Nuclear Physics B296 (1988) 710–716
North-Holland, Amsterdam

INTEGRAL CONSTRAINTS IN GENERAL RELATIVITY

L.F. ABBOTT*

Department of Physics, Boston University, Boston, MA 02215†, USA

J. TRASCHEN‡

Department of Astronomy and Astrophysics, University of Chicago, Chicago, IL 60633, USA

Rui-Ming XU

Department of Physics, Brandeis University, Waltham, MA 02254, USA

Received July 1987

Using a covariant formalism we derive the conditions for there to exist integral constraints on energy-momentum perturbations in an arbitrary background spacetime.

A conserved quantity corresponding to an asymptotic symmetry of an otherwise arbitrary spacetime manifold can be constructed in general relativity provided that a number of conditions are met. If we write the spacetime metric in the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ vanishes (sufficiently rapidly) at infinity but is otherwise arbitrary, then the background asymptotic metric $\bar{g}_{\mu\nu}$ must satisfy the Einstein equations with no energy-momentum source term,

$$\bar{R}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{R} = 0, \tag{1}$$

where $\bar{R}^{\mu\nu}$ and $\bar{R}$ are the Ricci tensor and scalar curvature corresponding to $\bar{g}_{\mu\nu}$. In addition, $\bar{g}_{\mu\nu}$ must allow a Killing vector $\bar{\xi}_{\mu}$. If these conditions are met, the procedure for constructing a conserved quantity is well-known [1]. Terms in the Einstein equation linear in $h_{\mu\nu}$ are separated from all terms containing higher powers of $h_{\mu\nu}$,

$$R_{\mathrm{L}}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}R_{\mathrm{L}} = (-\bar{g})^{-1/2}\mathscr{T}_{g}^{\mu\nu} . \tag{2}$$

* Supported in part by Department of Energy contract DE-AC02-76ER03230 and by an Alfred P. Sloan Foundation Fellowship.

† On leave from Brandeis University, Waltham, MA 02254, USA.

‡ Supported in part by Department of Energy grant AC02-80ER-10773. Address after Sept. 1, 1987: Department of Physics, University of California at Santa Barbara, Santa Barbara, CA 93106, USA.

0550-3213/88/$03.50

The sum of all non-linear terms, $\mathcal{T}_g^{\mu\nu}$, is the complete, general relativistic energy-momentum tensor density. It is covariantly conserved with respect to the background metric (that is $\bar{D}_\mu \mathcal{T}_g^{\mu\nu} = 0$ where $\bar{D}_\mu$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$). If the background metric allows a Killing vector $\bar{\xi}_\mu$ then $\mathcal{T}_g^{\mu\nu}\bar{\xi}_\mu$ is a conserved current and the quantity

$$\int \mathrm{d}^3x\, \mathcal{T}_g^{0\nu}\bar{\xi}_\nu \tag{3}$$

is conserved.

The quantity expressed in (3) as a volume integral can also be written as a flux integral over a two-dimensional surface at spatial infinity. More generally we can write

$$\int_{\mathrm{G}} \mathrm{d}^3x\, \mathcal{T}^{0\nu}\bar{\xi}_\nu = \oint_{\partial \mathrm{G}} \mathrm{d}^2S_i\, B^i \tag{4}$$

for any finite volume G with boundary ∂G. Eq. (4) is a general-relativistic analogue of Gauss' law. This sort of relation between a quantity defined by a volume integral and a two-dimensional flux-type integral is what we refer to as an integral constraint.

In most of the cases of interest to cosmology the above formalism is not directly applicable because we want to look at perturbations in background spacetimes which do not satisfy the necessary conditions. First, they are not solutions of the source-free Einstein equations but rather, since there is a background distribution of energy and momentum density $\bar{T}^{\mu\nu}$ the metric $\bar{g}_{\mu\nu}$ is a solution of the equation

$$\bar{R}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{R} = \bar{T}^{\mu\nu}. \tag{5}$$

Second, there may not be any appropriate Killing vectors for the background metric $\bar{g}_{\mu\nu}$. If we follow the procedure outlined above and write the full metric, including the deviation from the background, as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and the full matter energy-momentum tensor with perturbation $\delta T^{\mu\nu}$ as $T^{\mu\nu} = \bar{T}^{\mu\nu} + \delta T^{\mu\nu}$ we can again separate the Einstein equation into linear and non-linear parts (in $h_{\mu\nu}$),

$$R_{\mathrm{L}}^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}R_{\mathrm{L}} + \tfrac{1}{2}h^{\mu\nu}\bar{R} = (-\bar{g})^{-1/2}\mathcal{T}_g^{\mu\nu} + \delta T^{\mu\nu} \equiv (-\bar{g})^{-1/2}\mathcal{T}^{\mu\nu}\,. \tag{6}$$

However, in this case the resulting complete energy-momentum tensor density $\mathcal{T}^{\mu\nu}$ is not covariantly conserved with respect to the background metric, that is $\bar{D}_\mu \mathcal{T}^{\mu\nu} \neq 0$. Thus, it is not in general possible to construct conserved quantities and conserved currents in the usual manner. However, as we will demonstrate, it is still sometimes possible (even in the absence of a Killing vector) to construct an integral constraint equation of the form (4). The purpose of this paper is to show when and how this can be done.

