CHARGE DEFINITION IN NON-ABELIAN GAUGE THEORIES ☆

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Conserved gauge-invariant electric and magnetic charges are defined for non-abelian gauge theories in terms of the asymptotic symmetries of the field configurations. They are expressed as flux integrals. Illustrations include the magnetic charge of the 't Hooft–Polyakov monopole and the electric and magnetic charges of the Julia–Zee dyon.

1. Introduction. Despite the formal similarities between Yang–Mills and Maxwell theories, it is well-known that the definition of charge is much more subtle in the non-abelian case because the field strengths, rather than being invariants, transform as gauge vectors. The same situation occurs in gravity where the curvatures are gauge tensors and the "charges" include energy; but there is by now general agreement, both on intuitive and formal levels, as to the correct definition of gravitational energy. It is our purpose here to provide a framework for defining charges in Yang–Mills theory corresponding closely to that of gravity. We shall follow a recent treatment [1] of gravity and supergravity in which the role of asymptotic symmetries of field configurations (not necessarily those of flat space) was particularly emphasized in defining conserved quantities. Charges of both electric and magnetic type will be defined. Our principal explicit applications will be to the 't Hooft–Polyakov [2] monopole, and to the Julia–Zee [3] dyon as an illustration of the more straightforward notion of electric charge. The definitions can also be applied in euclidean formulation, although there the idea of a conservation law is less physical than in the hyperbolic case ‡1.

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‡1 An example would be instanton solutions, with respect to their asymptotic (pure vacuum) background, defining charge in terms of flux integrals over arbitrary two-spheres at infinity. Although instanton solutions do not themselves possess Killing vectors [4], it is the background's symmetries that are relevant.

2. Formalism. In matrix notation ($A_\mu = A_\mu^a T^a$ where T^a are group generators), the Yang–Mills field equations are

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = J^\nu \,, \tag{1}$$

where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \,.$$

Let us expand the gauge field A_μ about a background field $\bar{A}_\mu$ which is a solution of the source-free (for simplicity) field equations. The particular form of $\bar{A}_\mu$ is to be chosen so that A_μ approaches $\bar{A}_\mu$ at spatial infinity. Since we are interested in field configurations with bounded sources, this represents no real restriction. We write

$$A_\mu = \bar{A}_\mu + a_\mu \,.$$

The field a_μ is *not* necessarily a small perturbation but, of course, it does vanish at spatial infinity. Since $\bar{D}_\mu \bar{F}^{\mu\nu} \equiv \partial_\mu \bar{F}^{\mu\nu} - ig[\bar{A}_\mu, \bar{F}^{\mu\nu}] = 0$, the field equations (1) may be rearranged to read

$$\bar{D}_\mu f^{\mu\nu} - ig[a_\mu, \bar{F}^{\mu\nu}] = j^\nu \,, \tag{2}$$

where

$$f_{\mu\nu} = \bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu \,, \quad j^\nu = J^\nu - (D_\mu F^{\mu\nu})_{\rm N} \,.$$

Here $(D_\mu F^{\mu\nu})_{\rm N}$ represents the remaining terms in $D_\mu F^{\mu\nu}$ which are of quadratic or higher order in a_μ.

The left side of eq. (2) satisfies a background covariant conservation law since $\bar{D}_\nu \bar{D}_\mu f^{\mu\nu} = \frac{1}{2} ig[\bar{F}_{\mu\nu}, f^{\mu\nu}]$ and $\bar{D}_\nu [a_\mu, \bar{F}^{\mu\nu}] = -\frac{1}{2}[f_{\mu\nu}, \bar{F}^{\mu\nu}]$. (This conservation follows from expanding the identity $D_\mu D_\nu F^{\mu\nu} \equiv 0$ to first order in a_μ.) Thus, the total current j^ν is covariantly conserved,

$$\bar{D}_\nu j^\nu = 0 . \quad (3)$$

To obtain useful conservation laws, however, we need currents which are conserved in the ordinary, not in the covariant sense: they must be gauge-singlets. This can be done if (and only if) the background field $\bar{A}_\mu$ has symmetries. To each of these is associated a Killing vector, namely a covariantly constant matrix field $\bar{\xi}^s$

