

STRUCTURE OF THE YANG–MILLS VACUUM IN THE COULOMB GAUGE*

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The non-uniqueness of Yang–Mills potential in the Coulomb gauge leads to a non-trivial vacuum structure featuring vacuum fields of both integer and half-integer topological charge. Instantons fit in consistently with this picture and their interpretation is not changed. Integer and half-integer vacua are connected by certain meron solutions and the existence of half-integer charged states appears to be important for the confinement properties of the theory.

The discovery of instanton solutions to the Yang–Mills field equations [1] has revealed an interesting structure of the Yang–Mills vacuum [2]. This vacuum structure is most easily seen in the $A_0 = 0$ gauge [2] and arises from the non-uniqueness of vacuum-field potentials in this gauge. More recently, Gribov [3] has pointed out that non-uniqueness of Yang–Mills potential occurs even in the more restrictive Coulomb gauge, a so-called physical gauge. In this note, we study the Yang–Mills vacuum in the Coulomb gauge and find that Gribov's non-uniqueness leads to a non-trivial vacuum structure featuring vacuum potentials with both integer and half-integer topological charge. We find a similar non-uniqueness in the gauge transformation which takes the instanton solution into the Coulomb gauge and correlate this to our Coulomb gauge vacuum structure. The existence of half-integer charged vacuum fields does not change the effects of instantons in the theory, but rather is related to the presence of meron solutions of the field equations. Recently it has been realized by various authors that half-integer charged field configurations are those which are relevant to the confining properties of the theory. These fields give rise to a large Wilson vacuum-integral [4, 5] or a confining Coulomb force [3, 6].

We begin with a description of vacuum fields for SU(2) Yang–Mills theory in the Coulomb gauge. We will characterize these fields by the value of the topological charge

$$n = -(1/24\pi^2) \int d^3x \epsilon_{ijk} \text{Tr} \{A_i A_j A_k\}. \quad (1)$$

In the Coulomb gauge a vacuum-field potential A_μ is pure gauge

$$A = u^{-1} \nabla u, \quad A_0 = 0, \quad (2)$$

satisfying the condition

$$\nabla \cdot A = \nabla \cdot u^{-1} \nabla u = 0. \quad (3)$$

Following Gribov [3] and Wadia and Yoneya [7], we consider gauge matrices of the form

$$u = e^{i\alpha(r)} \mathbf{x} \cdot \boldsymbol{\tau} / r. \quad (4)$$

Substituting eqs. (2) and (4) into eq. (1), we find that the topological charge for such a potential is

$$n = -\pi^{-1} (\alpha(r) - \frac{1}{2} \sin 2\alpha(r)) \Big|_{r=0}^{r=\infty}. \quad (5)$$

In order to avoid singularities at $r = 0$ we must require

$$\alpha(0) = 0 \quad (\text{modulo } n\pi). \quad (6)$$

To find the value of $\alpha(\infty)$ as needed in eq. (5) we will analyse the Coulomb condition (3). For the ansatz (4), this condition becomes

$$\nabla^2 \alpha - r^{-2} \sin 2\alpha = 0. \quad (7)$$

With a change of variable $t = \ln r$, this becomes the equation of motion for a damped pendulum

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha = 0. \quad (8)$$

Condition (6) requires that we impose the boundary condition, consistent with the linearized form of the pendulum equation,

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$$\alpha(t) \xrightarrow[t \rightarrow -\infty]{} \delta e^t, \tag{9}$$

for arbitrary δ . This gives us three types of solutions to eq. (8). For $\delta = 0$, we have the trivial solution $\alpha(r) = 0$. For $\delta < 0$, we have the solution $\alpha(r) = \alpha_{1/2}(r)$ where $\alpha_{1/2}(0) = 0$ and $\alpha_{1/2}(\infty) = -\pi/2$. Finally, for $\delta > 0$ we have $\alpha(r) = -\alpha_{1/2}(r)$. Substituting these results into eq. (5) we find three types of Coulomb gauge vacuum field configurations,

$$\mathbf{A} = 0 \quad \text{with} \quad n = 0, \tag{10}$$

$$\mathbf{A} = e^{-i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \nabla e^{i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \quad \text{with} \quad n = 1/2, \tag{11}$$

and

$$\mathbf{A} = e^{i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \nabla e^{-i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \quad \text{with} \quad n = -1/2. \tag{12}$$

Note that since $A_0 = 0$ for vacuum fields in the Coulomb gauge, our Coulomb gauge vacuum potentials form a subset of those in the $A_0 = 0$ gauge. Let us define the gauge fields

$$\mathbf{A} = e^{-im\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \nabla e^{im\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r}, \tag{13}$$