Integral constraints in general relativity were first discussed in ref. [2] and they were used to constrain models of galaxy formation in ref. [3]. The original derivation used the non-covariant ADM [4] formalism. We present here an alternative covariant approach.

We construct integral constraints by relaxing the condition that there exists a conserved current while still demanding that eq. (4) be satisfied. In this case, the vector $\bar{\xi}_\mu$ is not necessarily a Killing vector but rather an integral constraint vector satisfying a more general condition which we will derive. Although an integral constraint vector does not have to be a Killing vector, a Killing vector may or may not be an integral constraint vector. The integral relation (4) becomes interesting for compact spatial surfaces or when we combine it with constraints coming from causality. Then, we see that the value of the quantity appearing on the l.h.s. of eq. (4) cannot change until a disturbance arising inside the integration volume G has had time to propagate to the surface ∂G. In this sense, the spatial integral on the left side of [4] behaves much like a conserved quantity even though it does not necessarily correspond to a spacetime symmetry and does not have a related conserved current. This is especially important in cosmology because Robertson-Walker spacetimes have ten integral constraint vectors but only six Killing vectors none of which has a component in the time direction.

We will now derive differential equations for $\bar{\xi}_\nu$ which must be satisfied if an integral constraint of the form (4) is to hold. Since we are dealing with vector and tensor densities eq. (4) is equivalent to

$$\mathcal{T}^{0\nu}\bar{\xi}_\nu = \partial_i B^i . \tag{7}$$

Before starting we must explain some of the conventions used in eq. (6) and below. The expressions $R_{\mathrm{L}}^{\mu\nu}$ and R_{L} refer to *all* terms in the complete $R^{\mu\nu}$ and R which are linear in $h_{\mu\nu}$ including those coming from raising, lowering and contracting indices. There are no terms in eq. (6) of zeroth order in $h_{\mu\nu}$ because $\bar{g}_{\mu\nu}$ satisfies the background field equation, (5). In what follows all indices are raised, lowered and contracted by the background metric $\bar{g}_{\mu\nu}$ alone. In particular, the integral constraint vector $\bar{\xi}_\nu$ is originally defined with an upper index which is then lowered using the background metric.

From eq. (6) if we expand the curvature to first order in $h_{\mu\nu}$ we find

$$(-\bar{g})^{-1/2}\mathcal{T}^{\mu\nu} = \bar{D}_\alpha \bar{D}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu} + Y^{\mu\nu} , \tag{8}$$

where

$$K^{\mu\alpha\nu\beta} = \tfrac{1}{2}\left(\bar{g}^{\mu\beta}H^{\nu\alpha} + \bar{g}^{\nu\alpha}H^{\mu\beta} - \bar{g}^{\mu\nu}H^{\alpha\beta} - \bar{g}^{\alpha\beta}H^{\mu\nu}\right), \tag{9}$$

$$X^{\mu\nu} = \tfrac{1}{2}\left[\bar{D}_\alpha, \bar{D}^\nu\right]H^{\mu\alpha}, \tag{10}$$

and

$$Y^{\mu\nu} = \tfrac{1}{2}h^{\mu\nu}\bar{R} - h^{\mu\alpha}\bar{R}^{\nu}_{\alpha} - h^{\nu\alpha}\bar{R}^{\mu}_{\alpha} + \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{R}^{\alpha\beta}h_{\alpha\beta}, \tag{11}$$

with

$$H^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}h^{\alpha}_{\alpha}. \tag{12}$$