$$\bar{D}_\mu \bar{\xi}^s = 0 , \quad (4)$$

where the label s distinguishes the various Killing vectors. Each of them is the parameter of an infinitesimal gauge transformation which leaves the field $\bar{A}_\mu$ invariant. Covariant differentiation of eq. (4) implies that $[\bar{F}_{\mu\nu}, \xi^s] = 0$. Eqs. (3) and (4) show that the quantity $\mathrm{tr}\{\bar{\xi}^s j^\nu\}$ is covariantly conserved, but since it is a background-gauge singlet, covariant and ordinary differentiations are identical

$$\partial_\nu \, \mathrm{tr}\{\bar{\xi}^s j^\nu\} = 0 .$$

Thus, the singlet currents $\mathrm{tr}\{\bar{\xi}^s j^\nu\}$ are conserved in the ordinary sense. With each of these is associated a charge

$$Q^s_\mathrm{E} = \frac{1}{4\pi}\int \mathrm{d}^3x \; \mathrm{tr}\{\bar{\xi}^s j^0\} .$$

This charge can be evaluated, using the field equation (2), as

$$Q^s_\mathrm{E} = \frac{1}{4\pi}\int \mathrm{d}^3x \; \mathrm{tr}\{\xi^s(\bar{D}_\mu f^{\mu 0} - \mathrm{i}g[a_\mu, \bar{F}^{\mu 0}])\}$$
$$= \frac{1}{4\pi}\int \mathrm{d}^3x \; \partial_i \, \mathrm{tr}\{\bar{\xi}^s f^{i0}\} , \quad (5)$$

where properties of the Killing vectors given above have been used. Finally, the integral in eq. (5) can be put into the flux integral form

$$Q^s_\mathrm{E} = -\frac{1}{4\pi}\int \mathrm{d}\boldsymbol{S} \cdot \mathrm{tr}[\boldsymbol{\nabla}\{\bar{\xi}^s a^0\} + \partial_0 \{\bar{\xi}^s \boldsymbol{a}\}] . \quad (6)$$

Note that this result is exactly like the definition of charge in electrodynamics with $\mathrm{tr}\{\bar{\xi}^s f^{i0}\}$ and $\mathrm{tr}\{\bar{\xi}^s a_\mu\}$ playing the role of the electric field and vector potential.

The number of charges which can be defined through eq. (6) depends on the number of Killing vectors in the background field $\bar{A}_\mu$. Physically, the most important case is that of vanishing background field strength when $\bar{A}_\mu$ is just a pure gauge

$$\bar{A}_\mu = (\mathrm{i}/g) G^{-1} \partial_\mu G .$$

Here the Killing vectors can be represented by

$$\bar{\xi}^s = G^{-1} T^s G ,$$

and their number equals the number of group generators T^s. In the quantum theory, the charges corresponding to these Killing vectors will be generators of symmetry transformations and, although background gauge singlets, will satisfy the non-abelian algebra

$$[Q^s_\mathrm{E}, Q^t_\mathrm{E}] = \mathrm{i} f^{stu} Q^u_\mathrm{E} ,$$

where

$$[\bar{\xi}^s, \bar{\xi}^t] = \mathrm{i} f^{stu} \bar{\xi}^u .$$

3. Magnetic charges, monopoles and dyons. Formally, magnetic charges can be defined for non-abelian gauge theories following the approach of section 2 but starting with the identity [‡2]

$$D_\mu {}^*F^{\mu\nu} = 0 \quad (7)$$

as the counterpart of (1), where

$${}^*F^{\mu\nu} = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} .$$

Expanding around a background $\bar{A}_\mu$ as before (with field strengths replaced by their duals) we find the covariantly conserved current $(\bar{D}_\mu {}^*f^{\mu\nu} - \mathrm{i}g[a_\mu, {}^*\bar{F}^{\mu\nu}])$. This can once again be transformed into an ordinary conservation law through the use of Killing vectors, leading to the conserved magnetic charges

$$Q^s_\mathrm{M} = \frac{1}{4\pi}\int \mathrm{d}^3x \; \partial_i \, \mathrm{tr}\{\bar{\xi}^{s*} f^{i0}\}$$
$$= \frac{1}{4\pi}\int \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{\nabla} x \; \mathrm{tr}\{\bar{\xi}^s \boldsymbol{a}\} . \quad (8)$$