$A_0 = 0$, where m is an integer. These are clearly in the $A_0 = 0$ gauge, but due to the non-linearity of eq. (7) they are not in the Coulomb gauge unless $m = 0, 1$ or -1 . For even m , these correspond to the $m/2$ vacuum fields of the $A_0 = 0$ gauge with integer topological charge, which have previously been discussed [2]. For m odd, eq. (13) generates a series of vacuum fields with half-integer topological charge. These were not considered in [2] due to the restrictive boundary condition $U \rightarrow_{r \rightarrow \infty} \pm 1$.

Conversely, we can transform $A_0 = 0$ gauge homotopically non-trivial vacua into the Coulomb gauge. A representative of an n th vacuum is given by [2]

$$\mathbf{A} = e^{-i\beta(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \nabla e^{i\beta(r)\mathbf{x}\cdot\boldsymbol{\tau}/r}, \tag{14}$$

$$\beta(r) = -n \tan^{-1}(2ar/(r^2 - a^2)). \tag{15}$$

After transforming by $U = \exp(i\alpha\mathbf{x}\cdot\boldsymbol{\tau}/r)$ and imposing the Coulomb condition, we obtain,

$$\nabla^2 \gamma - r^{-2} \sin 2\gamma = 0, \tag{16}$$

where

$$\gamma(r) = \alpha(r) + \beta(r). \tag{17}$$

Since

$$\beta(0) = 0, \quad \beta(\infty) = n\pi, \tag{18}$$

and

$$\gamma(0) = 0, \quad \gamma(\infty) = \pi/2, 0, -\pi/2, \tag{19}$$

we find

$$\alpha(0) = 0, \quad \alpha(\infty) = (\frac{1}{2} - n)\pi, -n\pi, -(\frac{1}{2} + n)\pi. \tag{20}$$

Thus the pendulum rotates exactly the right number of times to cancel the original homotopy up to $\pm 1/2$. Therefore all the homotopically non-trivial vacua in the $A_0 = 0$ gauge collapse into one of the cases $n = 0, 1/2$ or $-1/2$ when transformed into the Coulomb gauge.

Let us now consider how the instanton solutions fit in with our three vacuum fields of eqs. (10)–(12). Consider the instanton solution

$$A_\mu = (R^2/(R^2 + 1))g^{-1} \partial_\mu g, \tag{21}$$

with

$$R^2 = \sum_{i=1}^4 x_i^2 \tag{22}$$

and

$$g = (x_4 - i\mathbf{x}\cdot\boldsymbol{\tau})/R. \tag{23}$$

Note that as $R \rightarrow \infty$ the instanton field becomes a pure gauge and that

$$g \xrightarrow[t \rightarrow +\infty]{} 1 \tag{24}$$

and

$$g \xrightarrow[t \rightarrow -\infty]{} e^{-i\pi}. \tag{25}$$

We have made a detailed study of the transformation taking the above instanton solution into the Coulomb gauge. We find that this transformation is not unique but leads to three types of Coulomb gauge instantons which are gauge equivalent but topologically distinct. For large R , the Coulomb gauge instantons can be written in the form

$$A_\mu \xrightarrow[R \rightarrow \infty]{} U^{-1} \partial_\mu U, \tag{26}$$

with

$$U = e^{i\gamma(r,t)\mathbf{x}\cdot\boldsymbol{\tau}/r}. \tag{27}$$

In the three cases we find case I:

$$\gamma(r, +\infty) = 0, \quad \gamma(r, -\infty) = -\pi; \tag{28}$$

case II

$$\gamma(r, +\infty) = \alpha_{1/2}(r), \quad \gamma(r, -\infty) = -\alpha_{1/2}(r) - \pi; \quad (29)$$

and case III

$$\gamma(r, +\infty) = -\alpha_{1/2}(r), \quad \gamma(r, -\infty) = \alpha_{1/2}(r) - \pi, \quad (30)$$

defining the three types of instantons. The instanton winding number,

$$q = \frac{-1}{16\pi^2} \int d^4x \text{Tr} \{F_{\mu\nu} \tilde{F}_{\mu\nu}\} \quad (31)$$

can be written as a surface integral at large R in the form

$$q = n_+ - n_- + \Delta, \quad (32)$$

where

$$n_{\pm} = \frac{-1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} \{A_i A_j A_k\} \Big|_{t=\pm\infty} \quad (33)$$