If we multiply $\mathscr{T}^{\mu\nu}$ by $\bar{\xi}_\nu$ and use the fact that $K^{\mu\alpha\nu\beta}$ is antisymmetric in its first two and last two indices and symmetric with respect to interchange of the first two indices with the last two we find

$$\mathscr{T}^{0\nu}\bar{\xi}_\nu = \partial_i\left\{\sqrt{-\bar{g}}\left[\left(\bar{D}_\alpha K^{0i\nu\alpha}\right)\bar{\xi}_\nu - \tfrac{1}{2}K^{0j\nu i}W_{j\nu}\right]\right\} + \Lambda^0, \tag{13}$$

where

$$\Lambda^0 = \tfrac{1}{2}K^{0i\nu\alpha}\bar{D}_\alpha W_{i\nu} - \tfrac{1}{2}\left(\bar{D}_\alpha K^{0i\nu\alpha}\right)Z_{\nu i} + X^{0\nu}\bar{\xi}_\nu + Y^{0\nu}\bar{\xi}_\nu, \tag{14}$$

with

$$Z_{\mu\nu} = \bar{D}_\mu\bar{\xi}_\nu + \bar{D}_\nu\bar{\xi}_\mu, \tag{15}$$

$$W_{\mu\nu} = \bar{D}_\mu\bar{\xi}_\nu - \bar{D}_\nu\bar{\xi}_\mu. \tag{16}$$

Our problem is thus reduced to finding the conditions imposed on $\bar{\xi}_\mu$ by requiring that Λ^0 can be written as a total spatial derivative.

We must now choose a coordinate system which, among other things, defines the integration three-surface for eq. (4). We will choose the gauge conditions

$$g_{00} = -1, \tag{17}$$

$$h_{00} = h_{0i} = g_{0i} = 0. \tag{18}$$

In this gauge it is convenient to introduce the extrinsic curvature $\mathscr{K}_{ij} = -\Gamma^0_{ij} = -\Gamma^i_{0j}$. Then, Λ^0 can be written as

$$\Lambda^0 = \tfrac{1}{4}\partial_i\left\{\sqrt{-\bar{g}}\left(Z^i_\nu H^{0\nu} + Z^0_j H^{ij}\right)\right\} + \dot{h}^{ij}A_{ij} + h^{ij}B_{ij}, \tag{19}$$

where

$$A_{ij} = -\tfrac{1}{4}\left(\bar{g}_{ij}Z^k_k - Z_{ij}\right) \tag{20}$$

and

$$\begin{aligned} B_{ij} = -\tfrac{1}{4}\Big\{ &\bar{D}_j W^0_i - \bar{g}_{ij}\bar{D}^k W^0_k + \bar{D}_j Z^0_i - \bar{g}_{ij}\bar{D}_k Z^{0k} \\ &+ 2\left(\mathscr{K}_i{}^k Z_{kj} + \mathscr{K}_j{}^k Z_{ki}\right) - 3Z^k_k\mathscr{K}_{ij} - \bar{g}_{ij}\mathscr{K}_{kl}Z^{kl} \\ &+ \left(\bar{g}_{ij}\mathscr{K}^k_k - \mathscr{K}_{ij}\right)Z^0_0\Big\} + \tfrac{1}{2}\bar{R}^{0\nu}_{ij}\bar{\xi}_\nu + \tfrac{1}{2}\bar{R}_{ij}\bar{\xi}^0. \end{aligned} \tag{21}$$

The constraint (4) will hold for arbitrary $h_{\mu\nu}$ provided that A_{ij} and B_{ij} vanish. This gives the conditions required for $\bar{\xi}_\mu$ to be an integral constraint vector

$$\bar{D}_i\bar{\xi}_j + \bar{D}_j\bar{\xi}_i = 0 \tag{22}$$

and

$$\left\{\left(\bar{D}_i\bar{D}_j + \bar{D}_j\bar{D}_i\right) - 2\bar{g}_{ij}\bar{D}_k\bar{D}^k + 2\left(\bar{g}_{ij}\mathscr{K}^m_m - \mathscr{K}_{ij}\right)\bar{D}_0\right\}\bar{\xi}^0$$
$$= \bar{R}^{0\nu}_{ij}\bar{\xi}_\nu + \bar{R}^{0\nu}_{ji}\bar{\xi}_\nu + 2\bar{R}_{ij}\bar{\xi}^0 . \tag{23}$$

If a solution can be found to eqs. (22) and (23) then an identity of the form (4) exists with B_i given by

$$B_i = \sqrt{-\bar{g}}\left[\left(\bar{D}_\beta K^{0i\nu\beta}\right)\bar{\xi}_\nu - \tfrac{1}{2}K^{0j\nu i}W_{j\nu} + \tfrac{1}{4}\left(Z^i_\nu H^{0\nu} + Z^0_j H^{ij}\right)\right] . \tag{24}$$

Eqs. (22)–(24) are our final results. Note that although eq. (22) is just the spatial part of the equation for a Killing vector, eq. (23) is something different. If $\bar{\xi}_\mu$ is a Killing vector then eq. (23) reduces to an algebraic condition

$$\bar{g}_{ij}\bar{R}^\nu_0\bar{\xi}_\nu + \bar{R}_{ij}\bar{\xi}^0 = 0 . \tag{25}$$

If eq. (25) is not satisfied then the Killing vector will not give rise to an integral constraint on energy-momentum perturbations. Of course, if the background metric is a solution of the source-free Einstein equations, (25) becomes trivial and eqs. (4) and (24) reproduce the usual results for this case [1]. In the more general case of non-vacuum background spacetimes, results (22), (23) and (24) allow us to construct integral constraints even in the absence of any corresponding Killing vector. The results given above agree with those of refs. [2] and [5].