Note that this is exactly like magnetic charge in electrodynamics with $\mathrm{tr}\{\bar{\xi}^s \, {}^*f^{i0}\}$ and $\mathrm{tr}\{\bar{\xi}^s \boldsymbol{a}\}$ playing the role

‡2 The gravitational analog of (7) is the uncontracted Bianchi identity $D_\mu {}^*R^{*\mu\nu ab} \equiv 0$ where ${}^*R^*$ is the (double) dual of the curvature on its world and local indices. Summing over $\nu = a$ gives the contracted identity $D_\mu G^{\mu\nu} \equiv 0$ on which the energy definition is based [1]. It would be of interest to see whether there exist solutions for which the remaining (traceless) part of the Bianchi identity could define topological charge of the magnetic type.

of the magnetic field and vector potential. Just as in electrodynamics, eq. (8) will always lead to *vanishing* magnetic charges unless $\mathrm{tr}\{\bar{\xi}^s a\}$ has the singular properties of the vector potential for a magnetic monopole, namely a string. As we shall see this is exactly what happens in the 't Hooft–Polyakov solution [‡3].

In a spontaneously broken gauge theory, there is an additional consideration in defining a Killing vector. As we have noted, a background Killing vector is the parameter of a transformation which leaves the background gauge field $\bar{A}_\mu$ invariant. When a symmetry-breaking scalar vacuum expectation $\bar{\phi}$ is also present, this gauge transformation must also leave $\bar{\phi}$ unchanged for the Killing vector to correspond to a symmetry of the full vector–scalar system. Killing vectors which do not leave $\bar{\phi}$ invariant correspond to broken symmetries of the theory and hence will not lead to meaningful charges. If the scalar field ϕ is in the adjoint representation, then the further condition is just

$$[\bar{\xi}^s, \bar{\phi}] = 0 . \tag{9}$$

The original solution of 't Hooft and Polyakov [2] was in an SO(3) gauge theory with a triplet of scalars in the form

$$\phi = F(r)\, \boldsymbol{\sigma} \cdot \hat{r} , \quad F(r) \underset{r \to \infty}{\to} F .$$

Therefore, to satisfy eq. (9) we must choose the Killing vector to be parallel to ϕ. This can be obtained by gauge rotating the trivial Killling vector $\bar{\xi} = \sigma_3$ (or equivalently $\boldsymbol{\sigma} \cdot \hat{n}$) into the "hedgehog" configuration

$$\bar{\xi} = \boldsymbol{\sigma} \cdot \hat{r} . \tag{10}$$

This is necessarily a unit vector, since it results from rotating a unit vector. The background gauge field for the monopole is just the vacuum ($\bar{F}_{\mu\nu} = 0$); the pure gauge potential $\bar{A}_\mu$ must be written in a gauge which admits (10) as a Killing vector. This pure gauge field can be obtained from the trivial field $\bar{A}_\mu = 0$ (with Killing vector $\bar{\xi} = \sigma_3$) by performing the gauge rotation mentioned above [‡4],

$$G = \exp(\mathrm{i}\tfrac{1}{2}\phi\sigma_3) \exp(\mathrm{i}\tfrac{1}{2}\theta\sigma_2) \exp(-\mathrm{i}\tfrac{1}{2}\phi\sigma_3) ,$$

so that

$$\bar{\xi} = G^{-1}\sigma_3 G = \boldsymbol{\sigma} \cdot \hat{r}$$

and

$$\bar{A}_\mu = (\mathrm{i}/g)\, G^{-1}\partial_\mu G . \tag{11}$$

This $\bar{A}_\mu$ is singular along the negative z axis. In order to describe the background field everywhere, we must introduce a second field