represents the contributions from the two temporal boundaries of our surface and

$$\Delta = \frac{1}{24\pi^2} \int dt dS^i \epsilon_{i\mu\nu\alpha} \text{Tr} \{A_{\mu} A_{\nu} A_{\alpha}\} \Big|_{r=\infty} \quad (34)$$

gives the contribution from the spatial boundary. For potentials of the form (26) and (27), eqs. (33) and (34) can be explicitly evaluated with the results

$$n_{\pm} = -\pi^{-1} (\gamma(r, t) - \frac{1}{2} \sin 2\gamma(r, t)) \Big|_{r=0}^{r=\infty} \Big|_{t=\pm\infty} \quad (35)$$

and

$$\Delta = \pi^{-1} (\gamma(r, t) - \frac{1}{2} \sin 2\gamma(r, t)) \Big|_{t=-\infty}^{t=\infty} \Big|_{r=\infty}. \quad (36)$$

Substituting the three cases (28)–(30) into eqs. (35) and (36) and using the fact that $\alpha_{1/2}(0) = 0$ and $\alpha_{1/2}(\infty) = -\pi/2$ we can form the results summarized in table 1^{†1}. We can now correlate the non-uniqueness of the Coulomb gauge instanton with the vacuum structure discussed above. Clearly the type I instanton connects two $n = 0$ field configurations at $t = \pm\infty$ with all of the contribution to the winding number coming from the sides of the integration surface. The type II

^{†1} In the results of table 1 we take the limit $t \rightarrow \pm\infty$ before taking the $r \rightarrow \infty$ limit of eqs. (35) and (36). Results for the Pontryagin index q do not depend on which order these limits are taken provided that the same convention is used in both eqs. (35) and (36). This is because the terms at $r = \infty$ cancel when eqs. (35) and (36) are substituted into eq. (32).

Table 1

Instanton type	I	II	III
n_+	0	1/2	-1/2
n_-	0	-1/2	1/2
Δ	1	0	2
q	1	1	1

instanton connects an $n = -1/2$ vacuum field at $t = -\infty$ to an $n = +1/2$ potential at $t = \infty$, while the type III does just the inverse with a compensating term coming from the sides of the integration region once again.

Finally, we will discuss the relation of our half-integral vacuum fields to meron solutions [8, 4] of the Yang–Mills field equations. The $n = 1/2$ Coulomb gauge vacuum potential

$$\mathbf{A} = e^{-i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \boldsymbol{\nabla} e^{i\alpha_{1/2}(r)\mathbf{x}\cdot\boldsymbol{\tau}/r} \quad (37)$$

can be approximated, for large r , by the pure gauge field

$$\mathbf{A} = i(\boldsymbol{\tau} \times \mathbf{x})/r^2. \quad (38)$$

An important difference between eqs. (37) and (38), however, is that the singularity at $r = 0$ of eq. (38) is smoothed over in eq. (37) and does not appear there. Both eqs. (37) and (38) satisfy the Coulomb condition. A particular form of the meron solution is

$$A_{\mu} = -i\sigma_{\mu\nu} x_{\nu}/R^2. \quad (39)$$

When this is transformed into the Coulomb gauge we find the simple form

$$\mathbf{A} = \frac{1}{2} i(1 + t/R)(\boldsymbol{\tau} \times \mathbf{x})/r^2. \quad (40)$$

Thus, this meron can be considered as a transition field from the $n = 0$ potential $\mathbf{A} = 0$ at $t = -\infty$ to the potential of eq. (38) with $n = 1/2$ at $t = +\infty$. Thus, the meron represents a transition from integer to half-integer vacuum states. Note, however, that the above meron solution connects the $\mathbf{A} = 0$ vacuum to the singular $n = 1/2$ vacuum of eq. (38). We want a field which interpolates between $\mathbf{A} = 0$ and the non-singular field of eq. (37). Furthermore, this solution has an infinite action.

Now the important question is whether there are finite-action transitions between integer and half-integer worlds. At present it appears unlikely that

there exist finite-action solutions with a fractional index, however, even in this case there are trajectories in the functional space which start off one of the worlds at $t = -\infty$, comes close to the other world at finite t and returns to the original one at $t = +\infty$. These trajectories would have a finite action. Then the physical situation is very much like a resonance between different worlds. The Coulomb gauge seems to be a convenient one to describe this phenomenon.

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Note added. After the completion of this work we received a CERN preprint by S. Scuito in which our type II instanton is discussed. This subject has also

been discussed in a recent Brookhaven preprint by R. Jackiw, I. Muzinich and C. Rebbi.

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