One of the reasons that integral constraint vectors are interesting is that the Robertson-Walker spacetimes of importance to cosmology have four constraint vectors which are not Killing vectors. The six Killing vectors of the Robertson-Walker spacetimes are also constraint vectors but since they are tangent to the standard spatial sections they give us no information about density perturbations.

As an example of how integral constraints can be used, consider an energy-momentum tensor perturbation in a closed, open or flat Robertson-Walker universe which vanishes outside some finite-sized region. Then, the boundary term in (4) vanishes for large enough integration volumes. If the perturbation velocities are small, then in a closed Robertson-Walker universe with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2\right), \tag{26}$$

the integral constraints reduce to

$$\int \mathrm{d}V \cos\chi\, \delta\rho = 0\,, \tag{27}$$

$$\int \mathrm{d}V \sin\chi\, Y_{1m}(\Omega)\, \delta\rho = 0\,. \tag{28}$$

In the case of an open universe the factors $\cos\chi$ and $\sin\chi$ are replaced by $\cosh\chi$ and $\sinh\chi$, and for a spatially flat Robertson-Walker spacetime the constraints reduce to the usual special relativistic conditions [6]

$$\int \mathrm{d}^3x\, \delta\rho = 0\,, \tag{29}$$

$$\int \mathrm{d}^3x\, \delta\rho \boldsymbol{x} = 0\,. \tag{30}$$

There exists a simple example of a family of exact solutions to the Einstein equations illustrating integral constraints in a Robertson-Walker spacetime. Consider a distribution of dust with energy density $\rho(\boldsymbol{x}, \lambda)$ describing an overdense sphere surrounded by an empty shell lying in a closed Robertson-Walker universe. The parameter λ provides a measure of the difference between the densities inside the overdense sphere and in the surrounding spacetime and also of the thickness of the empty shell. In the limit $\lambda \to 0$, the two densities become equal and the width of the empty shell goes to zero leaving an ordinary, closed, Robertson-Walker universe. The solution to the Einstein equations for this situation is a closed Robertson-Walker metric inside the overdense sphere, a Schwarzchild solution describing the empty region and finally a closed Robertson-Walker metric for the surrounding spacetime. Of course, all of these solutions must be matched at the boundaries. From these matching conditions one finds the constraint

$$\int \mathrm{d}V \cos\chi\,(\rho(r, \lambda) - \rho(r, 0)) = 0\,. \tag{31}$$

This result is a statement about two exact solutions to the Einstein equations one with nonzero λ and the other with $\lambda = 0$. The derivative of this equation with respect to λ is identical to the integral constraint we have been discussing if we consider the overdense and vacuum regions for nonzero λ to be a perturbation on the Robertson-Walker spacetime corresponding to $\lambda = 0$.

Finally, we note that it is possible to find the conditions on an integral constraint vector density of weight w which can be used to construct an integral constraint on the quantity $(-\bar{g})^{w/2}\delta T^{\mu\nu}$ plus gravitational contributions. The derivation is essentially identical to that outline above except that everything gets multiplied by factors of $(-\bar{g})^{w/2}$. If we write the weight w constraint vector as $(-\bar{g})^{-w/2}\bar{\xi}_\mu$ then the result is that eq. (22) is unchanged while eq. (23) has an additional term

$$-w\left(\bar{R}^0_\nu - \tfrac{1}{2}\bar{g}^0_\nu\bar{R}\right)\bar{\xi}_\nu \tag{32}$$

on its right-hand side. There are no solutions to this equation with $w \neq 0$ in Robertson-Walker spacetimes but nonzero weight constraints may be of some interest in other spacetimes.

References

[1] L.F. Abbott and S. Deser, Nucl. Phys. B195 (1982) 76
[2] J. Traschen, Phys. Rev. D31 (1984) 283; D29 (1984) 1563
[3] L.F. Abbott and J. Traschen, Ap.J. 302 (1986) 39
[4] R. Arnowit, S. Deser and C. Misner, *in* Gravitation: An introduction to current research, ed. L. Witten (Wiley, NY, 1962)
[5] J. Traschen, Phys. Rev. errata, to appear
[6] J. Traschen and D. Eardley, Phys. Rev. D34 (1986) 1665