators of the matter-dominated era to those of the inflationary period, $\alpha_1\alpha_2 + \beta_1\beta_2^*$ and $\beta_1^*\alpha_2 + \alpha_1^*\beta_2^*$ of eq. (10). For waves which are well outside the horizon at the end of the radiation-dominated epoch, $k\tau_1 \gg 1$ and we find from (15) and (18)

$$\alpha_1\alpha_2 + \beta_1\beta_2^* \approx \beta_1\alpha_2^* + \alpha_1\beta_2 \approx \frac{-3i\chi}{2k^3\tau_0^2}. \tag{19}$$

We can now take this result and evaluate Δh^2 of eq. (13) using the ξ of eq. (6),

$$\Delta h^2(k) = \frac{9\chi^2 G}{2\pi^2}\left[\frac{j_1(k\tau_0)}{k\tau_0}\right]^2, \tag{20}$$

where j_1 is a spherical Bessel function. In eq. (20) we have evaluated Δh^2 at the present time $\tau = \tau_0$.

The results of eq. (19) or (15) and the fact that $|\psi\rangle$ can be taken to be the de Sitter vacuum can be used to compute the expectation value of any desired function of h_{ij}. Our results in eqs. (17) and (20) agree with those of refs. [1-4] but have been obtained in a purely quantum-mechanical formalism and shown to be independent of state over a wide range of initial states in the inflationary period.

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