$$\bar{A}'_\mu(x, y, z) = \bar{A}_\mu(x, y, -z) .$$

This field also admits (10) as a Killing vector and is singular along the positive z-axis but *regular* along the negative z-axis. We must use both gauge field patches to cover all of space with a background field which is non-singular. We will use the field $\bar{A}_\mu$ everywhere but in an infinitesimal region around the negative z-axis, which will be covered by $\bar{A}'_\mu$. There is another important singularity in this system, that of the Killing vector (10) at the origin. This results in a weak modification of the Killing equation at the origin,

$$\bar{D}_i \bar{\xi} = 4\pi\delta^3(r)\hat{r}_i r^2 . \tag{12}$$

Thus, in the case of the monopole, condition (9) has also forced us to use a Killing vector with singular properties at the origin. It is the presence of these singularities which allows a non-vanishing value of the magnetic charge for the 't Hooft–Polyakov solution. Note that the charge defined by eq. (8) is still conserved, despite (12), as may be checked explicitly.

Since we are using a background field $\bar{A}_\mu$ (or $\bar{A}'_\mu$) we must expand the monopole solution about this background so that

$$a_\mu = A_\mu^{(\mathrm{monopole})} - \bar{A}_\mu . \tag{13}$$

Since our background field is a pure gauge, $\bar{F}_{\mu\nu} = 0$, the current upon which we base our magnetic charge

‡3 Our formalism does not, of course, guarantee that there will be non-vanishing magnetic charge for a given configuration. Indeed, one of the important results in this area is that this can only occur when the symmetry is broken down to U(1). For a recent review and references to literature on monopoles, see ref. [5].

‡4 Since the charge is a background gauge invariant, one could also evaluate it in the ordinary $\bar{A}_i = 0$ gauge, with constant Killing vector $\bar{\xi} = \sigma_3$. The singularities would then reappear in the gauge transformed A_i, which would rotate according to the inverse of that defined by G of (11). Note that this procedure is *not* the same as using $A_i = 0$ together with the untransformed A_i; as explained in the text, this choice would not correspond to alignment with the physically unbroken direction.

is just $\bar{D}_\mu\, {}^*f^{\mu\nu}$. Using the definition of $f^{\mu\nu}$ given above one finds that this current vanishes identically, as follows from the original identity (7). However, the field we must consider for the magnetic conservation law is $\mathrm{tr}\{\bar{\xi}\, {}^*f^{\mu\nu}\}$ and from eq. (12) we find that

$$\partial_i \,\mathrm{tr}\{\bar{\xi}\, {}^*f^{i\nu}\} = 4\pi r^2 \delta^3(r)\, \mathrm{tr}\{\bar{\xi}\, {}^*f^{i\nu}\}\, \hat{r}_i \ . \tag{14}$$

Therefore, if the singularity in the Killing vector at the origin multiplies a $1/r^2$ singularity in the field $\mathrm{tr}\{\bar{\xi}\, {}^*f^{i\nu}\}$, then the right side of eq. (13) can act as a point-like monopole charge for this field. This is exactly how the 't Hooft–Polyakov solution develops its non-vanishing magnetic charge.

The value of the charge can be obtained from our definition (8) either by performing the volume integra. [taking (12) into account] or by using the surface integral. In the latter approach, one must recall that the background field $\bar{A}_\mu$ was defined on two patches, so the surface integral (which we take to be over the sphere at infinity) must correspondingly be divided into two regions. One is a small patch around the south pole of the sphere and the other the rest of the sphere. The surface integral can be expressed as a line integral around their boundary, which is an infinitesimal circle surrounding the negative z axis. Since the monopole field has asymptotic form

$$A^{(\mathrm{monopole})} \sim (\mathrm{i}/g)(\boldsymbol{\sigma} \times \hat{\boldsymbol{r}})/r \ ,$$

$\mathrm{tr}\{\bar{\xi} A_\mu^{(\mathrm{monopole})}\} = 0$ for $\bar{\xi}$ given by (10). Thus, curiously, the monopole field itself will not contribute directly to the charge flux integral [‡5]. Combining the above remarks with the definition (8), we find that

$$Q_\mathrm{M} = \frac{1}{4\pi} \oint \mathrm{d}\boldsymbol{l} \cdot \{\mathrm{tr}\{\bar{\xi} a'\} - \mathrm{tr}\{\bar{\xi} a\}\}|_{\theta\to\pi, r\to\infty}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}\phi\, \rho\, \mathrm{tr}\{\bar{\xi}(\bar{A}_\phi - \bar{A}'_\phi)\}|_{\theta\to\pi, r\to\infty} \ .$$

From eq. (11),

$$\mathrm{tr}\{\bar{\xi}\bar{A}_\phi\}|_{\theta\to\pi, r\to\infty} = 2/g\rho \ ,$$

$$\mathrm{tr}\{\bar{\xi}\bar{A}'_\phi\}|_{\theta\to\pi, r\to\infty} = 0 \ ,$$

‡5 Although this makes it appear that monopoles are present purely from the kinematics of the background "hedgehog" gauge choice, [or through the pure background Killing term in the form (16) below], this is of course not the case and dynamics is indeed critical. See the discussion in ref. [6] and also footnote 3.

so we obtain the usual monopole charge $Q_\mathrm{M} = 1/g$.

The evaluation of the magnetic charge of the Julia–Zee dyon [3] proceeds exactly as above. In addition, this solution has an electric charge arising from its non-vanishing scalar potential ($\bar{A}_0 = 0$ here),

$$A_0 = a_0 \underset{r\to\infty}{\to} [(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})/2gr](cr + d) \ ,$$

where c and d are constants.

Substituting a_0 together with the Killing vector (10) (which is Killing for the dyon as well as for the monopole) into the definition (6) results in the usual value

$$Q_\mathrm{E} = -\frac{1}{4\pi} \int \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{\nabla}(c/g + d/gr) = d/g \ .$$

We conclude with some remarks relating our formalism with some other approaches to the monopole charge appearing in the literature [2,5–7]. First, as has been noted in ref. [6], the conserved magnetic charge is not due to any additional U(1) symmetry in the theory. In our formulation, this is a natural consequence of the fact that magnetic charge conservation is based on the identity (7) and not on any dynamical equation of motion. Thus, there are no Noether currents or symmetries associated with it.

Second, our discussion of magnetic and electric charges for the monopole and dyon solutions has been based on the identification of the quantity

$$F^{\mu\nu}_\mathrm{EM} = \mathrm{tr}\{\bar{\xi} f^{\mu\nu}\} = \mathrm{tr}\{\partial^\mu(\bar{\xi} a^\nu) - \partial^\nu(\bar{\xi} a^\mu)\} \tag{15}$$

as the electromagnetic field tensor. In his original discussion [2], 't Hooft suggested using

$$F^{\mu\nu}_\mathrm{EM} = \mathrm{tr}\{\hat{\boldsymbol{\phi}} F^{\mu\nu} + (\mathrm{i}/4g)\hat{\boldsymbol{\phi}}[D^\mu\hat{\boldsymbol{\phi}}, D^\nu\hat{\boldsymbol{\phi}}]\} \ , \tag{16}$$

where $\hat{\boldsymbol{\phi}} = \phi/|\phi|$, which is also our $\bar{\xi}$. Eq. (15) can also be written as [6]

$$F^{\mu\nu}_\mathrm{EM} = \mathrm{tr}\{\partial^\mu(\hat{\boldsymbol{\phi}} A^\nu) - \partial^\nu(\hat{\boldsymbol{\phi}} A^\mu) + (\mathrm{i}/4g)\hat{\boldsymbol{\phi}}[\partial^\mu\hat{\boldsymbol{\phi}}, \partial^\nu\hat{\boldsymbol{\phi}}]\} \ . \tag{17}$$

The Killing equation (4) and the fact that $\bar{F}_{\mu\nu} = 0$ straightforwardly imply that our expression (15) is identical to (16) or (17). While all valid definitions of $F^{\mu\nu}_\mathrm{EM}$ must agree at infinity, they need not do so in the interior, where an unambiguous isolation of electromagnetic forces is impossible. It is amusing therefore that 't Hooft's original field definition for the monopole coincides everywhere with that based on the present framework.

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